Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/12S113

The Extended Log-Logistic Distribution: Properties, Inference, and Applications in Medicine and Geology

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Received: 7 Jun. 2023, Revised: 21 Sep. 2023, Accepted: 5 Nov. 2023
Published online: 1 Dec. 2023

Abstract: In this paper, a new flexible extension of the log-logistic model called the extended odd Weibull log-logistic (EOWLL) distribution is studied. The EOWLL density can be expressed as a mixture of Dagum densities. The EOWLL distribution provides decreasing, increasing, upside-down bathtub, and reversed-J-shaped hazard rates, and right-skewed, symmetrical, and left-skewed densities. Its mathematical properties are derived. The EOWLL parameters are estimated via eight classical methods of estimation. Additionally, extensive simulations are obtained to explore the performance of the proposed methods for small and large samples. Two real-life sets of data from medicine and geology are analyzed, showing the flexibility of the EOWLL distribution as compared to other log-logistic extensions. The results show that the EOWLL distribution is more appropriate as compared to the Kumaraswamy transmuted log-logistic, alpha-power transformed log-logistic, and additive Weibull log-logistic distributions, among others.

Keywords: log-logistic distribution; hazard function, maximum likelihood; statistical model; maximum product of spacing; real data; simulation

1 Introduction

Developing and proposing more flexible distributions by extending well-known classical distributions or introducing a new family has become a more crucial topic to model various real-life data sets such as financial returns, medicine, and geology, among others. Several authors have studied different generalized forms of the log-logistic (LL) distribution to improve its capability and flexibility. Some notable examples are the following: Kumaraswamy-LL [1], beta-LL [2], Marshall-Olkin-LL [3], McDonald-LL[4], Zografos-Balakrishnan-LL [5], odd Lomax-LL [6], extended-LL [7] and [8], Fréchet Topp–Leone LL [9], alpha power transformed-LL [11], generalized-LL [12], transmuted four-parameter generalized-LL [13], a new three-parameters-LL [14], exponentiated-LL geometric [15], the LL Weibul [16], transmuted-LL [17], exponentiated-LL [18] distributions.

In this paper, we introduce a new version of the LL distribution called the extended odd Weibull log-logistic (EOWLL) distribution, which can provide more flexibility in modeling medicine and geology data than other competing models. The EOWLL distribution is constructed based on the extended odd Weibull-G (EOW-G) family proposed by Alizadeh et al. [19]. The LL model is used as the baseline model in the EOW-G family to generate the new EOWLL distribution. Some mathematical properties of the EOWLL model are derived.

The EOWLL distribution exhibits increasing, decreasing, and unimodal hazard rates, whereas its density EOWLL model exhibits right-skewed, reversed-J shaped, or concave down and it can expressed as a mixture of Dagum densities. Two applications to real-life data from medicine and geology show that the EOWLL model performs better than seven other competing LL lifetime distributions, motivating its use in applied areas.

Furthermore, the behavior of the EOWLL unknown parameters for several sample sizes and for several parameter combinations is investigated using eight different estimation procedures. A guideline for selecting the optimum

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estimation method to estimate the EOWLL parameters is developed, which we believe applied statisticians and reliability engineers would find useful. These methods are compared based on numerical simulations. Numerical simulations are also used to examine the bias and mean square errors of the proposed estimators. The best estimation approach and ordering performance of these estimators are discussed using the partial and overall ranks of all methods of estimation for several parametric combinations.

Estimating the parameters of generalized models using classical estimation methods has been discussed by several authors. For example, the interested reader can see [20], [21], [22], and [23].

The rest of the paper is organized as follows: Section 2 introduces the EOWLL distribution. Mathematical properties of the model are derived in Section 3. Section 4 presents the estimators of the EOWLL parameters based on eight classical estimation methods. A simulation study is discussed in Section 5. Section 6 illustrates the applications of the EOWLL model in modeling real-life datasets. Section 7 gives some conclusions.

2 The EOWLL Distribution

In this section, we define the proposed EOWLL distribution. The probability density function (PDF) of the LL distribution has the form

$$f(x;\lambda,\alpha) = \alpha\lambda x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-2}, \; x>0, \tag{1}$$

where $\lambda > 0$ and $\alpha > 0$ are the scale and shape parameters, respectively.

The EOWLL distribution is constructed by using the LL model as a baseline model in the EOW-G family due to Alizadeh et al. [19]. The cumulative distribution function (CDF) of the EOW-G family is given by

$$F(x;\beta,\theta,\boldsymbol{\xi}) = 1-\left\{1+\beta\left[\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right]^{\theta}\right\}^{\frac{-1}{\beta}}, \; x\in\Re,\, \beta,\, \theta>0, \tag{2}$$

where $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi}) = dG(x;\boldsymbol{\xi})/dx$ denote the CDF and PDF of a baseline model with parameter vector $\boldsymbol{\xi}$. The corresponding PDF of (2) reduces to

$$f(x;\beta,\theta,\boldsymbol{\xi}) = \frac{\theta g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{\theta-1}}{[1-G(x;\boldsymbol{\xi})]^{\theta+1}}\left\{1+\beta\left[\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right]\right\}^{\frac{-1}{\beta}-1}. \tag{3}$$

By inserting the CDF of the LL model in Equation (2), the CDF of the EOWLL distribution follows as

$$F(x;\underline{\eta}) = 1-\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}, \quad x>0, \tag{4}$$

where $\underline{\eta} = (\alpha,\beta,\lambda,\theta)^{\intercal}$, α , β , and θ are positive shape parameters and $\lambda > 0$ is a scale parameter.

The PDF of the EOWLL distribution takes the form

$$\begin{aligned} f(x;\underline{\eta}) =& \theta\alpha\lambda x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{\theta-1}\left[1-\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-1}\right]^{-(\theta+1)} \\ &\times\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-(\frac{1}{\beta}+1)}. \end{aligned} \tag{5}$$

The survival function (SF) of the EOWLL distribution reduces to

$$S(x;\underline{\eta}) = \left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}. \tag{6}$$

The general form of a long term SF is $S_{Long-Term}(x;p,\underline{\eta}) = p + (1-p)S(x;\underline{\eta})$, where $S(x;\eta)$ is the survival function of any distribution and p denotes the probability of being cured. Based on the SF, the PDF (improper) can be easily derived as

$$f_{Long-Term}(x;p,\underline{\eta}) = -\frac{\partial}{\partial x} S_{Long-Term}(x;p,\underline{\eta}) = (1-p)f(x;p,\underline{\eta}),\ p \in (0,1). \tag{7}$$

Hence, the PDF of the long-term EOWLL distribution follows as

$$\begin{aligned} f_{Long-Term}(x;p,\underline{\eta}) =& \frac{\theta\alpha\lambda(1-p)}{x^{\alpha+1}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-(\theta+1)}} \left[1-\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-1}\right]^{-(\theta+1)} \\ & \times \left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-(\frac{1}{\beta}+1)},\ x>0. \end{aligned} \tag{8}$$

The hazard rate function (HRF) of the EOWLL distribution is derived as

$$h(x;\underline{\eta}) = \frac{\theta\alpha\lambda x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{\theta-1}\left[1-\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-1}\right]^{-(\theta+1)}}{\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}}. \tag{9}$$

The reversed HRF is

$$r(x;\underline{\eta}) = \frac{\theta\alpha\lambda x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{\theta-1}\left[1-\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-1}\right]^{-(\theta+1)}}{\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{(\frac{1}{\beta}+1)}\left[1-\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}\right]}. \tag{10}$$

The odds function of the EOWLL distribution reduces to

$$O(x;\underline{\eta}) = \frac{1-\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}}{\left\{1+\beta\left[\left(1+\frac{\lambda}{x^{\alpha}}\right)-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}}. \tag{11}$$

Figure 1 presents some shapes of the OEWLL PDF for different values of the parameters α, β, λ and θ. The shape of the PDF of the EOWLL model can be right-skewed, reversed-J shaped, or concave down.

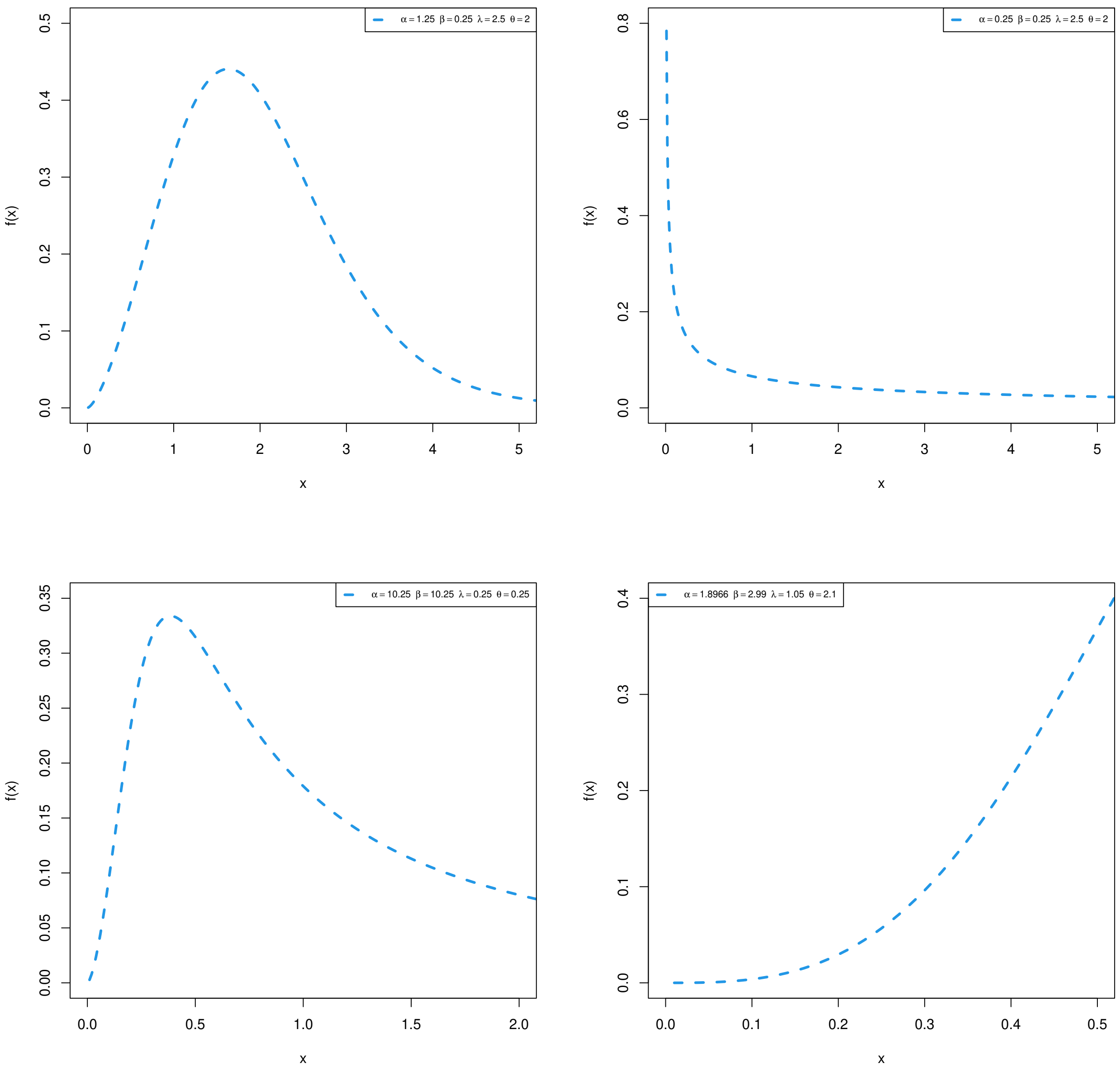

Fig. 1: Plots of the EOWLL density for different parametric values

Figure 2 displays some possible shapes of the HRF of the EOWLL for some selected values of α, β, λ, and θ. The shape of HRF can be increasing, decreasing, reversed-J shaped, J-shaped, or unimodal hazard rates.

3 Some Properties

In this section, we derive some mathematical properties of the EOWLL distribution including linear representation, moments, generating function, incomplete moments, mean residual life and mean inactivity time, entropies, Bonferroni and Lorenz curves, and order statistics.

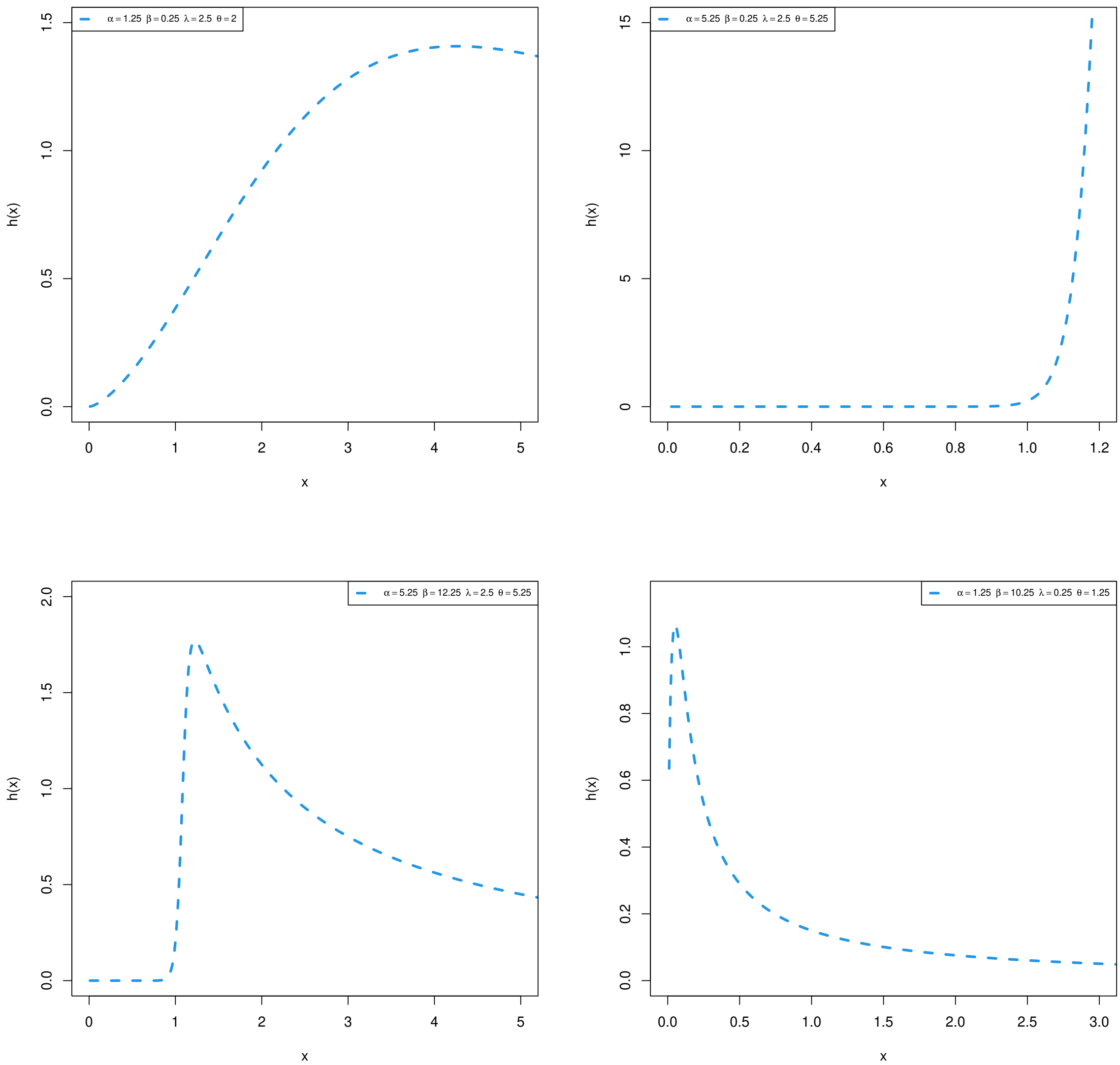

Fig. 2: Plots of the EOWLL HRF for different parametric values

3.1 Quantile Function

The quantile function (QF) of the EOWLL distribution (5) follows as

$$F^{-1}(u) = \lambda^{\frac{1}{\alpha}} \left(\frac{\beta^{\frac{1}{\theta}} + \left[(1-u)^{-\beta} - 1\right]^{\frac{1}{\theta}}}{\left[(1-u)^{-\beta} - 1\right]^{\frac{1}{\theta}}} - 1 \right)^{-\frac{1}{\alpha}}, 0 < u < 1. \tag{12}$$

The median of the EOWLL distribution follows, by substituting $u = 0.5$ in Equation (12), as

$$Median = \lambda^{\frac{1}{\alpha}} \left(\frac{\beta^{\frac{1}{\theta}} + \left[(0.5)^{-\beta} - 1\right]^{\frac{1}{\theta}}}{\left[(0.5)^{-\beta} - 1\right]^{\frac{1}{\theta}}} - 1 \right)^{-\frac{1}{\alpha}} \tag{13}$$

Other quantiles can be obtained from Equation (12) when the appropriate value of u is substituted.

3.2 Linear Representation

We provide a useful linear representation for the PDF of the EOWLL distribution. Alizadeh et al. [19] derived a mixture representation of the EOW-G density as follows

$$f(x) = \sum_{k,j=0}^{\infty} a_{k,j} h_{\theta k+j}(x), \tag{14}$$

where $a_{k,j} = -\beta^k \Gamma(\theta k + j)(-1/\beta)_k / k! j! \Gamma(\theta k)$ and $h_{\theta k+j}(x) = (\theta k + j) g(x) G(x)^{\theta k+j-1}$ is the Exp-G density with positive power parameter $(\theta k + j)$. Using the PDF and CDF of the LL distribution, Equation (14) can be rewritten as

$$\begin{aligned} f(x) &= \sum_{k,j=0}^{\infty} a_{k,j} (\theta k + j) \alpha \lambda x^{-(\alpha+1)} \left(1 + \frac{\lambda}{x^\alpha}\right)^{-2} \left[\left(1 + \frac{\lambda}{x^\alpha}\right)^{-1}\right]^{\theta k+j-1} \\ &= \sum_{k,j=0}^{\infty} a_{k,j} (\theta k + j) \alpha \lambda x^{-(\alpha+1)} \left(1 + \frac{\lambda}{x^\alpha}\right)^{-(\theta k+j)-1} \\ &= \sum_{k,j=0}^{\infty} a_{k,j} \delta \alpha \lambda x^{-(\alpha+1)} \left(1 + \frac{\lambda}{x^\alpha}\right)^{-\delta-1}, \end{aligned} \tag{15}$$

where $\delta = (\theta k + j)$. Equation (15) can be expressed as

$$f(x) = \sum_{k,j=0}^{\infty} a_{k,j}\, g(x) \tag{16}$$

where $g(x) = \delta \alpha \lambda x^{-(\alpha+1)} \left(1 + \frac{\lambda}{x^\alpha}\right)^{-\delta-1}$ denotes Dagum density with scale parameter λ and shape parameters α and δ. Then, the EOWLL PDF can be expressed as a double linear combination of Dagum densities. Let Z be a random variable having the Dagum distribution$(\delta, \alpha, \lambda)$. Then, the rth ordinary and MGF of Z are

$$\mu'_{r,z} = \lambda^{\frac{r}{\alpha}} \frac{\Gamma\left(1 - \frac{r}{\alpha}\right) \Gamma\left(\delta + \frac{r}{\alpha}\right)}{\Gamma(\delta)} \text{ and } M_z(t) = \lambda^{\frac{i}{\alpha}} \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{\Gamma\left(1 - \frac{i}{\alpha}\right) \Gamma\left(\delta + \frac{i}{\alpha}\right)}{\Gamma(\delta)}$$

3.3 Moments

The rth moment of X follows simply from Equation (16) as

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} a_{k,j} \lambda^{\frac{r}{\alpha}} \frac{\Gamma\left(1 - \frac{r}{\alpha}\right) \Gamma\left(\delta + \frac{r}{\alpha}\right)}{\Gamma(\delta)}.$$

Based on Equation (16), the moment generating function (MGF) of the EOXLL distribution takes the form

$$M(t) = \sum_{k,j,i=0}^{\infty} a_{k,j} \frac{x^i}{i!} \lambda^{\frac{r}{\alpha}} \frac{\Gamma\left(1 - \frac{r}{\alpha}\right) \Gamma\left(\delta + \frac{r}{\alpha}\right)}{\Gamma(\delta)}.$$

The sth incomplete moment, say $\varphi_s(t)$ of X can be expressed, from Equation (16), as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x) dx = \sum_{k,j=0}^{\infty} a_{k,j} \int_{-\infty}^{t} x^s g(x) dx \tag{17}$$

The first incomplete moment, say $\varphi_1(t)$, given by Equation (17) with $s = 1$. A general equation for $\varphi_1(t)$ can be derived from Equation (17) as

$$\varphi_1(t) = \sum_{k,j=0}^{\infty} a_{k,j} l(t), \tag{18}$$

where $l(t) = \int_{-\infty}^{t} x^s g(x) dx$ is the first incomplete moment of the Dagum distribution.

3.4 Mean Residual Life and Mean Inactivity Time

The mean residual life (MRL) represents the expected additional life length for a unit, which is alive at age t and is defined by $MRL_x(t) = E(X-t|X>t$, for $t>0$. The MRL of X is

$$MRL_X(t) = \frac{[1-\varphi_1(t)]}{S(t)} - t, \tag{19}$$

where $S(t)$ is the SF of the EOWLL distribution. By inserting (18) in Equation (19), we get

$$MRL_X(t) = \frac{1}{S(t)}\left[1-\sum_{k=0}^{\infty} a_k l(t)\right] - t.$$

The mean inactivity time (MIT) represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT is defined by $MIT_X(t) = E(t-X|X\le t$, for $t>0$. The MIT of X is

$$MIT_X(t) = t - \frac{\varphi_1(t)}{F(t)}. \tag{20}$$

Combining Equation (18) and Equation (20), the MIT of the EOWLL model follows as

$$MIT_X(t) = t - \frac{1}{F(t)}\sum_{k=0}^{\infty} a_k l(t).$$

3.5 Entropies

The entropy of a random variable X is a measure for the variation of the uncertainty. If X has the PDF $f(x)$, then the Shannon entropy ([24]) is defined by

$$H(X) = -E[\log f(x)] = -\int_0^{\infty} f(x)\log f(x)dx. \tag{21}$$

Inserting Equation (16) in Equation (21), we obtain the Shannon entropy of the EOWLL model

$$\begin{aligned}
H(X) &= \sum_{k,j=0}^{\infty} a_{k,j}\left[-\int_0^{\infty} f(x)\log(\delta\lambda\alpha)dx + (\alpha+1)\int_0^{\infty} f(x)\log(x)dx \right.\\
&\quad \left. +(\delta+1)\int_0^{\infty} f(x)\log\left(1+\frac{\lambda}{x^{\alpha}}\right)dx\right]\\
&= \sum_{k,j=0}^{\infty} a_{k,j}\left[-\log(\delta\lambda\alpha) + (\alpha+1)\int_0^{\infty} f(x)\log(x)dx \right.\\
&\quad \left. +(\delta+1)\int_0^{\infty} f(x)\log\left(1+\frac{\lambda}{x^{\alpha}}\right)dx\right]\\
&= \sum_{k,j=0}^{\infty} a_{k,j}\left\{-\log(\delta\lambda\alpha) + \frac{\delta}{\alpha}\left[\log(\lambda)\sum_{k=0}^{\infty}(-1)^k\binom{n-k-1}{k}\left(\frac{\lambda}{x^{\alpha}}\right)^k + \left(\frac{\psi(\delta)+\eta}{\delta}\right)\right]\right.\\
&\quad \left. +\frac{2(\delta+1)\delta}{(\delta-1)^2}\right\},
\end{aligned}$$

where $\psi(.)$ Is the digamma function. The Rényi entropy [25] can be defined as

$$H_\nu(X) = \sum_{k,j=0}^{\infty} a_{k,j}\left[\frac{1}{1-\nu}\log\left(\int_0^1 f^{\nu}(x)dx\right)\right], \quad \nu>0 \quad \text{and} \quad \nu\neq 1.$$

Hence, the Rényi entropy of the EOWLL model follows as

$$\begin{aligned}
H_\nu(X) &= \sum_{k,j=0}^{\infty} a_{k,j}\left\{\frac{1}{1-\nu}\log\left[\left(\frac{(\delta\lambda\alpha)^{\nu}\lambda^{\frac{1-\nu}{\alpha}-\nu}}{\alpha}\right)\int_0^1 z^{\delta\nu+\frac{1-\nu}{\alpha}-1}(1-z)^{\nu-\frac{1-\nu}{\alpha}-1}dz\right]\right\}\\
&= \sum_{k,j=0}^{\infty} a_{k,j}\left\{\frac{1}{1-\nu}\log\left[\left(\frac{(\delta\lambda\alpha)^{\nu}\lambda^{\frac{1-\nu}{\alpha}-\nu}}{\alpha}\right)\frac{\Gamma(\delta\nu+\frac{1-\nu}{\alpha})\Gamma(\nu-\frac{1-\nu}{\alpha})}{\Gamma(\delta\nu+\nu)}\right]\right\},
\end{aligned}$$

where $z = (1+\frac{\lambda}{x^{\alpha}})^{-1}$. When $\nu \to 1$, the Rényi entropy converges to the Shannon entropy.

3.6 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are proposed by Bonferroni [26]. The applications of these curves related to economics to study income and poverty and other fields such as demography, reliability, and medicine. They are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx \tag{22}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx, \tag{23}$$

where $\mu = E[X]$ and $q = F^{-1}(p)$.
Using Equation (16), Equations (22) and (23) can be expressed as

$$B(p) = \frac{\delta\lambda^{\frac{1}{\alpha}}(q^{-\alpha}\lambda)^{-\frac{1}{\alpha}}}{p\mu}\sum_{k,j,n=0}^{\infty} a_{k,j}(-1)^n\binom{\delta+n+1}{n}\frac{(q\lambda^{\frac{-1}{\alpha}})^{n+1}}{(n-\frac{1}{\alpha}+1)}$$

and

$$L(p) = \frac{\delta\lambda^{\frac{1}{\alpha}}(q^{-\alpha}\lambda)^{-\frac{1}{\alpha}}}{\mu}\sum_{k,j,n=0}^{\infty} a_{k,j}(-1)^n\binom{\delta+n+1}{n}\frac{(q\lambda^{\frac{-1}{\alpha}})^{n+1}}{(n-\frac{1}{\alpha}+1)}.$$

3.7 Order Statistics

According to [19], the PDF of ith order statistic of the EOW-G class, $X_{(i)}$ (for $i = 1,\dots,n$), can be expressed as

$$f_{X_{(i)}}(x) = \sum_{k,s=0}^{\infty} b_{k,s}\, h_{\theta(k+1)+s}(x), \tag{24}$$

where $h_{\theta(k+1)+s}(x) = [\theta(k+1)+s]g(x)G(x)^{-[\theta(k+1)+s]} - 1$ is the Exp-G density with positive power parameter $[\theta(k+1)+s]$ and

$$b_{k,s} = \sum_{j=0}^{n-i}\sum_{l=0}^{j+i-1}\frac{(-1)^{l+j}\theta\left(\frac{-l-1}{\beta}-1\right)_k\Gamma[\theta(k+1)+s]}{k!s!B(i,n-i+1)\beta^{-k}\Gamma[\theta(k+1)+1]}\binom{n-i}{j}\binom{j+i-1}{l}.$$

Using the PDF and CDF of the LL distribution, Equation (24) can be rewritten as

$$f_{X_{(i)}}(x) = \sum_{k,s=0}^{\infty} b_{k,s}[\theta(k+1)+s]\alpha\lambda x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-[\theta(k+1)+s]-1}. \tag{25}$$

Equation (25) can be expressed as

$$f_{X(i)}(x) = \sum_{k,s=0}^{\infty} b_{k,s}\,\omega\,\alpha\,\lambda\,x^{-(\alpha+1)}\left(1+\frac{\lambda}{x^{\alpha}}\right)^{-\omega-1}, \tag{26}$$

where $\omega = [\theta(k+1)+s]$. Equation (26) means that the PDF of ith order statistic of EOWLL distribution is a mixture of Dagum densities with scale parameter λ and shape parameters α and ω.

4 Estimation Methods

In this section, the EOWLL parameters α, λ, β, and θ are estimated using eight approaches of estimation. These methods are the maximum product of spacing estimators (MPSEs), weighted least-squares estimators (WLSEs), maximum likelihood estimators (MLEs), least-squares estimators (LSEs), Cramér–von Mises estimators (CRVMEs), Anderson–Darling estimators (ADEs), percentile based estimator (PCEs), right-tail Anderson–Darling estimators (RADEs).

Let $x_1,\ldots,x_n$ be a sample of size n from the EOWLL distribution given in (5) and let $x_{(1)}$, $x_{(2)}$, $\cdots$, $x_{(n)}$ be the associated order statistics. In this case, for $\underline{\eta}=(\alpha,\lambda,\beta,\theta)^{\intercal}$, the log-likelihood function from (5) is given by

$$l(\underline{\eta};\boldsymbol{x}) = n\log(\theta)+n\log(\alpha)+n\log(\lambda)-(\alpha+1)\sum_{i=1}^{n}\log(x_i)+(\theta-1)\sum_{i=1}^{n}\log(\varphi_{x_i})-(\theta-1)\sum_{i=1}^{n}\left[1-(\varphi_{x_i})^{-1}\right] \quad (27)$$
$$-(\frac{1}{\beta}+1)\sum_{i=1}^{n}\log\left\{1+\beta[(\varphi_{x_i})-1]^{-\theta}\right\},$$

where $\varphi_{x_i}=1+\frac{\lambda}{x_i^{\alpha}}$.

The likelihood equations are given by

$$\frac{\partial}{\partial\alpha}l(\underline{\eta};\boldsymbol{x}) = \frac{n}{\alpha}-\sum_{i=1}^{n}\log(x_i)-\alpha\lambda(\theta-1)\sum_{i=1}^{n}\frac{x_i^{-\alpha}\log(x_i)}{\varphi_{x_i}}=0,$$

$$\frac{\partial}{\partial\lambda}l(\underline{\eta};\boldsymbol{x}) = \frac{n}{\lambda}+(\theta-1)\sum_{i=1}^{n}\frac{x_i{}^{-\alpha}}{\varphi_{x_i}}-(\theta-1)\sum_{i=1}^{n}\frac{\varphi_{x_i}^{-2}x_i{}^{\alpha}}{1-\varphi_{x_i}^{-1}}=0,$$

$$\frac{\partial}{\partial\beta}l(\underline{\eta};\boldsymbol{x}) = -\left(\frac{1}{\beta}+1\right)\sum_{i=1}^{n}\frac{[\varphi_{x_i}-1]^{-\theta}}{1+\beta[\varphi_{x_i}-1]^{-\theta}}$$
$$+\beta^{-2}\sum_{i=1}^{n}\log\left(1+\beta[\varphi_{x_i}-1]^{-\theta}\right)=0$$

$$\frac{\partial}{\partial\theta}l(\underline{\eta};\boldsymbol{x}) = \frac{n}{\theta}+\sum_{i=1}^{n}\log(\varphi_{x_i})-\sum_{i=1}^{n}\log\left[1-\varphi_{x_i}^{-1}\right]$$
$$-(\beta+1)\sum_{i=1}^{n}\frac{[\varphi_{x_i}-1]^{-\theta}\ln[\varphi_{x_i}-1]}{1+\beta[\varphi_{x_i}-1]^{-\theta}}=0.$$

The MLEs of the parameters α , λ , β and θ of the EOWLL distribution can be obtained found by maximizing $l(\underline{\eta};\boldsymbol{x})$.

Swain [27] pioneered the methods of least-squares and weighted least-squares. The LSEs of the EOWLL parameters can be obtained by minimizing the following function

$$S(\underline{\eta})=\sum_{i=1}^{n}\left[F(x_{(i)})-\frac{i}{n+1}\right]^2,$$

with respect to α, λ, β and θ. Similarly, they can also be obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n}\left[\left(1-\left\{1+\beta\left[\varphi_{x_{(i)}}-1\right]^{-\theta}\right\}^{-\frac{1}{\beta}}\right)-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\underline{\eta}\right)=0,\ \ s=1,2,3,4,$$

where $\varphi_{x_{(i)}} = \left(1+\frac{\lambda}{x_{(i)}^{\alpha}}\right)$.

$$\begin{aligned}
\Delta_1\left(x_{(i)}|\underline{\eta}\right) &= \frac{\partial}{\partial\alpha}F\left(x_{(i)}|\underline{\eta}\right) = \frac{\theta\lambda\ln x_{(i)}}{x_{(i)}^{\alpha}}\left[\varphi_{x_{(i)}}-1\right]^{-(\theta+1)}\psi_{(i)},\\
\Delta_2\left(x_{(i)}|\underline{\eta}\right) &= \frac{\partial}{\partial\lambda}F\left(x_{(i)}|\underline{\eta}\right) = -\frac{\theta}{x_{(i)}^{\alpha}}\left[\varphi_{x_{(i)}}-1\right]^{-(\theta+1)}\psi_{(i)},\\
\Delta_3\left(x_{(i)}|\underline{\eta}\right) &= \frac{\partial}{\partial\beta}F\left(x_{(i)}|\underline{\eta}\right) = \frac{1}{\beta}\left[\varphi_{x_{(i)}}-1\right]^{-\theta}\psi_{(i)}
\end{aligned} \tag{28}$$

and

$$\Delta_4\left(x_{(i)}|\underline{\eta}\right) = \frac{\partial}{\partial\theta}F\left(x_{(i)}|\underline{\eta}\right) = -\frac{\ln\left[\varphi_{x_{(i)}}-1\right]}{\left[\varphi_{x_{(i)}}-1\right]^{\theta}}\psi_{(i)},$$

where $\psi_{(i)} = \left\{1+\beta\left[\varphi_{x_{(i)}}-1\right]^{-\theta}\right\}^{-\left(\frac{1}{\beta}+1\right)}$. Note that the solution of Δ_s for $s=1,2,3,4$ can be obtained numerically.

The WLSEs [27] of the EOWLL parameters can be obtained by minimizing the following equation:

$$W\left(\underline{\eta}\right) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)}|\underline{\eta}\right)-\frac{i}{n+1}\right]^2.$$

Further, the WLSEs can also be derived by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)}|\underline{\eta}\right)-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\underline{\eta}\right)=0,\quad s=1,2,3,4$$

where Δ_s for $s=1,2,3,4$are provided in (28).

The maximum product of the spacings method [28], [29], and [30], as an approximation of the Kullback–Leibler information measure is a good alternative to the ML method.

Let $D_i(\underline{\eta}) = F\left(x_{(i)}|\underline{\eta}\right) - F\left(x_{(i-1)}|\underline{\eta}\right)$, for $i=1,2,\ldots,n+1$, be the uniform spacing of a random sample from the EOWLL distribution, where $F\left(x_{(0)}|\underline{\eta}\right)=0$, $F\left(x_{(n+1)}|\underline{\eta}\right)=1$ and $\sum_{i=1}^{n+1}D_i(\underline{\eta})=1$. Generally, the MPSEs are obtained by maximizing the geometric mean (GcM) of spacings

$$G\left(\underline{\eta}\right) = \left[\prod_{i=1}^{n+1}D_i(\underline{\eta})\right]^{\frac{1}{n+1}}.$$

Or similarly, by maximizing the logarithm of the GcM of sample spacings

$$H\left(\underline{\eta}\right) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\underline{\eta}).$$

The MPSEs of the EOWLL parameters can be calculated by solving the nonlinear equations defined by

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\underline{\eta})}\left[\Delta_s(x_{(i)}|\underline{\eta})-\Delta_s(x_{(i-1)}|\underline{\eta})\right]=0,\quad s=1,2,3,4$$

where Δ_s are defined in (28).

The CVMEs [31] are obtained as the difference between the estimates of the CDF and the empirical CDF [32]. The CVMEs of the EOWLL parameters can be obtained by minimizing

$$C(\underline{\eta})=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}|\underline{\eta}\right)-\frac{2i-1}{2n}\right]^{2}.$$

Additionally, the CVMEs can follow by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\underline{\eta}\right)-\frac{2i-1}{2n}\right]\Delta_s\left(x_{(i)}|\underline{\eta}\right)=0,\quad s=1,2,3,4$$

where Δ_s are defined in (28).

The ADEs of the EOWLL parameters are obtained by minimizing

$$A(\underline{\eta})=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)}|\underline{\eta}\right)+\log S\left(x_{(i)}|\underline{\eta}\right)\right],$$

with respect to α, β, λ and θ. These ADEs are also obtained by solving the non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\underline{\eta}_s\left(x_{(i)}|\underline{\eta}\right)}{F\left(x_{(i)}|\underline{\eta}\right)}-\frac{\Delta_j\left(x_{(n+1-i)}|\underline{\eta}\right)}{S\left(x_{(n+1-i)}|\underline{\eta}\right)}\right]=0,\ \ s=1,2,3,4.$$

The RADEs of the EOWLL parameters follow by minimizing

$$R(\underline{\eta})=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{i:n}|\underline{\eta}\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n}|\underline{\eta}\right).$$

The RADEs can also follow by solving the non-linear equations

$$-2\sum_{i=1}^{n}\Delta_s\left(x_{i:n}|\underline{\eta}\right)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s\left(x_{n+1-i:n}|\underline{\eta}\right)}{S\left(x_{n+1-i:n}|\underline{\eta}\right)}=0,\ \ s=1,2,3,4.$$

where Δ_s are given in (28).

The percentile method was originally introduced by [33] and [34]. Let $u_i=i/(1+n)$ be an unbiased estimator of $F\left(x_{(i)}|\eta\right)$. Hence, the PCEs of the EOWLL parameters follow by minimizing the following function

$$P(\underline{\eta})=\sum_{i=1}^{n}\left(x_{(i)}-\lambda^{\frac{1}{\alpha}}\left(\frac{\beta^{\frac{1}{\theta}}+\left[(1-u_i)^{-\beta}-1\right]^{\frac{1}{\theta}}}{\left[(1-u_i)^{-\beta}-1\right]^{\frac{1}{\theta}}}-1\right)^{-\frac{1}{\alpha}}\right)^{2},$$

with respect to α, β, λ and θ.

5 Simulation Analysis

The simulation results are conducted to explore the behavior of the proposed estimators with respect to their: average absolute value of biases (|BIAS|),$|BIAS(\widehat{\underline{\eta}})|=\frac{1}{n}\sum_{i=1}^{n}|\widehat{\underline{\eta}}-\underline{\eta}|$, average mean square errors (MSE), $MSE=\frac{1}{n}\sum_{i=1}^{n}(\widehat{\underline{\eta}}-\underline{\eta})^2$, and average mean relative errors of the estimates (MRE), $MRE=\frac{1}{n}\sum_{i=1}^{n}|\widehat{\underline{\eta}}-\underline{\eta}|/\underline{\eta}$.

We generated $N=5000$ random samples $x_1,x_2,\ldots,x_n$ of sizes $n=50$, 150, 350, and 500 from the EOWLL distribution by using its QF (12). The parametric values of $\alpha=\{0.5,1.5,1.75,3.00\}$, $\beta=\{0.25,0.50,2.00\}$, $\lambda=\{0.25,1.50\}$ and $\theta=\{1.25,2.50\}$ are considered. The results are obtained by using the **R** software (version 4.0.2) [35].

For each parameter combination and each sample, the EOWLL parameters α, β, λ, and θ are estimated using eight estimators including WLSEs, LSEs, MLEs, MPSEs, CRVMEs, ADEs, RADEs, and PCEs. Then, the MSE, $|Bias|$, and MRE of the parameters are also computed.

All simulated outcomes are reported in Tables 1–8. Furthermore, these tables display the rank of all estimators in each row, where the superscripts are the indicators, and the $\sum Ranks$ represents the partial sum of the ranks for each column in a specific sample size. Table 9 reports the partial and overall ranks of the addressed estimators.

Tables 1–8 illustrate that all estimation approaches show the consistency property, that is, the MSEs and MREs decrease as sample size increases, for all parametric combinations. Additionally, Table 9 shows that the PCEs, MLEs, and ADEs outperform all other estimators with overall scores of 63, 68, and 76, respectively. Hence, based on our study, we confirm the superiority of PCEs, MLEs, and ADEs for the EOWLL parameters.

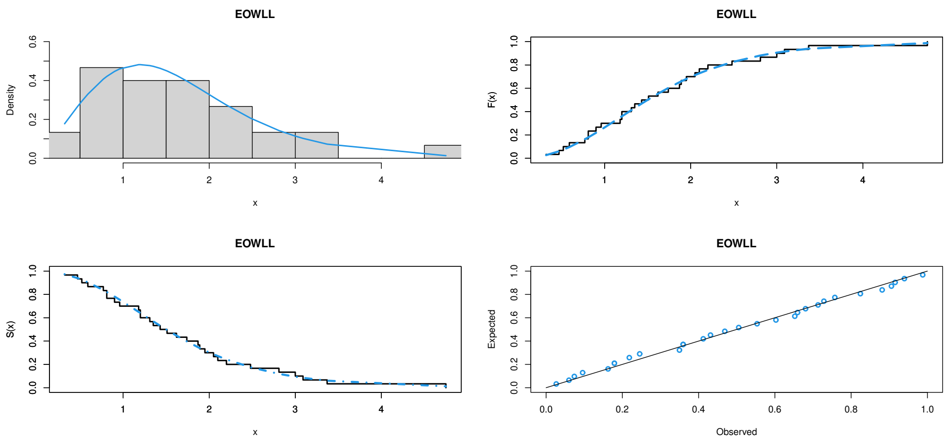

Fig. 3: The fitted EOWLL PDF, CDF, SF, and PP plots for the first data.

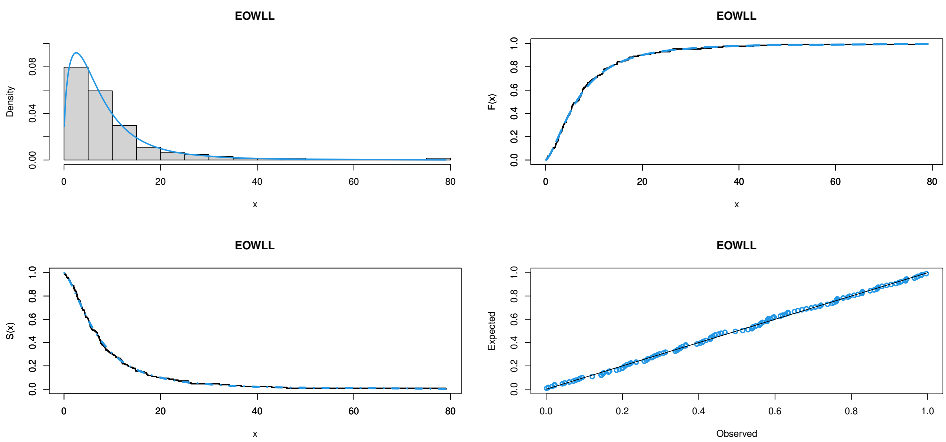

Fig. 4: The fitted EOWLL PDF, CDF, SF, and PP plots for the second data.

Table 1: Simulation results for $\underline{\eta} = (\alpha = 0.5, \beta = 0.5, \lambda = 0.25, \theta = 1.25)^{\intercal}$

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.04922 $^{\{4\}}$	0.04975 $^{\{5\}}$	0.04608 $^{\{2\}}$	0.05096 $^{\{6\}}$	0.06003 $^{\{7\}}$	0.15341 $^{\{8\}}$	0.04080 $^{\{1\}}$	0.04704 $^{\{3\}}$
		$\hat{\beta}$	0.33408 $^{\{2\}}$	0.47669 $^{\{7\}}$	0.39094 $^{\{5\}}$	0.47270 $^{\{6\}}$	0.10918 $^{\{1\}}$	1.08676 $^{\{8\}}$	0.36243 $^{\{3\}}$	0.38319 $^{\{4\}}$
		$\hat{\lambda}$	0.07055 $^{\{4\}}$	0.07846 $^{\{7\}}$	0.06059 $^{\{2\}}$	0.07634 $^{\{5\}}$	0.07660 $^{\{6\}}$	0.14029 $^{\{8\}}$	0.06006 $^{\{1\}}$	0.06416 $^{\{3\}}$
		$\hat{\theta}$	0.11071 $^{\{2\}}$	0.16489 $^{\{4\}}$	0.18947 $^{\{7\}}$	0.16692 $^{\{5\}}$	0.05752 $^{\{1\}}$	1.11643 $^{\{8\}}$	0.16206 $^{\{3\}}$	0.18682 $^{\{6\}}$
	MSEs	$\hat{\alpha}$	0.00385 $^{\{5\}}$	0.00369 $^{\{4\}}$	0.00340 $^{\{2\}}$	0.00389 $^{\{6\}}$	0.00572 $^{\{7\}}$	0.03700 $^{\{8\}}$	0.00262 $^{\{1\}}$	0.00346 $^{\{3\}}$
		$\hat{\beta}$	0.23053 $^{\{2\}}$	0.48889 $^{\{7\}}$	0.27613 $^{\{4\}}$	0.47870 $^{\{6\}}$	0.19510 $^{\{1\}}$	2.76477 $^{\{8\}}$	0.24511 $^{\{3\}}$	0.28978 $^{\{5\}}$
		$\hat{\lambda}$	0.00762 $^{\{4\}}$	0.00878 $^{\{6\}}$	0.00603 $^{\{2\}}$	0.00829 $^{\{5\}}$	0.00906 $^{\{7\}}$	0.03010 $^{\{8\}}$	0.00570 $^{\{1\}}$	0.00621 $^{\{3\}}$
		$\hat{\theta}$	0.04913 $^{\{2\}}$	0.11378 $^{\{5\}}$	0.08582 $^{\{4\}}$	0.12186 $^{\{7\}}$	0.00748 $^{\{1\}}$	3.57667 $^{\{8\}}$	0.07130 $^{\{3\}}$	0.11685 $^{\{6\}}$
	MREs	$\hat{\alpha}$	0.09845 $^{\{4\}}$	0.09950 $^{\{5\}}$	0.09216 $^{\{2\}}$	0.10191 $^{\{6\}}$	0.12006 $^{\{7\}}$	0.30682 $^{\{8\}}$	0.08160 $^{\{1\}}$	0.09408 $^{\{3\}}$
		$\hat{\beta}$	0.66817 $^{\{2\}}$	0.95337 $^{\{7\}}$	0.78187 $^{\{5\}}$	0.94540 $^{\{6\}}$	0.65759 $^{\{1\}}$	2.17353 $^{\{8\}}$	0.72486 $^{\{3\}}$	0.76638 $^{\{4\}}$
		$\hat{\lambda}$	0.28222 $^{\{4\}}$	0.31383 $^{\{7\}}$	0.24235 $^{\{2\}}$	0.30537 $^{\{5\}}$	0.30639 $^{\{6\}}$	0.56115 $^{\{8\}}$	0.24023 $^{\{1\}}$	0.25664 $^{\{3\}}$
		$\hat{\theta}$	0.08857 $^{\{2\}}$	0.13191 $^{\{4\}}$	0.15158 $^{\{7\}}$	0.13354 $^{\{5\}}$	0.04601 $^{\{1\}}$	0.89315 $^{\{8\}}$	0.12965 $^{\{3\}}$	0.14946 $^{\{6\}}$
		$\sum Ranks$	37 $^{\{2\}}$	68 $^{\{6.5\}}$	44 $^{\{3\}}$	68 $^{\{6.5\}}$	46 $^{\{4\}}$	96 $^{\{8\}}$	24 $^{\{1\}}$	49 $^{\{5\}}$
150	|BIAS|	$\hat{\alpha}$	0.03100 $^{\{6\}}$	0.02990 $^{\{4\}}$	0.02975 $^{\{3\}}$	0.02993 $^{\{5\}}$	0.03526 $^{\{7\}}$	0.18314 $^{\{8\}}$	0.02580 $^{\{1\}}$	0.02771 $^{\{2\}}$
		$\hat{\beta}$	0.18105 $^{\{2\}}$	0.26313 $^{\{6\}}$	0.21109 $^{\{5\}}$	0.26735 $^{\{7\}}$	0.17412 $^{\{1\}}$	0.89899 $^{\{8\}}$	0.19858 $^{\{3\}}$	0.20758 $^{\{4\}}$
		$\hat{\lambda}$	0.04236 $^{\{4\}}$	0.04805 $^{\{7\}}$	0.03814 $^{\{2\}}$	0.04754 $^{\{6\}}$	0.04395 $^{\{5\}}$	0.12616 $^{\{8\}}$	0.03652 $^{\{1\}}$	0.03835 $^{\{3\}}$
		$\hat{\theta}$	0.04695 $^{\{2\}}$	0.08346 $^{\{4\}}$	0.09042 $^{\{6\}}$	0.08638 $^{\{5\}}$	0.02810 $^{\{1\}}$	0.87339 $^{\{8\}}$	0.07947 $^{\{3\}}$	0.09457 $^{\{7\}}$
	MSE	$\hat{\alpha}$	0.00153 $^{\{6\}}$	0.00134 $^{\{3\}}$	0.00142 $^{\{5\}}$	0.00136 $^{\{4\}}$	0.00192 $^{\{7\}}$	0.05033 $^{\{8\}}$	0.00105 $^{\{1\}}$	0.00120 $^{\{2\}}$
		$\hat{\beta}$	0.05237 $^{\{2\}}$	0.11143 $^{\{6\}}$	0.07154 $^{\{4\}}$	0.11632 $^{\{7\}}$	0.05003 $^{\{1\}}$	1.74183 $^{\{8\}}$	0.06417 $^{\{3\}}$	0.07269 $^{\{5\}}$
		$\hat{\lambda}$	0.00283 $^{\{4\}}$	0.00342 $^{\{7\}}$	0.00238 $^{\{3\}}$	0.00332 $^{\{6\}}$	0.00304 $^{\{5\}}$	0.02123 $^{\{8\}}$	0.00214 $^{\{1\}}$	0.00224 $^{\{2\}}$
		$\hat{\theta}$	0.00553 $^{\{2\}}$	0.01591 $^{\{4\}}$	0.01876 $^{\{6\}}$	0.01777 $^{\{5\}}$	0.00136 $^{\{1\}}$	2.48379 $^{\{8\}}$	0.01398 $^{\{3\}}$	0.02231 $^{\{7\}}$
	MRE	$\hat{\alpha}$	0.06201 $^{\{6\}}$	0.05981 $^{\{4\}}$	0.05949 $^{\{3\}}$	0.05987 $^{\{5\}}$	0.07052 $^{\{7\}}$	0.36628 $^{\{8\}}$	0.05160 $^{\{1\}}$	0.05541 $^{\{2\}}$
		$\hat{\beta}$	0.36210 $^{\{2\}}$	0.52625 $^{\{6\}}$	0.42218 $^{\{5\}}$	0.53470 $^{\{7\}}$	0.34824 $^{\{1\}}$	1.79799 $^{\{8\}}$	0.39716 $^{\{3\}}$	0.41517 $^{\{4\}}$
		$\hat{\lambda}$	0.16943 $^{\{4\}}$	0.19222 $^{\{7\}}$	0.15254 $^{\{2\}}$	0.19014 $^{\{6\}}$	0.17580 $^{\{5\}}$	0.50464 $^{\{8\}}$	0.14607 $^{\{1\}}$	0.15342 $^{\{3\}}$
		$\hat{\theta}$	0.03756 $^{\{2\}}$	0.06677 $^{\{4\}}$	0.07234 $^{\{6\}}$	0.06911 $^{\{5\}}$	0.02248 $^{\{1\}}$	0.69871 $^{\{8\}}$	0.06357 $^{\{3\}}$	0.07565 $^{\{7\}}$
		$\sum Ranks$	42 $^{\{2.5\}}$	62 $^{\{6\}}$	50 $^{\{5\}}$	68 $^{\{7\}}$	42 $^{\{2.5\}}$	96 $^{\{8\}}$	24 $^{\{1\}}$	48 $^{\{4\}}$
350	|BIAS|	$\hat{\alpha}$	0.02150 $^{\{5\}}$	0.01985 $^{\{3\}}$	0.02159 $^{\{6\}}$	0.01991 $^{\{4\}}$	0.02266 $^{\{7\}}$	0.22208 $^{\{8\}}$	0.01798 $^{\{1\}}$	0.01909 $^{\{2\}}$
		$\hat{\beta}$	0.11401 $^{\{2\}}$	0.17114 $^{\{6\}}$	0.13305 $^{\{4\}}$	0.17214 $^{\{7\}}$	0.10918 $^{\{1\}}$	0.77778 $^{\{8\}}$	0.12993 $^{\{3\}}$	0.13759 $^{\{5\}}$
		$\hat{\lambda}$	0.02848 $^{\{5\}}$	0.03116 $^{\{6.5\}}$	0.02738 $^{\{3\}}$	0.03116 $^{\{6.5\}}$	0.02825 $^{\{4\}}$	0.12203 $^{\{8\}}$	0.02536 $^{\{1\}}$	0.02607 $^{\{2\}}$
		$\hat{\theta}$	0.02540 $^{\{2\}}$	0.05429 $^{\{5\}}$	0.04979 $^{\{4\}}$	0.05468 $^{\{6\}}$	0.01697 $^{\{1\}}$	0.73479 $^{\{8\}}$	0.04560 $^{\{3\}}$	0.05957 $^{\{7\}}$
	MSE	$\hat{\alpha}$	0.00073 $^{\{5\}}$	0.00060 $^{\{3\}}$	0.00080 $^{\{6\}}$	0.00061 $^{\{4\}}$	0.00081 $^{\{7\}}$	0.07425 $^{\{8\}}$	0.00052 $^{\{1\}}$	0.00057 $^{\{2\}}$
		$\hat{\beta}$	0.02050 $^{\{2\}}$	0.04759 $^{\{7\}}$	0.02836 $^{\{4\}}$	0.04748 $^{\{6\}}$	0.01949 $^{\{1\}}$	1.21710 $^{\{8\}}$	0.02695 $^{\{3\}}$	0.03026 $^{\{5\}}$
		$\hat{\lambda}$	0.00127 $^{\{5\}}$	0.00145 $^{\{7\}}$	0.00124 $^{\{3\}}$	0.00143 $^{\{6\}}$	0.00125 $^{\{4\}}$	0.01956 $^{\{8\}}$	0.00100 $^{\{1\}}$	0.00104 $^{\{2\}}$
		$\hat{\theta}$	0.00150 $^{\{2\}}$	0.00696 $^{\{6\}}$	0.00574 $^{\{4\}}$	0.00687 $^{\{5\}}$	0.00047 $^{\{1\}}$	1.86345 $^{\{8\}}$	0.00473 $^{\{3\}}$	0.00831 $^{\{7\}}$
	MRE	$\hat{\alpha}$	0.04299 $^{\{5\}}$	0.03971 $^{\{3\}}$	0.04318 $^{\{6\}}$	0.03983 $^{\{4\}}$	0.04531 $^{\{7\}}$	0.44417 $^{\{8\}}$	0.03596 $^{\{1\}}$	0.03817 $^{\{2\}}$
		$\hat{\beta}$	0.22801 $^{\{2\}}$	0.34229 $^{\{6\}}$	0.26610 $^{\{4\}}$	0.34428 $^{\{7\}}$	0.21837 $^{\{1\}}$	1.55556 $^{\{8\}}$	0.25986 $^{\{3\}}$	0.27517 $^{\{5\}}$
		$\hat{\lambda}$	0.11392 $^{\{5\}}$	0.12464 $^{\{7\}}$	0.10952 $^{\{3\}}$	0.12462 $^{\{6\}}$	0.11300 $^{\{4\}}$	0.48814 $^{\{8\}}$	0.10144 $^{\{1\}}$	0.10429 $^{\{2\}}$
		$\hat{\theta}$	0.02032 $^{\{2\}}$	0.04343 $^{\{5\}}$	0.03983 $^{\{4\}}$	0.04375 $^{\{6\}}$	0.01358 $^{\{1\}}$	0.58783 $^{\{8\}}$	0.03648 $^{\{3\}}$	0.04765 $^{\{7\}}$
		$\sum Ranks$	42 $^{\{3\}}$	64.5 $^{\{6\}}$	51 $^{\{5\}}$	67.5 $^{\{7\}}$	39 $^{\{2\}}$	96 $^{\{8\}}$	24 $^{\{1\}}$	48 $^{\{4\}}$
500	|BIAS|	$\hat{\alpha}$	0.01813 $^{\{5\}}$	0.01652 $^{\{3\}}$	0.01861 $^{\{6\}}$	0.01687 $^{\{4\}}$	0.01944 $^{\{7\}}$	0.23352 $^{\{8\}}$	0.01586 $^{\{1\}}$	0.01638 $^{\{2\}}$
		$\hat{\beta}$	0.09334 $^{\{2\}}$	0.14054 $^{\{6\}}$	0.11096 $^{\{4\}}$	0.14369 $^{\{7\}}$	0.09292 $^{\{1\}}$	0.74459 $^{\{8\}}$	0.10833 $^{\{3\}}$	0.11281 $^{\{5\}}$
		$\hat{\lambda}$	0.02343 $^{\{4\}}$	0.02597 $^{\{7\}}$	0.02329 $^{\{3\}}$	0.02581 $^{\{6\}}$	0.02421 $^{\{5\}}$	0.11844 $^{\{8\}}$	0.02193 $^{\{1\}}$	0.02205 $^{\{2\}}$
		$\hat{\theta}$	0.01983 $^{\{2\}}$	0.04372 $^{\{5\}}$	0.04247 $^{\{4\}}$	0.04568 $^{\{6\}}$	0.01443 $^{\{1\}}$	0.70692 $^{\{8\}}$	0.03599 $^{\{3\}}$	0.04644 $^{\{7\}}$
	MSE	$\hat{\alpha}$	0.00052 $^{\{5\}}$	0.00042 $^{\{2\}}$	0.00059 $^{\{6.5\}}$	0.00044 $^{\{4\}}$	0.00059 $^{\{6.5\}}$	0.08237 $^{\{8\}}$	0.00040 $^{\{1\}}$	0.00043 $^{\{3\}}$
		$\hat{\beta}$	0.01380 $^{\{1\}}$	0.03132 $^{\{6\}}$	0.01942 $^{\{4\}}$	0.03280 $^{\{7\}}$	0.01413 $^{\{2\}}$	1.08103 $^{\{8\}}$	0.01872 $^{\{3\}}$	0.02025 $^{\{5\}}$
		$\hat{\lambda}$	0.00087 $^{\{3\}}$	0.00100 $^{\{7\}}$	0.00090 $^{\{4\}}$	0.00098 $^{\{6\}}$	0.00092 $^{\{5\}}$	0.01854 $^{\{8\}}$	0.00075 $^{\{1.5\}}$	0.00075 $^{\{1.5\}}$
		$\hat{\theta}$	0.00093 $^{\{2\}}$	0.00440 $^{\{5\}}$	0.00419 $^{\{4\}}$	0.00478 $^{\{6\}}$	0.00034 $^{\{1\}}$	1.69640 $^{\{8\}}$	0.00309 $^{\{3\}}$	0.00517 $^{\{7\}}$
	MRE	$\hat{\alpha}$	0.03626 $^{\{5\}}$	0.03304 $^{\{3\}}$	0.03722 $^{\{6\}}$	0.03375 $^{\{4\}}$	0.03888 $^{\{7\}}$	0.46705 $^{\{8\}}$	0.03172 $^{\{1\}}$	0.03276 $^{\{2\}}$
		$\hat{\beta}$	0.18667 $^{\{2\}}$	0.28107 $^{\{6\}}$	0.22192 $^{\{4\}}$	0.28739 $^{\{7\}}$	0.18583 $^{\{1\}}$	1.48919 $^{\{8\}}$	0.21667 $^{\{3\}}$	0.22563 $^{\{5\}}$
		$\hat{\lambda}$	0.09374 $^{\{4\}}$	0.10389 $^{\{7\}}$	0.09316 $^{\{3\}}$	0.10323 $^{\{6\}}$	0.09685 $^{\{5\}}$	0.47377 $^{\{8\}}$	0.08773 $^{\{1\}}$	0.08820 $^{\{2\}}$
		$\hat{\theta}$	0.01586 $^{\{2\}}$	0.03498 $^{\{5\}}$	0.03398 $^{\{4\}}$	0.03654 $^{\{6\}}$	0.01155 $^{\{1\}}$	0.56554 $^{\{8\}}$	0.02880 $^{\{3\}}$	0.03715 $^{\{7\}}$
		$\sum Ranks$	37 $^{\{2\}}$	62 $^{\{6\}}$	52.5 $^{\{5\}}$	69 $^{\{7\}}$	42.5 $^{\{3\}}$	96 $^{\{8\}}$	24.5 $^{\{1\}}$	48.5 $^{\{4\}}$

Table 2: Simulation results for $\underline{\eta} = (\alpha = 1.75, \beta = 0.5, \lambda = 0.25, \theta = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.07753 {3}	0.08430 {6}	0.07888 {5}	0.08638 {7}	0.09015 {8}	0.07876 {4}	0.06380 {1}	0.07525 {2}
		$\hat{\beta}$	0.33422 {2}	0.47217 {6}	0.38508 {4}	0.47779 {7}	0.11058 {1}	0.71076 {8}	0.35389 {3}	0.39417 {5}
		$\hat{\lambda}$	0.05784 {4}	0.06658 {7}	0.05293 {1}	0.06508 {6}	0.05917 {5}	0.06768 {8}	0.05377 {2}	0.05684 {3}
		$\hat{\theta}$	0.16944 {2}	0.21957 {4}	0.22628 {5}	0.22699 {6}	0.14849 {1}	0.43289 {8}	0.19796 {3}	0.24367 {7}
	MSE	$\hat{\alpha}$	0.00988 {3}	0.01077 {4}	0.01098 {6}	0.01162 {7}	0.01354 {8}	0.01082 {5}	0.00689 {1}	0.00934 {2}
		$\hat{\beta}$	0.21113 {1}	0.46014 {6}	0.27153 {4}	0.50308 {7}	0.24488 {3}	4.20486 {8}	0.23522 {2}	0.31072 {5}
		$\hat{\lambda}$	0.00527 {4}	0.00651 {7}	0.00451 {1}	0.00626 {6}	0.00558 {5}	0.00683 {8}	0.00452 {2}	0.00501 {3}
		$\hat{\theta}$	0.06872 {2}	0.12624 {5}	0.11216 {4}	0.14913 {6}	0.04471 {1}	2.42182 {8}	0.09228 {3}	0.15655 {7}
	MRE	$\hat{\alpha}$	0.04430 {3}	0.04817 {6}	0.04508 {5}	0.04936 {7}	0.05152 {8}	0.04500 {4}	0.03646 {1}	0.04300 {2}
		$\hat{\beta}$	0.66843 {1}	0.94435 {6}	0.77017 {4}	0.95557 {7}	0.67774 {2}	1.42152 {8}	0.70778 {3}	0.78834 {5}
		$\hat{\lambda}$	0.23135 {4}	0.26633 {7}	0.21173 {1}	0.26033 {6}	0.23668 {5}	0.27072 {8}	0.21506 {2}	0.22738 {3}
		$\hat{\theta}$	0.13555 {2}	0.17566 {4}	0.18103 {5}	0.18160 {6}	0.11879 {1}	0.34631 {8}	0.15837 {3}	0.19494 {7}
		$\sum Ranks$	31 {2}	68 {6}	45 {3}	78 {7}	48 {4}	85 {8}	26 {1}	51 {5}
150	|BIAS|	$\hat{\alpha}$	0.04852 {5}	0.04753 {3}	0.05526 {7}	0.04818 {4}	0.05014 {6}	0.05926 {8}	0.03724 {1}	0.04197 {2}
		$\hat{\beta}$	0.18015 {2}	0.26249 {7}	0.20718 {4}	0.25707 {6}	0.17846 {1}	0.40377 {8}	0.19605 {3}	0.20956 {5}
		$\hat{\lambda}$	0.03424 {5}	0.04018 {7}	0.03201 {2}	0.03999 {6}	0.03297 {3}	0.04652 {8}	0.03103 {1}	0.03347 {4}
		$\hat{\theta}$	0.08568 {2}	0.11694 {6}	0.11465 {4}	0.11550 {5}	0.07841 {1}	0.24686 {8}	0.10603 {3}	0.12439 {7}
	MSE	$\hat{\alpha}$	0.00388 {5}	0.00332 {3}	0.00589 {7}	0.00347 {4}	0.00403 {6}	0.00624 {8}	0.00235 {1}	0.00279 {2}
		$\hat{\beta}$	0.05201 {2}	0.11275 {7}	0.06948 {4}	0.10744 {6}	0.05139 {1}	0.77215 {8}	0.06235 {3}	0.07208 {5}
		$\hat{\lambda}$	0.00186 {5}	0.00246 {7}	0.00166 {2}	0.00244 {6}	0.00172 {3.5}	0.00335 {8}	0.00152 {1}	0.00172 {3.5}
		$\hat{\theta}$	0.01233 {2}	0.02518 {6}	0.02484 {5}	0.02433 {4}	0.00992 {1}	0.36629 {8}	0.02070 {3}	0.02942 {7}
	MRE	$\hat{\alpha}$	0.02773 {5}	0.02716 {3}	0.03158 {7}	0.02753 {4}	0.02865 {6}	0.03387 {8}	0.02128 {1}	0.02398 {2}
		$\hat{\beta}$	0.36030 {2}	0.52498 {7}	0.41436 {4}	0.51414 {6}	0.35692 {1}	0.80755 {8}	0.39211 {3}	0.41912 {5}
		$\hat{\lambda}$	0.13694 {5}	0.16072 {7}	0.12802 {2}	0.15995 {6}	0.13187 {3}	0.18608 {8}	0.12413 {1}	0.13390 {4}
		$\hat{\theta}$	0.06854 {2}	0.09355 {6}	0.09172 {4}	0.09240 {5}	0.06273 {1}	0.19749 {8}	0.08482 {3}	0.09952 {7}
		$\sum Ranks$	42 {3}	69 {7}	52 {4}	62 {6}	33.5 {2}	96 {8}	24 {1}	53.5 {5}
350	|BIAS|	$\hat{\alpha}$	0.03309 {6}	0.03125 {4}	0.04224 {7}	0.03121 {3}	0.03244 {5}	0.05465 {8}	0.02710 {1}	0.02801 {2}
		$\hat{\beta}$	0.11308 {2}	0.17218 {6}	0.13254 {4}	0.17314 {7}	0.11058 {1}	0.27498 {8}	0.12890 {3}	0.13620 {5}
		$\hat{\lambda}$	0.02226 {5}	0.02693 {6}	0.02196 {3}	0.02706 {7}	0.02105 {1}	0.03637 {8}	0.02180 {2}	0.02198 {4}
		$\hat{\theta}$	0.05347 {2}	0.07516 {5}	0.07088 {4}	0.07628 {6}	0.05016 {1}	0.16636 {8}	0.06618 {3}	0.07849 {7}
	MSE	$\hat{\alpha}$	0.00183 {6}	0.00145 {3}	0.00361 {7}	0.00147 {4}	0.00165 {5}	0.00503 {8}	0.00115 {1}	0.00123 {2}
		$\hat{\beta}$	0.02025 {2}	0.04675 {6}	0.02782 {4}	0.04737 {7}	0.01981 {1}	0.26313 {8}	0.02650 {3}	0.02960 {5}
		$\hat{\lambda}$	0.00079 {4}	0.00111 {6}	0.00081 {5}	0.00112 {7}	0.00069 {1}	0.00215 {8}	0.00074 {2.5}	0.00074 {2.5}
		$\hat{\theta}$	0.00459 {2}	0.00963 {5}	0.00916 {4}	0.01027 {6}	0.00397 {1}	0.12604 {8}	0.00756 {3}	0.01114 {7}
	MRE	$\hat{\alpha}$	0.01891 {6}	0.01786 {4}	0.02414 {7}	0.01784 {3}	0.01854 {5}	0.03123 {8}	0.01549 {1}	0.01601 {2}
		$\hat{\beta}$	0.22616 {2}	0.34435 {6}	0.26508 {4}	0.34628 {7}	0.22116 {1}	0.54995 {8}	0.25780 {3}	0.27240 {5}
		$\hat{\lambda}$	0.08902 {5}	0.10771 {6}	0.08783 {3}	0.10824 {7}	0.08421 {1}	0.14549 {8}	0.08719 {2}	0.08791 {4}
		$\hat{\theta}$	0.04278 {2}	0.06013 {5}	0.05671 {4}	0.06103 {6}	0.04013 {1}	0.13309 {8}	0.05295 {3}	0.06279 {7}
		$\sum Ranks$	44 {3}	62 {6}	56 {5}	70 {7}	24 {1}	96 {8}	27.5 {2}	52.5 {4}
500	|BIAS|	$\hat{\alpha}$	0.02702 {5}	0.02613 {3}	0.03778 {7}	0.02660 {4}	0.02757 {6}	0.05367 {8}	0.02358 {1}	0.02487 {2}
		$\hat{\beta}$	0.09448 {2}	0.14262 {7}	0.10725 {3}	0.14064 {6}	0.09282 {1}	0.23374 {8}	0.10866 {4}	0.11379 {5}
		$\hat{\lambda}$	0.01831 {3}	0.02246 {7}	0.01818 {2}	0.02221 {6}	0.01782 {1}	0.03292 {8}	0.01848 {4}	0.01888 {5}
		$\hat{\theta}$	0.04369 {2}	0.06332 {6}	0.05813 {4}	0.06224 {5}	0.04249 {1}	0.13825 {8}	0.05385 {3}	0.06382 {7}
	MSE	$\hat{\alpha}$	0.00121 {6}	0.00103 {3}	0.00286 {7}	0.00106 {4}	0.00120 {5}	0.00520 {8}	0.00090 {1}	0.00097 {2}
		$\hat{\beta}$	0.01411 {2}	0.03319 {7}	0.01844 {3}	0.03136 {6}	0.01404 {1}	0.14166 {8}	0.01876 {4}	0.02064 {5}
		$\hat{\lambda}$	0.00053 {2.5}	0.00077 {7}	0.00055 {5}	0.00076 {6}	0.00050 {1}	0.00180 {8}	0.00053 {2.5}	0.00054 {4}
		$\hat{\theta}$	0.00305 {2}	0.00723 {7}	0.00635 {4}	0.00655 {5}	0.00287 {1}	0.06193 {8}	0.00496 {3}	0.00719 {6}
	MRE	$\hat{\alpha}$	0.01544 {5}	0.01493 {3}	0.02159 {7}	0.01520 {4}	0.01575 {6}	0.03067 {8}	0.01347 {1}	0.01421 {2}
		$\hat{\beta}$	0.18896 {2}	0.28524 {7}	0.21451 {3}	0.28127 {6}	0.18565 {1}	0.46748 {8}	0.21732 {4}	0.22757 {5}
		$\hat{\lambda}$	0.07326 {3}	0.08985 {7}	0.07273 {2}	0.08883 {6}	0.07126 {1}	0.13168 {8}	0.07393 {4}	0.07552 {5}
		$\hat{\theta}$	0.03495 {2}	0.05066 {6}	0.04650 {4}	0.04979 {5}	0.03399 {1}	0.11060 {8}	0.04308 {3}	0.05106 {7}
		$\sum Ranks$	36.5 {3}	70 {7}	51 {4}	63 {6}	26 {1}	96 {8}	34.5 {2}	55 {5}

Table 3: Simulation results for $\underline{\eta} = (\alpha = 1.5, \beta = 0.25, \lambda = 0.25, \theta = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.07284 {4}	0.07974 {6}	0.06809 {2}	0.08259 {7}	0.08341 {8}	0.07530 {5}	0.06125 {1}	0.06998 {3}
		$\hat{\beta}$	0.23806 {2}	0.37289 {7}	0.29781 {5}	0.37242 {6}	0.09145 {1}	0.37833 {8}	0.27946 {3}	0.29380 {4}
		$\hat{\lambda}$	0.04720 {3}	0.05599 {8}	0.04437 {1}	0.05515 {7}	0.05050 {5}	0.05511 {6}	0.04535 {2}	0.04794 {4}
		$\hat{\theta}$	0.13197 {2}	0.18906 {4}	0.19140 {5}	0.19695 {6}	0.11913 {1}	0.22050 {8}	0.17685 {3}	0.20190 {7}
	MSE	$\hat{\alpha}$	0.00882 {5}	0.00933 {6}	0.00797 {3}	0.01027 {7}	0.01118 {8}	0.00876 {4}	0.00613 {1}	0.00758 {2}
		$\hat{\beta}$	0.10426 {1}	0.49802 {8}	0.16942 {5}	0.45693 {7}	0.11842 {2}	0.43973 {6}	0.15436 {3}	0.16824 {4}
		$\hat{\lambda}$	0.00335 {3}	0.00453 {8}	0.00300 {1}	0.00441 {7}	0.00384 {5}	0.00439 {6}	0.00305 {2}	0.00337 {4}
		$\hat{\theta}$	0.04798 {2}	0.27182 {8}	0.08765 {4}	0.24458 {7}	0.02625 {1}	0.20193 {6}	0.08035 {3}	0.10811 {5}
	MRE	$\hat{\alpha}$	0.04856 {4}	0.05316 {6}	0.04539 {2}	0.05506 {7}	0.05560 {8}	0.05020 {5}	0.04083 {1}	0.04666 {3}
		$\hat{\beta}$	0.95226 {1}	1.49155 {7}	1.19124 {5}	1.48966 {6}	1.01928 {2}	1.51331 {8}	1.11785 {3}	1.17519 {4}
		$\hat{\lambda}$	0.18879 {3}	0.22398 {8}	0.17749 {1}	0.22062 {7}	0.20200 {5}	0.22045 {6}	0.18141 {2}	0.19175 {4}
		$\hat{\theta}$	0.10558 {2}	0.15124 {4}	0.15312 {5}	0.15756 {6}	0.09531 {1}	0.17640 {8}	0.14148 {3}	0.16152 {7}
		$\sum Ranks$	32 {2}	80 {7.5}	39 {3}	80 {7.5}	47 {4}	76 {6}	27 {1}	51 {5}
150	|BIAS|	$\hat{\alpha}$	0.04723 {5}	0.04667 {3}	0.04935 {6.5}	0.04705 {4}	0.04935 {6.5}	0.05393 {8}	0.03842 {1}	0.04180 {2}
		$\hat{\beta}$	0.13872 {1}	0.21096 {7}	0.16676 {5}	0.21072 {6}	0.14189 {2}	0.22896 {8}	0.16075 {3}	0.16664 {4}
		$\hat{\lambda}$	0.03052 {4}	0.03608 {7}	0.02965 {3}	0.03584 {6}	0.03056 {5}	0.03700 {8}	0.02928 {1}	0.02964 {2}
		$\hat{\theta}$	0.06932 {2}	0.09974 {5}	0.09759 {4}	0.10085 {6}	0.06674 {1}	0.13652 {8}	0.09235 {3}	0.10834 {7}
	MSE	$\hat{\alpha}$	0.00369 {5}	0.00324 {3}	0.00429 {7}	0.00335 {4}	0.00384 {6}	0.00441 {8}	0.00249 {1}	0.00279 {2}
		$\hat{\beta}$	0.02962 {1}	0.07372 {7}	0.04414 {4}	0.07351 {6}	0.03248 {2}	0.10891 {8}	0.04211 {3}	0.04534 {5}
		$\hat{\lambda}$	0.00141 {4}	0.00189 {7}	0.00137 {3}	0.00186 {6}	0.00143 {5}	0.00204 {8}	0.00130 {1}	0.00132 {2}
		$\hat{\theta}$	0.00812 {2}	0.01972 {5}	0.01830 {4}	0.02067 {6}	0.00715 {1}	0.04614 {8}	0.01667 {3}	0.02373 {7}
	MRE	$\hat{\alpha}$	0.03149 {5}	0.03111 {3}	0.03290 {6.5}	0.03137 {4}	0.03290 {6.5}	0.03596 {8}	0.02562 {1}	0.02787 {2}
		$\hat{\beta}$	0.55488 {1}	0.84383 {7}	0.66705 {5}	0.84288 {6}	0.56756 {2}	0.91583 {8}	0.64301 {3}	0.66655 {4}
		$\hat{\lambda}$	0.12206 {4}	0.14432 {7}	0.11861 {3}	0.14337 {6}	0.12224 {5}	0.14802 {8}	0.11711 {1}	0.11857 {2}
		$\hat{\theta}$	0.05546 {2}	0.07979 {5}	0.07807 {4}	0.08068 {6}	0.05339 {1}	0.10922 {8}	0.07388 {3}	0.08668 {7}
		$\sum Ranks$	36 {2}	66 {6.5}	55 {5}	66 {6.5}	43 {3}	96 {8}	24 {1}	46 {4}
350	|BIAS|	$\hat{\alpha}$	0.03183 {5}	0.03097 {3}	0.03886 {7}	0.03125 {4}	0.03247 {6}	0.04412 {8}	0.02700 {1}	0.02855 {2}
		$\hat{\beta}$	0.09101 {1}	0.14367 {7}	0.10919 {4}	0.14361 {6}	0.09145 {2}	0.15142 {8}	0.10676 {3}	0.10926 {5}
		$\hat{\lambda}$	0.02048 {4}	0.02521 {7}	0.02057 {5}	0.02513 {6}	0.02018 {1}	0.02739 {8}	0.02044 {3}	0.02019 {2}
		$\hat{\theta}$	0.04504 {2}	0.06629 {5}	0.06313 {4}	0.06656 {6}	0.04366 {1}	0.08785 {8}	0.05721 {3}	0.06838 {7}
	MSE	$\hat{\alpha}$	0.00165 {5}	0.00143 {3}	0.00279 {7}	0.00148 {4}	0.00167 {6}	0.00313 {8}	0.00117 {1}	0.00130 {2}
		$\hat{\beta}$	0.01293 {1}	0.03178 {7}	0.01861 {4}	0.03174 {6}	0.01334 {2}	0.04439 {8}	0.01800 {3}	0.01910 {5}
		$\hat{\lambda}$	0.00066 {4}	0.00093 {6.5}	0.00070 {5}	0.00093 {6.5}	0.00064 {2.5}	0.00116 {8}	0.00064 {2.5}	0.00062 {1}
		$\hat{\theta}$	0.00326 {2}	0.00768 {5}	0.00738 {4}	0.00781 {6}	0.00303 {1}	0.01706 {8}	0.00576 {3}	0.00867 {7}
	MRE	$\hat{\alpha}$	0.02122 {5}	0.02065 {3}	0.02590 {7}	0.02083 {4}	0.02165 {6}	0.02942 {8}	0.01800 {1}	0.01903 {2}
		$\hat{\beta}$	0.36402 {1}	0.57468 {7}	0.43675 {4}	0.57442 {6}	0.36580 {2}	0.60567 {8}	0.42706 {3}	0.43703 {5}
		$\hat{\lambda}$	0.08192 {4}	0.10085 {7}	0.08229 {5}	0.10053 {6}	0.08072 {1}	0.10955 {8}	0.08174 {3}	0.08075 {2}
		$\hat{\theta}$	0.03603 {2}	0.05303 {5}	0.05050 {4}	0.05325 {6}	0.03492 {1}	0.07028 {8}	0.04577 {3}	0.05470 {7}
		$\sum Ranks$	36 {3}	65.5 {6}	60 {5}	66.5 {7}	31.5 {2}	96 {8}	29.5 {1}	47 {4}
500	|BIAS|	$\hat{\alpha}$	0.02720 {5}	0.02620 {3}	0.03495 {7}	0.02635 {4}	0.02757 {6}	0.04090 {8}	0.02400 {1}	0.02500 {2}
		$\hat{\beta}$	0.07555 {2}	0.12196 {7}	0.09292 {4}	0.12195 {6}	0.07549 {1}	0.12890 {8}	0.09087 {3}	0.09297 {5}
		$\hat{\lambda}$	0.01723 {2}	0.02134 {7}	0.01791 {5}	0.02130 {6}	0.01694 {1}	0.02408 {8}	0.01765 {4}	0.01759 {3}
		$\hat{\theta}$	0.03786 {2}	0.05736 {6}	0.05333 {4}	0.05744 {7}	0.03680 {1}	0.07455 {8}	0.04764 {3}	0.05660 {5}
	MSE	$\hat{\alpha}$	0.00120 {5.5}	0.00102 {3}	0.00232 {7}	0.00104 {4}	0.00120 {5.5}	0.00285 {8}	0.00094 {1}	0.00099 {2}
		$\hat{\beta}$	0.00903 {1}	0.02306 {7}	0.01349 {4}	0.02304 {6}	0.00920 {2}	0.03272 {8}	0.01304 {3}	0.01379 {5}
		$\hat{\lambda}$	0.00046 {2}	0.00068 {7}	0.00053 {5}	0.00067 {6}	0.00045 {1}	0.00092 {8}	0.00048 {4}	0.00047 {3}
		$\hat{\theta}$	0.00229 {2}	0.00577 {5}	0.00530 {4}	0.00582 {6}	0.00215 {1}	0.01272 {8}	0.00400 {3}	0.00587 {7}
	MRE	$\hat{\alpha}$	0.01814 {5}	0.01746 {3}	0.02330 {7}	0.01757 {4}	0.01838 {6}	0.02727 {8}	0.01600 {1}	0.01667 {2}
		$\hat{\beta}$	0.30219 {2}	0.48785 {7}	0.37167 {4}	0.48779 {6}	0.30195 {1}	0.51558 {8}	0.36348 {3}	0.37188 {5}
		$\hat{\lambda}$	0.06891 {2}	0.08534 {7}	0.07164 {5}	0.08518 {6}	0.06777 {1}	0.09633 {8}	0.07060 {4}	0.07037 {3}
		$\hat{\theta}$	0.03029 {2}	0.04589 {6}	0.04267 {4}	0.04595 {7}	0.02944 {1}	0.05964 {8}	0.03812 {3}	0.04528 {5}
		$\sum Ranks$	32.5 {2}	68 {6.5}	60 {5}	68 {6.5}	27.5 {1}	96 {8}	33 {3}	47 {4}

Table 4: Simulation results for $\underline{\eta} = (\alpha = 1.5, \beta = 0.25, \lambda = 1.5, \theta = 2.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.15921 {1}	0.21936 {7}	0.18233 {4}	0.22740 {8}	0.16775 {3}	0.16558 {2}	0.18292 {5}	0.21308 {6}
		$\hat{\beta}$	0.23806 {2}	0.37038 {7}	0.29780 {6}	0.37069 {8}	0.09164 {1}	0.27836 {3}	0.27949 {4}	0.29394 {5}
		$\hat{\lambda}$	0.10796 {3}	0.12167 {7}	0.11233 {4}	0.12325 {8}	0.10466 {1}	0.10642 {2}	0.11496 {6}	0.11360 {5}
		$\hat{\theta}$	0.11036 {2}	0.12669 {5}	0.12765 {6}	0.13313 {7}	0.10094 {1}	0.11875 {3}	0.12153 {4}	0.14234 {8}
	MSE	$\hat{\alpha}$	0.05188 {3}	0.13515 {7}	0.06616 {4}	0.14410 {8}	0.05041 {2}	0.04851 {1}	0.06922 {5}	0.09387 {6}
		$\hat{\beta}$	0.10426 {1}	0.36638 {7}	0.16942 {6}	0.36819 {8}	0.11867 {2}	0.14470 {3}	0.15438 {4}	0.16832 {5}
		$\hat{\lambda}$	0.01947 {3}	0.02480 {7}	0.02075 {4}	0.02548 {8}	0.01764 {1}	0.01859 {2}	0.02219 {6}	0.02178 {5}
		$\hat{\theta}$	0.02294 {2}	0.02911 {5}	0.03297 {6}	0.03351 {7}	0.01688 {1}	0.02455 {3}	0.02554 {4}	0.03643 {8}
	MRE	$\hat{\alpha}$	0.10614 {1}	0.14624 {7}	0.12156 {4}	0.15160 {8}	0.11183 {3}	0.11038 {2}	0.12195 {5}	0.14205 {6}
		$\hat{\beta}$	0.95226 {1}	1.48152 {7}	1.19120 {6}	1.48275 {8}	1.02034 {2}	1.11343 {3}	1.11796 {4}	1.17575 {5}
		$\hat{\lambda}$	0.07197 {3}	0.08111 {7}	0.07489 {4}	0.08217 {8}	0.06977 {1}	0.07095 {2}	0.07664 {6}	0.07573 {5}
		$\hat{\theta}$	0.04414 {2}	0.05068 {5}	0.05106 {6}	0.05325 {7}	0.04038 {1}	0.04750 {3}	0.04861 {4}	0.05694 {8}
		$\sum Ranks$	24 {2}	78 {7}	60 {5}	93 {8}	19 {1}	29 {3}	57 {4}	72 {6}
150	|BIAS|	$\hat{\alpha}$	0.09351 {1}	0.12025 {7}	0.10383 {5}	0.12156 {8}	0.09531 {2}	0.09980 {3}	0.10234 {4}	0.11958 {6}
		$\hat{\beta}$	0.13872 {1}	0.21097 {8}	0.16676 {6}	0.21074 {7}	0.14196 {2}	0.16135 {4}	0.16078 {3}	0.16669 {5}
		$\hat{\lambda}$	0.06227 {2}	0.07187 {7}	0.06628 {5}	0.07216 {8}	0.06143 {1}	0.06357 {3}	0.06672 {6}	0.06491 {4}
		$\hat{\theta}$	0.06075 {2}	0.07583 {6}	0.07245 {5}	0.07633 {7}	0.05849 {1}	0.06764 {3}	0.06798 {4}	0.07679 {8}
	MSE	$\hat{\alpha}$	0.01412 {1}	0.02482 {7}	0.01795 {5}	0.02585 {8}	0.01455 {2}	0.01619 {3}	0.01757 {4}	0.02435 {6}
		$\hat{\beta}$	0.02962 {1}	0.07372 {8}	0.04414 {5}	0.07352 {7}	0.03250 {2}	0.04361 {4}	0.04212 {3}	0.04537 {6}
		$\hat{\lambda}$	0.00612 {2}	0.00834 {7}	0.00700 {5}	0.00839 {8}	0.00593 {1}	0.00641 {3}	0.00719 {6}	0.00671 {4}
		$\hat{\theta}$	0.00616 {2}	0.00938 {5}	0.01032 {8}	0.00967 {6}	0.00544 {1}	0.00782 {3}	0.00804 {4}	0.00995 {7}
	MRE	$\hat{\alpha}$	0.06234 {1}	0.08017 {7}	0.06922 {5}	0.08104 {8}	0.06354 {2}	0.06654 {3}	0.06823 {4}	0.07972 {6}
		$\hat{\beta}$	0.55488 {1}	0.84387 {8}	0.66704 {6}	0.84295 {7}	0.56783 {2}	0.64539 {4}	0.64310 {3}	0.66675 {5}
		$\hat{\lambda}$	0.04151 {2}	0.04792 {7}	0.04419 {5}	0.04810 {8}	0.04096 {1}	0.04238 {3}	0.04448 {6}	0.04327 {4}
		$\hat{\theta}$	0.02430 {2}	0.03033 {6}	0.02898 {5}	0.03053 {7}	0.02339 {1}	0.02706 {3}	0.02719 {4}	0.03072 {8}
		$\sum Ranks$	18 {1.5}	83 {7}	65 {5}	89 {8}	18 {1.5}	39 {3}	51 {4} 69 {6}	
350	|BIAS|	$\hat{\alpha}$	0.06256 {2}	0.08158 {7}	0.07087 {5}	0.08206 {8}	0.06249 {1}	0.06702 {3}	0.06871 {4}	0.07863 {6}
		$\hat{\beta}$	0.09101 {1}	0.14368 {8}	0.10919 {5}	0.14363 {7}	0.09164 {2}	0.10480 {3}	0.10676 {4}	0.10930 {6}
		$\hat{\lambda}$	0.03957 {1}	0.04538 {7}	0.04231 {6}	0.04545 {8}	0.03972 {2}	0.04126 {4}	0.04170 {5}	0.04096 {3}
		$\hat{\theta}$	0.03931 {2}	0.04874 {6}	0.04951 {8}	0.04888 {7}	0.03834 {1}	0.04297 {4}	0.04259 {3}	0.04823 {5}
	MSE	$\hat{\alpha}$	0.00617 {1}	0.01051 {7}	0.00821 {5}	0.01072 {8}	0.00620 {2}	0.00722 {3}	0.00753 {4}	0.00996 {6}
		$\hat{\beta}$	0.01293 {1}	0.03179 {8}	0.01861 {5}	0.03175 {7}	0.01337 {2}	0.01813 {4}	0.01800 {3}	0.01912 {6}
		$\hat{\lambda}$	0.00248 {1.5}	0.00332 {7}	0.00285 {6}	0.00334 {8}	0.00248 {1.5}	0.00268 {3}	0.00281 {5}	0.00270 {4}
		$\hat{\theta}$	0.00250 {2}	0.00382 {5}	0.00473 {8}	0.00389 {7}	0.00233 {1}	0.00315 {4}	0.00307 {3}	0.00383 {6}
	MRE	$\hat{\alpha}$	0.04170 {2}	0.05439 {7}	0.04725 {5}	0.05471 {8}	0.04166 {1}	0.04468 {3}	0.04581 {4}	0.05242 {6}
		$\hat{\beta}$	0.36402 {1}	0.57474 {8}	0.43674 {5}	0.57452 {7}	0.36655 {2}	0.41919 {3}	0.42706 {4}	0.43721 {6}
		$\hat{\lambda}$	0.02638 {1}	0.03025 {7}	0.02821 {6}	0.03030 {8}	0.02648 {2}	0.02751 {4}	0.02780 {5}	0.02731 {3}
		$\hat{\theta}$	0.01573 {2}	0.01950 {6}	0.01980 {8}	0.01955 {7}	0.01534 {1}	0.01719 {4}	0.01704 {3}	0.01929 {5}
		$\sum Ranks$	17.5 {1}	83 {7}	72 {6}	90 {8}	18.5 {2}	42 {3}	47 {4}	62 {5}
500	|BIAS|	$\hat{\alpha}$	0.05294 {2}	0.07037 {7.5}	0.06119 {5}	0.07037 {7.5}	0.05276 {1}	0.05689 {3}	0.05872 {4}	0.06619 {6}
		$\hat{\beta}$	0.07555 {1}	0.12197 {8}	0.09292 {5}	0.12196 {7}	0.07565 {2}	0.08673 {3}	0.09087 {4}	0.09297 {6}
		$\hat{\lambda}$	0.03344 {2}	0.03809 {7}	0.03590 {6}	0.03824 {8}	0.03340 {1}	0.03448 {3}	0.03518 {5}	0.03475 {4}
		$\hat{\theta}$	0.03323 {2}	0.04117 {6}	0.04319 {8}	0.04143 {7}	0.03253 {1}	0.03588 {3}	0.03605 {4}	0.04083 {5}
	MSE	$\hat{\alpha}$	0.00442 {1.5}	0.00783 {7}	0.00621 {5}	0.00786 {8}	0.00442 {1.5}	0.00518 {3}	0.00549 {4}	0.00706 {6}
		$\hat{\beta}$	0.00903 {1}	0.02306 {8}	0.01349 {5}	0.02304 {7}	0.00923 {2}	0.01238 {3}	0.01304 {4}	0.01379 {6}
		$\hat{\lambda}$	0.00175 {1.5}	0.00232 {7}	0.00203 {6}	0.00233 {8}	0.00175 {1.5}	0.00187 {3}	0.00196 {5}	0.00191 {4}
		$\hat{\theta}$	0.00176 {2}	0.00271 {5}	0.00374 {8}	0.00280 {6}	0.00167 {1}	0.00216 {3.5}	0.00216 {3.5}	0.00282 {7}
	MRE	$\hat{\alpha}$	0.03529 {2}	0.04691 {7.5}	0.04079 {5}	0.04691 {7.5}	0.03517 {1}	0.03792 {3}	0.03914 {4}	0.04413 {6}
		$\hat{\beta}$	0.30219 {1}	0.48787 {8}	0.37169 {5}	0.48784 {7}	0.30259 {2}	0.34693 {3}	0.36347 {4}	0.37186 {6}
		$\hat{\lambda}$	0.02229 {2}	0.02539 {7}	0.02393 {6}	0.02549 {8}	0.02226 {1}	0.02299 {3}	0.02345 {5}	0.02317 {4}
		$\hat{\theta}$	0.01329 {2}	0.01647 {6}	0.01728 {8}	0.01657 {7}	0.01301 {1}	0.01435 {3}	0.01442 {4}	0.01633 {5}
		$\sum Ranks$	20 {2}	84 {7}	72 {6}	88 {8}	16 {1}	36.5 {3}	50.5 {4}	65 {5}

Table 5: Simulation results for $\underline{\eta} = (\alpha = 3, \beta = 0.5, \lambda = 0.25, \theta = 2.5)^\intercal$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.15550 {3}	0.21373 {7}	0.11634 {1}	0.21815 {8}	0.20063 {5}	0.20296 {6}	0.13830 {2}	0.18967 {4}
		$\hat{\beta}$	0.33514 {2}	0.47835 {7}	0.38624 {5}	0.48054 {8}	0.11167 {1}	0.36529 {4}	0.35842 {3}	0.38766 {6}
		$\hat{\lambda}$	0.03759 {3}	0.04722 {8}	0.03042 {1}	0.04688 {7}	0.04161 {5}	0.04328 {6}	0.03538 {2}	0.04084 {4}
		$\hat{\theta}$	0.30712 {3}	0.36417 {5}	0.42023 {8}	0.37823 {6}	0.25133 {1}	0.26538 {2}	0.34327 {4}	0.39361 {7}
	MSE	$\hat{\alpha}$	0.03733 {3}	0.06902 {7}	0.02464 {1}	0.07358 {8}	0.06603 {5}	0.06633 {6}	0.02713 {2}	0.05206 {4}
		$\hat{\beta}$	0.19965 {2}	0.59795 {7}	0.26913 {5}	0.65032 {8}	0.19937 {1}	0.24763 {4}	0.23810 {3}	0.28363 {6}
		$\hat{\lambda}$	0.00207 {3}	0.00323 {8}	0.00152 {1}	0.00318 {7}	0.00265 {5}	0.00283 {6}	0.00182 {2}	0.00238 {4}
		$\hat{\theta}$	0.23363 {3}	0.59262 {7}	0.37099 {6}	0.71814 {8}	0.10861 {1}	0.12269 {2}	0.28740 {4}	0.36544 {5}
	MRE	$\hat{\alpha}$	0.05183 {3}	0.07124 {7}	0.03878 {1}	0.07272 {8}	0.06688 {5}	0.06765 {6}	0.04610 {2}	0.06322 {4}
		$\hat{\beta}$	0.67028 {2}	0.95669 {7}	0.77247 {5}	0.96108 {8}	0.66924 {1}	0.73058 {4}	0.71684 {3}	0.77533 {6}
		$\hat{\lambda}$	0.15035 {3}	0.18890 {8}	0.12167 {1}	0.18752 {7}	0.16644 {5}	0.17313 {6}	0.14151 {2}	0.16335 {4}
		$\hat{\theta}$	0.12285 {3}	0.14567 {5}	0.16809 {8}	0.15129 {6}	0.10053 {1}	0.10615 {2}	0.13731 {4}	0.15744 {7}
		$\sum Ranks$	33 {1.5}	83 {7}	43 {4}	89 {8}	36 {3}	54 {5}	33 {1.5}	61 {6}
150	|BIAS|	$\hat{\alpha}$	0.09979 {3}	0.11812 {7}	0.08934 {2}	0.11870 {8}	0.11013 {5}	0.11251 {6}	0.07875 {1}	0.10388 {4}
		$\hat{\beta}$	0.17628 {2}	0.26209 {7}	0.20580 {5}	0.26213 {8}	0.17561 {1}	0.19978 {4}	0.19699 {3}	0.21028 {6}
		$\hat{\lambda}$	0.02238 {3}	0.02823 {8}	0.01799 {1}	0.02813 {7}	0.02299 {4}	0.02455 {6}	0.02069 {2}	0.02353 {5}
		$\hat{\theta}$	0.15032 {3}	0.20081 {5}	0.23479 {8}	0.20267 {6}	0.13629 {1}	0.14834 {2}	0.18576 {4}	0.21147 {7}
	MSE	$\hat{\alpha}$	0.01583 {4}	0.01987 {6}	0.01488 {2}	0.02025 {8}	0.01948 {5}	0.02012 {7}	0.00950 {1}	0.01547 {3}
		$\hat{\beta}$	0.05039 {1}	0.11275 {8}	0.06894 {5}	0.11263 {7}	0.05127 {2}	0.06877 {4}	0.06376 {3}	0.07312 {6}
		$\hat{\lambda}$	0.00078 {3}	0.00118 {8}	0.00057 {1}	0.00117 {7}	0.00084 {5}	0.00094 {6}	0.00064 {2}	0.00080 {4}
		$\hat{\theta}$	0.03999 {3}	0.07356 {5}	0.09963 {8}	0.07611 {6}	0.03012 {1}	0.03715 {2}	0.06701 {4}	0.08509 {7}
	MRE	$\hat{\alpha}$	0.03326 {3}	0.03937 {7}	0.02978 {2}	0.03957 {8}	0.03671 {5}	0.03750 {6}	0.02625 {1}	0.03463 {4}
		$\hat{\beta}$	0.35257 {2}	0.52418 {7}	0.41159 {5}	0.52426 {8}	0.35122 {1}	0.39956 {4}	0.39398 {3}	0.42056 {6}
		$\hat{\lambda}$	0.08950 {3}	0.11294 {8}	0.07194 {1}	0.11254 {7}	0.09196 {4}	0.09821 {6}	0.08275 {2}	0.09411 {5}
		$\hat{\theta}$	0.06013 {3}	0.08032 {5}	0.09392 {8}	0.08107 {6}	0.05452 {1}	0.05934 {2}	0.07431 {4}	0.08459 {7}
		$\sum Ranks$	33 {2}	81 {7}	48 {4}	86 {8}	35 {3}	55 {5}	30 {1}	64 {6}
350	|BIAS|	$\hat{\alpha}$	0.06757 {3}	0.07649 {7}	0.07490 {6}	0.07676 {8}	0.07173 {4}	0.07453 {5}	0.05345 {1}	0.06519 {2}
		$\hat{\beta}$	0.11175 {2}	0.16931 {7.5}	0.12949 {5}	0.16931 {7.5}	0.11167 {1}	0.12818 {4}	0.12703 {3}	0.13362 {6}
		$\hat{\lambda}$	0.01470 {3}	0.01853 {8}	0.01438 {2}	0.01852 {7}	0.01493 {4}	0.01619 {6}	0.01373 {1}	0.01511 {5}
		$\hat{\theta}$	0.09404 {2}	0.12950 {5}	0.13719 {8}	0.12989 {6}	0.08831 {1}	0.09656 {3}	0.11778 {4}	0.13438 {7}
	MSE	$\hat{\alpha}$	0.00747 {3}	0.00821 {5}	0.01011 {8}	0.00830 {6}	0.00817 {4}	0.00873 {7}	0.00423 {1}	0.00619 {2}
		$\hat{\beta}$	0.01982 {1}	0.04549 {8}	0.02675 {4}	0.04547 {7}	0.02006 {2}	0.02743 {5}	0.02576 {3}	0.02885 {6}
		$\hat{\lambda}$	0.00034 {2.5}	0.00051 {7.5}	0.00035 {4.5}	0.00051 {7.5}	0.00035 {4.5}	0.00041 {6}	0.00028 {1}	0.00034 {2.5}
		$\hat{\theta}$	0.01475 {2}	0.02912 {5}	0.03325 {8}	0.02942 {6}	0.01244 {1}	0.01544 {3}	0.02514 {4}	0.03295 {7}
	MRE	$\hat{\alpha}$	0.02252 {3}	0.02550 {7}	0.02497 {6}	0.02559 {8}	0.02391 {4}	0.02484 {5}	0.01782 {1}	0.02173 {2}
		$\hat{\beta}$	0.22349 {2}	0.33861 {7}	0.25898 {5}	0.33863 {8}	0.22334 {1}	0.25635 {4}	0.25406 {3}	0.26725 {6}
		$\hat{\lambda}$	0.05880 {3}	0.07411 {8}	0.05751 {2}	0.07408 {7}	0.05972 {4}	0.06475 {6}	0.05494 {1}	0.06045 {5}
		$\hat{\theta}$	0.03762 {2}	0.05180 {5}	0.05488 {8}	0.05196 {6}	0.03532 {1}	0.03862 {3}	0.04711 {4}	0.05375 {7}
		$\sum Ranks$	28.5 {2}	80 {7}	66.5 {6}	84 {8}	31.5 {3}	57 {4}	27 {1}	57.5 {5}
500	|BIAS|	$\hat{\alpha}$	0.05718 {3}	0.06497 {6}	0.06988 {8}	0.06512 {7}	0.06054 {4}	0.06315 {5}	0.04669 {1}	0.05514 {2}
		$\hat{\beta}$	0.09316 {2}	0.14046 {7}	0.10910 {5}	0.14048 {8}	0.09274 {1}	0.10629 {3}	0.10739 {4}	0.11317 {6}
		$\hat{\lambda}$	0.01231 {2}	0.01547 {7.5}	0.01247 {3}	0.01547 {7.5}	0.01249 {4}	0.01363 {6}	0.01171 {1}	0.01281 {5}
		$\hat{\theta}$	0.07898 {2}	0.10906 {5}	0.11815 {8}	0.10919 {6}	0.07437 {1}	0.08105 {3}	0.09888 {4}	0.11372 {7}
	MSE	$\hat{\alpha}$	0.00531 {3}	0.00593 {5}	0.00889 {8}	0.00597 {6}	0.00583 {4}	0.00624 {7}	0.00326 {1}	0.00448 {2}
		$\hat{\beta}$	0.01381 {1}	0.03133 {7.5}	0.01879 {4}	0.03133 {7.5}	0.01394 {2}	0.01880 {5}	0.01827 {3}	0.02050 {6}
		$\hat{\lambda}$	0.00024 {2.5}	0.00035 {7.5}	0.00027 {5}	0.00035 {7.5}	0.00025 {4}	0.00029 {6}	0.00020 {1}	0.00024 {2.5}
		$\hat{\theta}$	0.01042 {2}	0.02059 {5}	0.02473 {8}	0.02071 {6}	0.00881 {1}	0.01072 {3}	0.01763 {4}	0.02348 {7}
	MRE	$\hat{\alpha}$	0.01906 {3}	0.02166 {6}	0.02329 {8}	0.02171 {7}	0.02018 {4}	0.02105 {5}	0.01556 {1}	0.01838 {2}
		$\hat{\beta}$	0.18631 {2}	0.28092 {7}	0.21820 {5}	0.28096 {8}	0.18548 {1}	0.21258 {3}	0.21478 {4}	0.22634 {6}
		$\hat{\lambda}$	0.04922 {2}	0.06189 {8}	0.04987 {3}	0.06186 {7}	0.04998 {4}	0.05452 {6}	0.04684 {1}	0.05124 {5}
		$\hat{\theta}$	0.03159 {2}	0.04362 {5}	0.04726 {8}	0.04368 {6}	0.02975 {1}	0.03242 {3}	0.03955 {4}	0.04549 {7}
		$\sum Ranks$	26.5 {1}	76.5 {7}	73 {6}	83.5 {8}	31 {3}	55 {4}	29 {2}	57.5 {5}

Table 6: Simulation results for $\underline{\eta} = (\alpha = 3, \beta = 0.25, \lambda = 1.5, \theta = 2.5)^\intercal$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.18909 {2}	0.25679 {7}	0.24424 {5}	0.26975 {8}	0.18843 {1}	0.19339 {3}	0.23149 {4}	0.25619 {6}
		$\hat{\beta}$	0.23806 {2}	0.37063 {7}	0.29784 {6}	0.37093 {8}	0.09169 {1}	0.25924 {3}	0.27965 {4}	0.29398 {5}
		$\hat{\lambda}$	0.10810 {3}	0.12521 {7}	0.11414 {4}	0.12654 {8}	0.10710 {2}	0.10609 {1}	0.11688 {6}	0.11455 {5}
		$\hat{\theta}$	0.21253 {1}	0.26865 {6}	0.21935 {3}	0.27783 {7}	0.21894 {2}	0.22364 {4}	0.22793 {5}	0.27800 {8}
	MSE	$\hat{\alpha}$	0.08246 {3}	0.18749 {7}	0.14239 {6}	0.22755 {8}	0.06215 {2}	0.06133 {1}	0.11519 {4}	0.13617 {5}
		$\hat{\beta}$	0.10426 {1}	0.37676 {7}	0.16943 {6}	0.38763 {8}	0.11851 {3}	0.11411 {2}	0.15443 {4}	0.16833 {5}
		$\hat{\lambda}$	0.01899 {3}	0.02581 {7}	0.02109 {4}	0.02726 {8}	0.01834 {2}	0.01803 {1}	0.02298 {6}	0.02242 {5}
		$\hat{\theta}$	0.08003 {2}	0.13277 {6}	0.07996 {1}	0.14709 {8}	0.08136 {4}	0.08029 {3}	0.08520 {5}	0.13775 {7}
	MRE	$\hat{\alpha}$	0.06303 {2}	0.08560 {7}	0.08141 {5}	0.08992 {8}	0.06281 {1}	0.06446 {3}	0.07716 {4}	0.08540 {6}
		$\hat{\beta}$	0.95226 {1}	1.48253 {7}	1.19135 {6}	1.48371 {8}	1.02002 {2}	1.03695 {3}	1.11859 {4}	1.17593 {5}
		$\hat{\lambda}$	0.07207 {3}	0.08347 {7}	0.07609 {4}	0.08436 {8}	0.07140 {2}	0.07072 {1}	0.07792 {6}	0.07637 {5}
		$\hat{\theta}$	0.08501 {1}	0.10746 {6}	0.08774 {3}	0.11113 {7}	0.08758 {2}	0.08946 {4}	0.09117 {5}	0.11120 {8}
		$\sum Ranks$	24 {1.5}	81 {7}	53 {4}	94 {8}	24 {1.5}	29 {3}	57 {5}	70 {6}
150	|BIAS|	$\hat{\alpha}$	0.10768 {1}	0.14188 {7}	0.12522 {5}	0.14346 {8}	0.10814 {2}	0.11176 {3}	0.12479 {4}	0.14179 {6}
		$\hat{\beta}$	0.13872 {1}	0.21097 {8}	0.16675 {6}	0.21073 {7}	0.14209 {2}	0.14909 {3}	0.16080 {4}	0.16668 {5}
		$\hat{\lambda}$	0.06438 {2}	0.07706 {8}	0.06904 {6}	0.07701 {7}	0.06353 {1}	0.06480 {3}	0.06888 {5}	0.06613 {4}
		$\hat{\theta}$	0.12525 {1}	0.15533 {7}	0.13531 {5}	0.15651 {8}	0.12575 {2}	0.12945 {3}	0.13272 {4}	0.15517 {6}
	MSE	$\hat{\alpha}$	0.01918 {2}	0.03559 {7}	0.02949 {5}	0.03703 {8}	0.01877 {1}	0.01991 {3}	0.02732 {4}	0.03517 {6}
		$\hat{\beta}$	0.02962 {1}	0.07372 {8}	0.04414 {5}	0.07352 {7}	0.03253 {2}	0.03504 {3}	0.04214 {4}	0.04537 {6}
		$\hat{\lambda}$	0.00648 {2}	0.00946 {8}	0.00753 {5}	0.00945 {7}	0.00633 {1}	0.00652 {3}	0.00762 {6}	0.00694 {4}
		$\hat{\theta}$	0.02484 {1}	0.03887 {6}	0.03144 {5}	0.04012 {8}	0.02512 {2}	0.02632 {3}	0.02804 {4}	0.03899 {7}
	MRE	$\hat{\alpha}$	0.03589 {1}	0.04729 {7}	0.04174 {5}	0.04782 {8}	0.03605 {2}	0.03725 {3}	0.04160 {4}	0.04726 {6}
		$\hat{\beta}$	0.55488 {1}	0.84388 {8}	0.66701 {6}	0.84293 {7}	0.56836 {2}	0.59635 {3}	0.64322 {4}	0.66673 {5}
		$\hat{\lambda}$	0.04292 {2}	0.05138 {8}	0.04603 {6}	0.05134 {7}	0.04235 {1}	0.04320 {3}	0.04592 {5}	0.04409 {4}
		$\hat{\theta}$	0.05010 {1}	0.06213 {7}	0.05412 {5}	0.06260 {8}	0.05030 {2}	0.05178 {3}	0.05309 {4}	0.06207 {6}
		$\sum Ranks$	16 {1}	89 {7}	64 {5}	90 {8}	20 {2}	36 {3}	52 {4}	65 {6}
350	|BIAS|	$\hat{\alpha}$	0.07171 {2}	0.09596 {7}	0.08369 {5}	0.09643 {8}	0.07072 {1}	0.07420 {3}	0.08040 {4}	0.09257 {6}
		$\hat{\beta}$	0.09101 {1}	0.14368 {8}	0.10919 {5}	0.14363 {7}	0.09169 {2}	0.09692 {3}	0.10677 {4}	0.10929 {6}
		$\hat{\lambda}$	0.04151 {2}	0.05026 {7}	0.04482 {6}	0.05027 {8}	0.04128 {1}	0.04235 {3}	0.04418 {5}	0.04291 {4}
		$\hat{\theta}$	0.08290 {2}	0.10345 {7}	0.09165 {5}	0.10384 {8}	0.08252 {1}	0.08602 {3}	0.08821 {4}	0.10073 {6}
	MSE	$\hat{\alpha}$	0.00816 {2}	0.01486 {7}	0.01276 {5}	0.01514 {8}	0.00797 {1}	0.00874 {3}	0.01062 {4}	0.01417 {6}
		$\hat{\beta}$	0.01293 {1}	0.03179 {8}	0.01861 {5}	0.03175 {7}	0.01338 {2}	0.01496 {3}	0.01801 {4}	0.01911 {6}
		$\hat{\lambda}$	0.00274 {2}	0.00400 {7.5}	0.00320 {6}	0.00400 {7.5}	0.00270 {1}	0.00284 {3}	0.00310 {5}	0.00291 {4}
		$\hat{\theta}$	0.01085 {2}	0.01654 {7}	0.01420 {5}	0.01679 {8}	0.01081 {1}	0.01174 {3}	0.01225 {4}	0.01592 {6}
	MRE	$\hat{\alpha}$	0.02390 {2}	0.03199 {7}	0.02790 {5}	0.03214 {8}	0.02357 {1}	0.02473 {3}	0.02680 {4}	0.03086 {6}
		$\hat{\beta}$	0.36402 {1}	0.57473 {8}	0.43675 {5}	0.57453 {7}	0.36677 {2}	0.38766 {3}	0.42708 {4}	0.43715 {6}
		$\hat{\lambda}$	0.02767 {2}	0.03350 {7}	0.02988 {6}	0.03351 {8}	0.02752 {1}	0.02823 {3}	0.02945 {5}	0.02861 {4}
		$\hat{\theta}$	0.03316 {2}	0.04138 {7}	0.03666 {5}	0.04153 {8}	0.03301 {1}	0.03441 {3}	0.03528 {4}	0.04029 {6}
		$\sum Ranks$	21 {2}	87.5 {7}	63 {5}	92.5 {8}	15 {1}	36 {3}	51 {4}	66 {6}
500	|BIAS|	$\hat{\alpha}$	0.06055 {2}	0.08252 {7}	0.07269 {5}	0.08293 {8}	0.05955 {1}	0.06258 {3}	0.06855 {4}	0.07802 {6}
		$\hat{\beta}$	0.07555 {1}	0.12197 {7}	0.09292 {5}	0.12199 {8}	0.07572 {2}	0.08045 {3}	0.09086 {4}	0.09297 {6}
		$\hat{\lambda}$	0.03477 {2}	0.04234 {7.5}	0.03810 {6}	0.04234 {7.5}	0.03467 {1}	0.03555 {3}	0.03717 {5}	0.03652 {4}
		$\hat{\theta}$	0.07037 {2}	0.08896 {7}	0.07993 {5}	0.08906 {8}	0.06982 {1}	0.07276 {3}	0.07536 {4}	0.08549 {6}
	MSE	$\hat{\alpha}$	0.00583 {2}	0.01102 {7}	0.01005 {5}	0.01121 {8}	0.00567 {1}	0.00616 {3}	0.00773 {4}	0.01011 {6}
		$\hat{\beta}$	0.00903 {1}	0.02306 {7.5}	0.01349 {5}	0.02306 {7.5}	0.00922 {2}	0.01030 {3}	0.01304 {4}	0.01379 {6}
		$\hat{\lambda}$	0.00190 {2}	0.00283 {7.5}	0.00232 {6}	0.00283 {7.5}	0.00188 {1}	0.00197 {3}	0.00217 {5}	0.00209 {4}
		$\hat{\theta}$	0.00779 {2}	0.01219 {7}	0.01118 {5}	0.01229 {8}	0.00772 {1}	0.00827 {3}	0.00884 {4}	0.01155 {6}
	MRE	$\hat{\alpha}$	0.02018 {2}	0.02751 {7}	0.02423 {5}	0.02764 {8}	0.01985 {1}	0.02086 {3}	0.02285 {4}	0.02601 {6}
		$\hat{\beta}$	0.30219 {1}	0.48789 {7}	0.37167 {5}	0.48794 {8}	0.30289 {2}	0.32181 {3}	0.36346 {4}	0.37187 {6}
		$\hat{\lambda}$	0.02318 {2}	0.02823 {7.5}	0.02540 {6}	0.02823 {7.5}	0.02311 {1}	0.02370 {3}	0.02478 {5}	0.02435 {4}
		$\hat{\theta}$	0.02815 {2}	0.03558 {7}	0.03197 {5}	0.03562 {8}	0.02793 {1}	0.02910 {3}	0.03014 {4}	0.03420 {6}
		$\sum Ranks$	21 {2}	86 {7}	63 {5}	94 {8}	15 {1}	36 {3}	51 {4}	66 {6}

Table 7: Simulation results for $\underline{\eta} = (\alpha = 0.5, \beta = 2, \lambda = 1.5, \theta = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.14212 {5}	0.15744 {6}	0.12109 {4}	0.16708 {7}	0.11923 {3}	0.10216 {1}	0.11787 {2}	0.17031 {8}
		$\hat{\beta}$	0.96245 {5}	1.12938 {6}	0.91866 {4}	1.16325 {8}	0.24324 {1}	0.56191 {2}	0.85392 {3}	1.14080 {7}
		$\hat{\lambda}$	0.18655 {5}	0.18815 {6}	0.17745 {4}	0.19432 {8}	0.16874 {2}	0.14197 {1}	0.17505 {3}	0.19179 {7}
		$\hat{\theta}$	0.07076 {3}	0.07596 {4}	0.11332 {7}	0.08033 {5}	0.04799 {1}	0.59033 {8}	0.06880 {2}	0.09420 {6}
	MSE	$\hat{\alpha}$	0.06905 {5}	0.07505 {6}	0.04114 {3}	0.09181 {8}	0.04343 {4}	0.02640 {1}	0.03129 {2}	0.07650 {7}
		$\hat{\beta}$	2.98910 {5}	3.44121 {7}	2.01134 {3}	3.88222 {8}	2.35070 {4}	1.65170 {2}	1.51514 {1}	3.37338 {6}
		$\hat{\lambda}$	0.06355 {7}	0.06295 {5}	0.05355 {4}	0.06701 {8}	0.05116 {3}	0.03787 {1}	0.05057 {2}	0.06352 {6}
		$\hat{\theta}$	0.01127 {3}	0.01212 {4}	0.03197 {7}	0.01424 {5}	0.00837 {1}	5.69884 {8}	0.01020 {2}	0.01925 {6}
	MRE	$\hat{\alpha}$	0.28423 {5}	0.31488 {6}	0.24219 {4}	0.33416 {7}	0.23846 {3}	0.20433 {1}	0.23574 {2}	0.34061 {8}
		$\hat{\beta}$	0.48123 {4}	0.56469 {5}	0.45933 {3}	0.58163 {7}	0.41758 {1}	2.24763 {8}	0.42696 {2}	0.57040 {6}
		$\hat{\lambda}$	0.12437 {5}	0.12544 {6}	0.11830 {4}	0.12954 {8}	0.11249 {2}	0.09465 {1}	0.11670 {3}	0.12786 {7}
		$\hat{\theta}$	0.02830 {3}	0.03038 {4}	0.04533 {7}	0.03213 {5}	0.01919 {1}	0.23613 {8}	0.02752 {2}	0.03768 {6}
		$\sum Ranks$	55 {5}	65 {6}	54 {4}	84 {8}	26 {1.5}	42 {3}	26 {1.5}	80 {7}
150	|BIAS|	$\hat{\alpha}$	0.06071 {3}	0.07368 {5}	0.05897 {2}	0.07447 {6}	0.05750 {1}	0.07727 {7}	0.06117 {4}	0.08073 {8}
		$\hat{\beta}$	0.42339 {3}	0.53885 {6}	0.44823 {5}	0.54121 {7}	0.39697 {2}	0.32374 {1}	0.43676 {4}	0.55118 {8}
		$\hat{\lambda}$	0.10043 {3}	0.10590 {7}	0.10381 {5}	0.10707 {8}	0.09607 {2}	0.08418 {1}	0.10092 {4}	0.10457 {6}
		$\hat{\theta}$	0.04055 {3}	0.04112 {4}	0.09383 {7}	0.04143 {5}	0.02380 {1}	0.24914 {8}	0.03422 {2}	0.04970 {6}
	MSE	$\hat{\alpha}$	0.00669 {4}	0.00973 {5}	0.00647 {3}	0.01007 {6}	0.00571 {1}	0.01336 {8}	0.00638 {2}	0.01209 {7}
		$\hat{\beta}$	0.31255 {3}	0.51753 {6}	0.34730 {5}	0.52195 {7}	0.28345 {1}	0.30878 {2}	0.32559 {4}	0.55341 {8}
		$\hat{\lambda}$	0.01606 {3}	0.01785 {7}	0.01756 {5}	0.01817 {8}	0.01463 {2}	0.01294 {1}	0.01631 {4}	0.01758 {6}
		$\hat{\theta}$	0.00375 {5}	0.00327 {3}	0.02355 {7}	0.00336 {4}	0.00093 {1}	0.45362 {8}	0.00237 {2}	0.00484 {6}
	MRE	$\hat{\alpha}$	0.12142 {3}	0.14737 {5}	0.11795 {2}	0.14893 {6}	0.11500 {1}	0.15454 {7}	0.12233 {4}	0.16147 {8}
		$\hat{\beta}$	0.21170 {2}	0.26942 {5}	0.22411 {4}	0.27060 {6}	0.19849 {1}	1.29496 {8}	0.21838 {3}	0.27559 {7}
		$\hat{\lambda}$	0.06695 {3}	0.07060 {7}	0.06921 {5}	0.07138 {8}	0.06405 {2}	0.05612 {1}	0.06728 {4}	0.06971 {6}
		$\hat{\theta}$	0.01622 {3}	0.01645 {4}	0.03753 {7}	0.01657 {5}	0.00952 {1}	0.09965 {8}	0.01369 {2}	0.01988 {6}
		$\sum Ranks$	38 {2}	64 {6}	57 {4}	76 {7}	16 {1}	60 {5}	39 {3}	82 {8}
350	|BIAS|	$\hat{\alpha}$	0.03796 {2}	0.04663 {5}	0.03911 {3}	0.04689 {6}	0.03694 {1}	0.06995 {8}	0.03958 {4}	0.05074 {7}
		$\hat{\beta}$	0.26317 {3}	0.34095 {6}	0.28123 {5}	0.34153 {7}	0.24324 {2}	0.23731 {1}	0.27852 {4}	0.34286 {8}
		$\hat{\lambda}$	0.06203 {4}	0.06579 {7}	0.06494 {6}	0.06603 {8}	0.06089 {2}	0.05846 {1}	0.06191 {3}	0.06417 {5}
		$\hat{\theta}$	0.02884 {6}	0.02249 {4}	0.07042 {7}	0.02241 {3}	0.01490 {1}	0.15059 {8}	0.02226 {2}	0.02867 {5}
	MSE	$\hat{\alpha}$	0.00242 {2}	0.00352 {5}	0.00281 {4}	0.00359 {6}	0.00223 {1}	0.01210 {8}	0.00252 {3}	0.00421 {7}
		$\hat{\beta}$	0.11426 {2}	0.19063 {6}	0.12937 {4}	0.19114 {7}	0.10834 {1}	0.15284 {5}	0.12658 {3}	0.19269 {8}
		$\hat{\lambda}$	0.00619 {3}	0.00691 {7}	0.00688 {6}	0.00696 {8}	0.00592 {1}	0.00635 {4}	0.00614 {2}	0.00656 {5}
		$\hat{\theta}$	0.00191 {6}	0.00099 {3}	0.01215 {7}	0.00098 {2}	0.00035 {1}	0.13937 {8}	0.00104 {4}	0.00170 {5}
	MRE	$\hat{\alpha}$	0.07592 {2}	0.09326 {5}	0.07822 {3}	0.09378 {6}	0.07388 {1}	0.13990 {8}	0.07916 {4}	0.10148 {7}
		$\hat{\beta}$	0.13158 {2}	0.17048 {5}	0.14061 {4}	0.17076 {6}	0.12162 {1}	0.94924 {8}	0.13926 {3}	0.17143 {7}
		$\hat{\lambda}$	0.04135 {4}	0.04386 {7}	0.04330 {6}	0.04402 {8}	0.04060 {2}	0.03897 {1}	0.04127 {3}	0.04278 {5}
		$\hat{\theta}$	0.01153 {6}	0.00900 {4}	0.02817 {7}	0.00896 {3}	0.00596 {1}	0.06024 {8}	0.00890 {2}	0.01147 {5}
		$\sum Ranks$	42 {3}	64 {5}	62 {4}	70 {7}	15 {1}	68 {6}	37 {2}	74 {8}
500	|BIAS|	$\hat{\alpha}$	0.03140 {2}	0.03871 {5}	0.03325 {4}	0.03886 {6}	0.03078 {1}	0.06318 {8}	0.03298 {3}	0.04281 {7}
		$\hat{\beta}$	0.22261 {3}	0.27996 {6}	0.23553 {5}	0.28028 {7}	0.20141 {2}	0.20103 {1}	0.23390 {4}	0.28829 {8}
		$\hat{\lambda}$	0.05310 {4}	0.05474 {6}	0.05591 {8}	0.05491 {7}	0.05202 {2}	0.05094 {1}	0.05271 {3}	0.05440 {5}
		$\hat{\theta}$	0.02527 {6}	0.01740 {2}	0.06399 {7}	0.01755 {3}	0.01256 {1}	0.12197 {8}	0.01951 {4}	0.02268 {5}
	MSE	$\hat{\alpha}$	0.00163 {2}	0.00240 {5}	0.00200 {4}	0.00243 {6}	0.00155 {1}	0.00960 {8}	0.00173 {3}	0.00294 {7}
		$\hat{\beta}$	0.08011 {2}	0.12655 {6}	0.08929 {4}	0.12677 {7}	0.07618 {1}	0.11316 {5}	0.08819 {3}	0.13376 {8}
		$\hat{\lambda}$	0.00448 {3}	0.00478 {5}	0.00509 {7}	0.00479 {6}	0.00425 {1}	0.00557 {8}	0.00440 {2}	0.00468 {4}
		$\hat{\theta}$	0.00156 {6}	0.00057 {2}	0.01039 {7}	0.00059 {3}	0.00025 {1}	0.12057 {8}	0.00083 {4}	0.00107 {5}
	MRE	$\hat{\alpha}$	0.06280 {2}	0.07743 {5}	0.06649 {4}	0.07772 {6}	0.06156 {1}	0.12636 {8}	0.06596 {3}	0.08563 {7}
		$\hat{\beta}$	0.11130 {2}	0.13998 {5}	0.11776 {4}	0.14014 {6}	0.10070 {1}	0.80412 {8}	0.11695 {3}	0.14415 {7}
		$\hat{\lambda}$	0.03540 {4}	0.03649 {6}	0.03727 {8}	0.03660 {7}	0.03468 {2}	0.03396 {1}	0.03514 {3}	0.03627 {5}
		$\hat{\theta}$	0.01011 {6}	0.00696 {2}	0.02560 {7}	0.00702 {3}	0.00502 {1}	0.04879 {8}	0.00780 {4}	0.00907 {5}
		$\sum Ranks$	42 {3}	55 {4}	69 {6}	67 {5}	15 {1}	72 {7}	39 {2}	73 {8}

Table 8: Simulation results for $\underline{\eta} = (\alpha = 0.5, \beta = 0.25, \lambda = 1.5, \theta = 2.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
50	|BIAS|	$\hat{\alpha}$	0.07083 {1}	0.09564 {6}	0.07798 {3}	0.10017 {7}	0.07303 {2}	0.10359 {8}	0.07897 {4}	0.09058 {5}
		$\hat{\beta}$	0.23806 {2}	0.36984 {6}	0.29777 {5}	0.37012 {7}	0.09111 {1}	0.54506 {8}	0.27951 {3}	0.29401 {4}
		$\hat{\lambda}$	0.10930 {2}	0.12000 {6}	0.11320 {3}	0.12267 {7}	0.10688 {1}	0.14229 {8}	0.11331 {4}	0.11528 {5}
		$\hat{\theta}$	0.04771 {3}	0.04858 {4}	0.07908 {7}	0.05204 {5}	0.03420 {1}	0.52003 {8}	0.04640 {2}	0.06502 {6}
	MSE	$\hat{\alpha}$	0.01183 {2}	0.02616 {6}	0.01209 {3}	0.02900 {7}	0.01012 {1}	0.03274 {8}	0.01257 {4}	0.01710 {5}
		$\hat{\beta}$	0.10426 {1}	0.34701 {6}	0.16940 {5}	0.34765 {7}	0.11878 {2}	1.28538 {8}	0.15439 {3}	0.16834 {4}
		$\hat{\lambda}$	0.02009 {2}	0.02421 {6}	0.02130 {3}	0.02543 {7}	0.01839 {1}	0.03784 {8}	0.02131 {4}	0.02150 {5}
		$\hat{\theta}$	0.00507 {4}	0.00480 {2}	0.01742 {7}	0.00574 {5}	0.00188 {1}	2.89437 {8}	0.00502 {3}	0.01021 {6}
	MRE	$\hat{\alpha}$	0.14166 {1}	0.19128 {6}	0.15596 {3}	0.20034 {7}	0.14605 {2}	0.20718 {8}	0.15793 {4}	0.18117 {5}
		$\hat{\beta}$	0.95226 {1}	1.47936 {6}	1.19108 {5}	1.48047 {7}	1.02022 {2}	2.18023 {8}	1.11805 {3}	1.17604 {4}
		$\hat{\lambda}$	0.07287 {2}	0.08000 {6}	0.07547 {3}	0.08178 {7}	0.07125 {1}	0.09486 {8}	0.07554 {4}	0.07685 {5}
		$\hat{\theta}$	0.01909 {3}	0.01943 {4}	0.03163 {7}	0.02082 {5}	0.01368 {1}	0.20801 {8}	0.01856 {2}	0.02601 {6}
		$\sum Ranks$	24 {2}	64 {6}	54 {4}	78 {7}	16 {1}	96 {8}	40 {3}	60 {5}
150	|BIAS|	$\hat{\alpha}$	0.04020 {1}	0.05166 {6}	0.04422 {3}	0.05224 {7}	0.04103 {2}	0.07907 {8}	0.04476 {4}	0.05070 {5}
		$\hat{\beta}$	0.13872 {1}	0.21097 {7}	0.16676 {5}	0.21075 {6}	0.14176 {2}	0.35514 {8}	0.16079 {3}	0.16675 {4}
		$\hat{\lambda}$	0.06287 {2}	0.06983 {6}	0.06532 {3}	0.07043 {7}	0.06261 {1}	0.08745 {8}	0.06581 {4}	0.06638 {5}
		$\hat{\theta}$	0.02561 {3}	0.02568 {4}	0.05304 {7}	0.02617 {5}	0.01898 {1}	0.29648 {8}	0.02502 {2}	0.03225 {6}
	MSE	$\hat{\alpha}$	0.00264 {1}	0.00459 {6}	0.00340 {4}	0.00480 {7}	0.00273 {2}	0.01418 {8}	0.00337 {3}	0.00435 {5}
		$\hat{\beta}$	0.02962 {1}	0.07373 {7}	0.04414 {4}	0.07353 {6}	0.03256 {2}	0.52144 {8}	0.04213 {3}	0.04540 {5}
		$\hat{\lambda}$	0.00628 {2}	0.00774 {6}	0.00681 {3}	0.00790 {7}	0.00613 {1}	0.01467 {8}	0.00691 {4}	0.00696 {5}
		$\hat{\theta}$	0.00148 {5}	0.00122 {2}	0.00749 {7}	0.00128 {3}	0.00058 {1}	1.36571 {8}	0.00143 {4}	0.00229 {6}
	MRE	$\hat{\alpha}$	0.08039 {1}	0.10332 {6}	0.08844 {3}	0.10449 {7}	0.08206 {2}	0.15814 {8}	0.08951 {4}	0.10140 {5}
		$\hat{\beta}$	0.55488 {1}	0.84388 {7}	0.66703 {5}	0.84298 {6}	0.56705 {2}	1.42057 {8}	0.64316 {3}	0.66702 {4}
		$\hat{\lambda}$	0.04191 {2}	0.04655 {6}	0.04354 {3}	0.04695 {7}	0.04174 {1}	0.05830 {8}	0.04387 {4}	0.04426 {5}
		$\hat{\theta}$	0.01024 {3}	0.01027 {4}	0.02122 {7}	0.01047 {5}	0.00759 {1}	0.11859 {8}	0.01001 {2}	0.01290 {6}
		$\sum Ranks$	23 {2}	67 {6}	54 {4}	73 {7}	18 {1}	96 {8}	40 {3}	61 {5}
350	|BIAS|	$\hat{\alpha}$	0.02667 {1}	0.03449 {6}	0.02951 {4}	0.03465 {7}	0.02690 {2}	0.06882 {8}	0.02947 {3}	0.03317 {5}
		$\hat{\beta}$	0.09100 {1}	0.14368 {7}	0.10918 {4}	0.14364 {6}	0.09111 {2}	0.23419 {8}	0.10678 {3}	0.10931 {5}
		$\hat{\lambda}$	0.03995 {1}	0.04348 {6}	0.04134 {4}	0.04379 {7}	0.04028 {2}	0.05899 {8}	0.04126 {3}	0.04195 {5}
		$\hat{\theta}$	0.01474 {2}	0.01603 {4}	0.03931 {7}	0.01669 {5}	0.01196 {1}	0.14576 {8}	0.01518 {3}	0.01789 {6}
	MSE	$\hat{\alpha}$	0.00113 {1}	0.00188 {6}	0.00145 {4}	0.00192 {7}	0.00115 {2}	0.01174 {8}	0.00139 {3}	0.00176 {5}
		$\hat{\beta}$	0.01293 {1}	0.03178 {7}	0.01861 {4}	0.03176 {6}	0.01335 {2}	0.16608 {8}	0.01801 {3}	0.01912 {5}
		$\hat{\lambda}$	0.00255 {1.5}	0.00302 {6}	0.00275 {4}	0.00306 {7}	0.00255 {1.5}	0.00699 {8}	0.00274 {3}	0.00285 {5}
		$\hat{\theta}$	0.00045 {2}	0.00052 {3}	0.00411 {7}	0.00065 {5}	0.00023 {1}	0.18925 {8}	0.00056 {4}	0.00071 {6}
	MRE	$\hat{\alpha}$	0.05335 {1}	0.06899 {6}	0.05902 {4}	0.06930 {7}	0.05380 {2}	0.13765 {8}	0.05893 {3}	0.06634 {5}
		$\hat{\beta}$	0.36402 {1}	0.57472 {7}	0.43673 {4}	0.57457 {6}	0.36445 {2}	0.93675 {8}	0.42711 {3}	0.43723 {5}
		$\hat{\lambda}$	0.02663 {1}	0.02899 {6}	0.02756 {4}	0.02919 {7}	0.02685 {2}	0.03933 {8}	0.02751 {3}	0.02796 {5}
		$\hat{\theta}$	0.00590 {2}	0.00641 {4}	0.01572 {7}	0.00668 {5}	0.00479 {1}	0.05831 {8}	0.00607 {3}	0.00715 {6}
		$\sum Ranks$	15.5 {1}	68 {6}	57 {4}	75 {7}	20.5 {2}	96 {8}	37 {3}	63 {5}
500	|BIAS|	$\hat{\alpha}$	0.02258 {1}	0.02972 {6}	0.02535 {4}	0.02979 {7}	0.02270 {2}	0.06337 {8}	0.02515 {3}	0.02824 {5}
		$\hat{\beta}$	0.07555 {2}	0.12198 {7}	0.09292 {4}	0.12196 {6}	0.07513 {1}	0.20680 {8}	0.09087 {3}	0.09298 {5}
		$\hat{\lambda}$	0.03373 {1}	0.03666 {6}	0.03567 {5}	0.03689 {7}	0.03387 {2}	0.05308 {8}	0.03501 {3}	0.03559 {4}
		$\hat{\theta}$	0.01229 {2}	0.01330 {4}	0.03373 {7}	0.01390 {5}	0.01005 {1}	0.13936 {8}	0.01263 {3}	0.01421 {6}
	MSE	$\hat{\alpha}$	0.00081 {1}	0.00139 {6}	0.00108 {4}	0.00141 {7}	0.00082 {2}	0.01076 {8}	0.00101 {3}	0.00128 {5}
		$\hat{\beta}$	0.00903 {1}	0.02306 {7}	0.01349 {4}	0.02304 {6}	0.00922 {2}	0.13657 {8}	0.01304 {3}	0.01380 {5}
		$\hat{\lambda}$	0.00180 {1.5}	0.00211 {6}	0.00202 {5}	0.00215 {7}	0.00180 {1.5}	0.00607 {8}	0.00193 {3}	0.00201 {4}
		$\hat{\theta}$	0.00031 {2}	0.00036 {4}	0.00316 {7}	0.00049 {6}	0.00016 {1}	0.24949 {8}	0.00034 {3}	0.00041 {5}
	MRE	$\hat{\alpha}$	0.04515 {1}	0.05945 {6}	0.05069 {4}	0.05958 {7}	0.04541 {2}	0.12674 {8}	0.05029 {3}	0.05648 {5}
		$\hat{\beta}$	0.30219 {2}	0.48791 {7}	0.37168 {4}	0.48784 {6}	0.30052 {1}	0.82718 {8}	0.36348 {3}	0.37194 {5}
		$\hat{\lambda}$	0.02248 {1}	0.02444 {6}	0.02378 {5}	0.02459 {7}	0.02258 {2}	0.03539 {8}	0.02334 {3}	0.02372 {4}
		$\hat{\theta}$	0.00492 {2}	0.00532 {4}	0.01349 {7}	0.00556 {5}	0.00402 {1}	0.05575 {8}	0.00505 {3}	0.00568 {6}
		$\sum Ranks$	17.5 {1}	69 {6}	60 {5}	76 {7}	18.5 {2}	96 {8}	36 {3}	59 {4}

Table 9: Partial and overall ranks of all estimation methods for various combinations of $\underline{\eta}$.

$\underline{\eta}^{\intercal}$	n	MLEs	MPSEs	LSEs	WLSEs	PCEs	CRVMEs	ADEs	RADEs
($\hat{\alpha}$=0.5, $\hat{\beta}$=0.5, $\hat{\lambda}$=0.25, $\hat{\theta}$=1.25)	50	2	6.5	3	6.5	4	8	1	5
	150	2.5	6	5	7	2.5	8	1	4
	350	3	6	5	7	2	8	1	4
	500	2	6	5	7	3	8	1	4
($\hat{\alpha}$=1.75, $\hat{\beta}$=0.5, $\hat{\lambda}$=0.25, $\hat{\theta}$=1.25)	50	2	6	3	7	4	8	1	5
	150	3	7	4	6	2	8	1	5
	350	3	6	5	7	1	8	2	4
	500	3	6	4	5	1	8	3	7
($\hat{\alpha}$=1.5, $\hat{\beta}$=0.25, $\hat{\lambda}$=0.25,$\hat{\theta}$=1.25)	50	2	7.5	3	7.5	4	6	1	5
	150	2	6.5	5	6.5	3	8	1	4
	350	3	6	5	7	2	8	1	4
	500	2	6.5	5	6.5	1	8	3	4
($\hat{\alpha}$=1.5, $\hat{\beta}$=0.25, $\hat{\lambda}$=1.5, $\hat{\theta}$=2.5)	50	2	7	5	8	1	3	4	6
	150	1.5	7	5	8	1.5	3	4	6
	350	1	7	6	8	2	3	4	5
	500	2	7	6	8	1	3	4	5
($\hat{\alpha}$=3, $\hat{\beta}$=0.5, $\hat{\lambda}$=0.25, $\hat{\theta}$=2.5)	50	1.5	7	4	8	3	5	1.5	6
	150	2	7	4	8	3	5	1	6
	350	2	7	6	8	3	4	1	5
	500	1	7	6	8	3	4	2	5
($\hat{\alpha}$=3, $\hat{\beta}$=0.25, $\hat{\lambda}$=1.5,$\hat{\theta}$=2.5)	50	1.5	7	4	8	1.5	3	5	6
	150	1	7	5	8	2	3	4	6
	350	2	7	5	8	1	3	4	6
	500	2	7	5	8	1	3	4	6
($\hat{\alpha}$=0.5, $\hat{\beta}$=2, $\hat{\lambda}$=1.5, $\hat{\theta}$=1.25)	50	5	6	4	8	1.5	3	1.5	7
	150	2	6	4	7	1	5	3	8
	350	3	5	4	7	1	6	2	8
	500	3	4	6	5	1	7	2	8
($\hat{\alpha}$=0.5, $\hat{\beta}$=0.25, $\hat{\lambda}$=1.5, $\hat{\theta}$=2.5)	50	2	6	4	7	1	8	3	5
	150	2	6	4	7	1	8	3	5
	350	1	6	4	7	2	8	3	5
	500	1	6	5	7	2	8	3	4
$\sum Ranks$		68	205	148	231	63	189	76	173
Overall Rank		2	7	4	8	1	6	3	5

6 Real-Life Data Analysis

In this section, the importance of the EOWLL distribution in medicine and geology fields is illustrated by fitting two real-life data sets. The first data was originally used by Usman et al. [36]. This data consists of 30 observations of March precipitation (in inches) in Minneapolis/St Paul. The observations are: 0.47, 0.77, 0.81, 1.74, 1.20, 1.20, 1.95, 1.43, 3.37, 2.20, 3.00, 1.51, 3.09, 2.10, 0.52, 1.31, 1.62, 0.32, 0.59, 0.81, 1.87, 2.81, 1.18, 1.35, 2.48, 0.96, 4.75, 1.89, 0.90, 2.05. The second data represents the remission times (in months) of 128 bladder cancer patients as mentioned in Lee and Wang [37]. The observations are: 3.48, 0.08, 2.09, 6.94, 4.87, 8.66, 23.63, 13.11, 0.20, 3.52, 2.23, 4.98, 25.74, 9.02, 6.97, 13.29, 2.26, 0.40, 3.57, 7.09, 9.22, 5.06, 13.80, 0.50, 3.64, 0.81, 2.64, 2.46, 5.09, 9.47, 7.26, 14.24, 0.51, 25.82, 2.54, 5.17, 3.70, 7.28, 14.76, 9.74, 26.31, 2.62, 5.32, 3.82, 7.32, 12.07, 14.77, 10.06, 32.15, 3.88, 7.39, 5.32, 10.34, 34.26, 14.83, 0.90, 4.18, 2.69, 5.34, 10.66, 7.59, 17.14, 4.26, 36.66, 15.96, 1.05, 4.23, 2.69, 5.41, 8.65, 10.75, 7.62, 16.62, 1.19,43.01, 2.75, 11.25, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 21.73, 3.36, 22.69, 12.63, 6.93, 2.07, 79.05, 8.37, 2.87, 1.35, 5.62, 11.64, 7.87, 17.36, 3.02, 1.40, 4.34, 7.93, 5.71, 6.25, 6.76, 11.79, 12.02, 18.10, 2.02, 1.46, 3.31, 4.40, 4.51, 5.85, 6.54, 8.26, 8.53, 11.98, 12.03, 19.13, 20.28, 1.76, 2.02, 3.25, 3.36, 4.50.

We compare the fits of the EWOLL distribution and other competitive models namely: the additive Weibull log-logistic (AWLL)[38], Kumaraswamy transmuted log-logistic (KwTLL) [39], Fréchet Topp–Leone Burr XII (FTLBXII) [9], generalized odd log-logistic BXII (GOLLBXII) [10], alpha power transformed log-logistic (APTLL) [11], Topp–Leone BXII (TLBXII) [40], and LL distributions.

Some goodness-of-fit analytical measures are adopted to compare the fitted distributions. These measures include the W^* (Cramér–Von Mises), A^* (Anderson–Darling), and KS (Kolmogorov–Smirnov) statistics with the KS p-value.

The parameters of the fitted models are estimated using the ML methods and the standard errors (SEs) of the estimates are also obtained. Tables 10 and 11 report the parameters' estimates, SEs, and goodness-of-fit analytical measures, for both datasets. The figures in these tables illustrate that the EOWLL distribution has the lowest values for all goodness-of-fit statistics among all fitted models and the highest values for p-value.

The fitted EOWLL PDF, CDF, SF, and probability-probability (P-P) for both data sets are displayed in Figures 3 and 4. The PP plots of the EOWLL model and other its competing models are displayed in Figures 5 and 6, for both data sets, respectively. These plots support the results provided in Tables 10 and 11. The EOWLL model provides a consistently better fit for both data sets.

Table 10: The estimates, their SEs, and goodness-of-fit measures of the fitted distributions for the first data.

Model	Param.	Estimates	SEs	W^*	A^*	KS	p-value
EOWLL	$\hat{\alpha}$	1.5130771	(162.9429587)	0.01453006	0.1108988	0.05456922	0.9999916
	$\hat{\beta}$	0.3124447	(0.4602645)				
	$\hat{\lambda}$	2.2050794	(187.7782201)				
	$\hat{\theta}$	1.4344619	(154.4767052)				
AWLL	$\hat{\alpha}$	0.8973661	(79.61166)	0.02196099	0.1693153	0.06894252	0.9988398
	$\hat{\lambda}$	7.5958742	(1705.04837)				
	$\hat{a}$	9.3547619	(40949.27174)				
	$\hat{b}$	2.0157636	(178.84476)				
	$\hat{c}$	2.9992230	(37693.36566)				
	$\hat{d}$	2.0157661	(178.77794)				
KwTLL	$\hat{\alpha}$	0.02017801	(0.02291894)	0.01467574	0.1049755	0.06334347	0.9997443
	$\hat{\beta}$	0.39101743	(0.46832970)				
	$\hat{\delta}$	0.06445104	(7.66180374)				
	$\hat{a}$	32.31511258	(201.826442)				
	$\hat{b}$	127.06851752	(531.38860943)				
FTLBXII	$\hat{\alpha}$	0.13784861	(0.02000754)	0.02479906	0.16493750	0.07332679	0.9970238
	$\hat{\beta}$	299.643220	(478.2712203)				
	$\hat{\lambda}$	3.29159012	(0.05226548)				
	$\hat{a}$	2.97750150	(0.02427089)				
	$\hat{b}$	3.39224593	(0.02427208)				
GOLLBXII	$\hat{\alpha}$	12.6904032	(87.3875)	0.03161599	0.2067355	0.07428283	0.9964274
	$\hat{\beta}$	1.3684297	(6.803861)				
	$\hat{a}$	0.1855423	(1.288253)				
	$\hat{b}$	1.2683195	(5.344326)				
APTLL	$\hat{\alpha}$	0.9999735	(4.3429372)	0.03142086	0.2056157	0.07415133	0.9965145
	$\hat{a}$	1.4407538	(1.1343252)				
	$\hat{b}$	2.7879079	(0.4215438)				
TLBXII	$\hat{\alpha}$	0.9777833	(0.9061276)	0.05149547	0.3226386	0.1097417	0.8628496
	$\hat{\beta}$	1.4200724	(1.0211569)				
	$\hat{\lambda}$	3.9509996	(5.6976210)				
LL	$\hat{\alpha}$	2.787908	(0.4215438)	0.03142087	0.2056157	0.07415145	0.9965144
	$\hat{\lambda}$	2.767755	(0.9972921)				

Table 11: The estimates, their SEs and goodness-of-fit measures of the fitted distributions for the second data.

Model	Param.	Estimates	SEs	W^*	A^*	KS	p-value
EOWLL	$\hat{\alpha}$	0.4624981	(8.2108498)	0.01945769	0.1307742	0.03505583	0.9975193
	$\hat{\beta}$	0.4830693	(0.2259435)				
	$\hat{\lambda}$	2.4963320	(40.5426549)				
	$\hat{\theta}$	3.0867259	(54.7986454)				
AWLL	$\hat{\alpha}$	1.1929517	(60.22699)	0.1313741	0.7864834	0.07001688	0.5569649
	$\hat{\lambda}$	16.3968415	(1802.43237)				
	$\hat{a}$	1.0341931	(6490.25232)				
	$\hat{b}$	0.8783549	(44.34378)				
	$\hat{c}$	0.7256618	(6504.31449)				
	$\hat{d}$	0.8783548	(44.34597)				
KwTLL	$\hat{\alpha}$	0.03889348	(0.07528510)	0.04896754	0.3216464	0.04848026	0.9243237
	$\hat{\beta}$	0.22607394	(0.09111364)				
	$\hat{\delta}$	-0.6185948	(2.33433556)				
	$\hat{a}$	13.212721	(21.56758502)				
	$\hat{b}$	219.153301	(534.22748580)				
FTLBXII	$\hat{\alpha}$	0.2442945	(0.09103486)	0.1937228	1.26832	0.07885305	0.4037042
	$\hat{\beta}$	4230.8201974	(8590.028)				
	$\hat{\lambda}$	0.4003174	(0.6355659)				
	$\hat{a}$	1.215823	(0.3501235)				
	$\hat{b}$	2.0473523	(0.7479278)				
GOLLBXII	$\hat{\alpha}$	39.51970395	(44.966740)	0.04352989	0.3147213	0.03997653	0.9866817
	$\hat{\beta}$	2.473694	(5.3743930)				
	$\hat{a}$	0.02712606	(0.02376913)				
	$\hat{b}$	1.96262451	(2.63961296)				
APTLL	$\hat{\alpha}$	2.63961296	(6.8425099)	0.04291863	0.3098658	0.03999176	0.9866255
	$\hat{a}$	4.917469	(4.6253246)				
	$\hat{b}$	1.711875	(0.1708215)				
TLBXII	$\hat{\alpha}$	0.9507123	(0.3158292)	0.3348713	2.150697	0.09827943	0.1686183
	$\hat{\beta}$	0.6536834	(0.1667003)				
	$\hat{\lambda}$	9.2555625	(4.7545499)				
LL	$\hat{\alpha}$	1.725159	(0.127925)	0.04300956	0.3111341	0.03988524	0.9870151
	$\hat{\lambda}$	22.571891	(6.315329)				

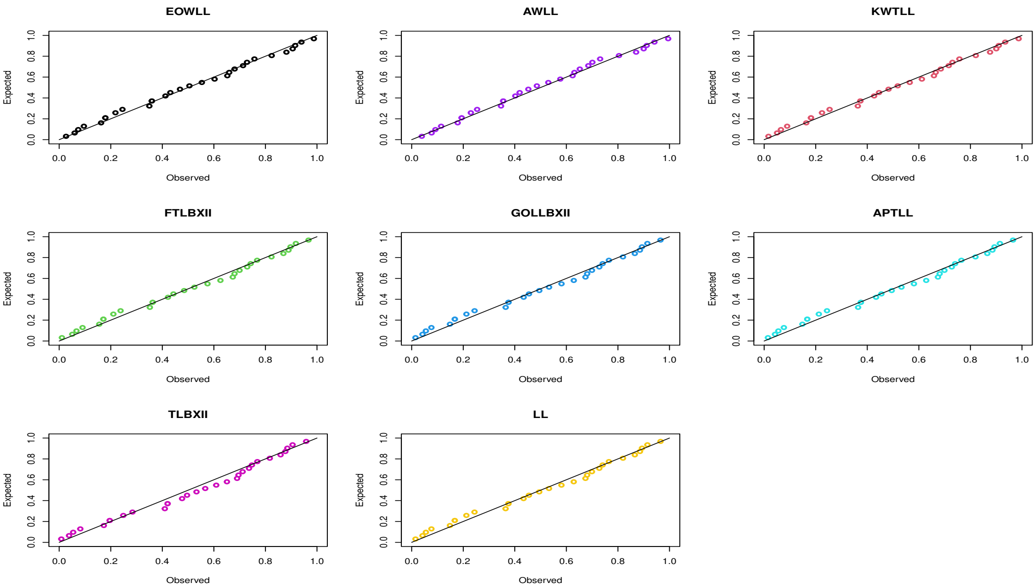

Fig. 5: The PP plots of the EOWLL model and other models for the first data.

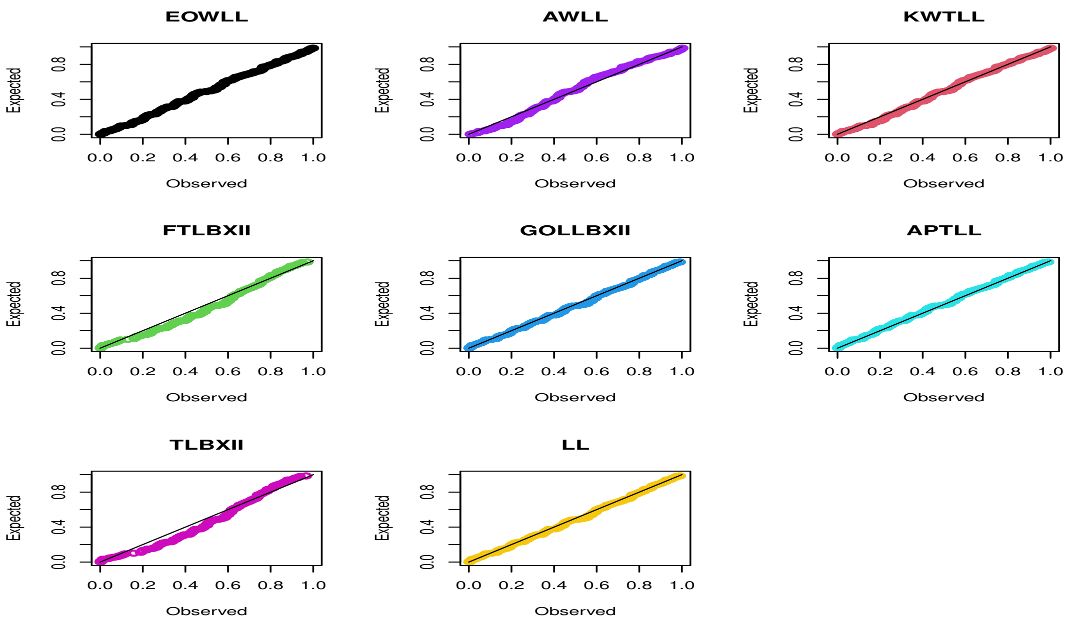

Fig. 6: The PP plots of the EOWLL model and other models for the second data.

7 Concluding Remarks

In this paper, we introduce a new four-parameter distribution called the extended odd Weibull log-logistic (EOWLL) distribution. The EOWLL density is expressed as a mixture of the Dagum densities. The mathematical properties of the EOWLL model are derived. Further, the EOWLL parameters are estimated by eight methods of estimation—namely, the maximum product of spacing, weighted least-squares, maximum likelihood, least squares, Anderson–Darling, Cramér–von Mises, right-tail Anderson–Darling, and percentile estimators. A simulation study is conducted to compare these approaches and showed that the percentile method outperforms all other methods of estimation. Hence, based on our study, we confirm the superiority of the percentile method for estimating the EOWLL parameters. Finally, the practical importance of the EOWLL distribution is studied by modeling two real-life data applications. Goodness-of-fit statistics for the analyzed data sets showed that our EOWLL distribution provides a better fit in comparison with other rival distributions.

For a possible direction of future works, the work in this article can be extended in some ways. For example,a bivariate EOWLL model may be established. The EOWLL parameters can also be estimated under different censoring schemes. A discrete version of the EOWLL model may also be studied. The construction of quantile regression models by exploiting the flexibility of the EOWLL distribution.

Acknowledgement

Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

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