

Parametrized Hyperbolic Tangent Induced Banach Space Valued Ordinary and Fractional Neural Network Approximation

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Abstract: Here we examine the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson-type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative of fractional derivatives. Our operators are defined by using a density function generated by a parametrized hyperbolic tangent function, which is a sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

Keywords: Parametrized hyperbolic tangent function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation.

1 Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson-type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [2] he gives the N th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

Again the first author inspired by [3], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent-type which resulted into [4]-[8], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [9], [10], [11].

The authors here perform parametrized hyperbolic tangent function, activated neural network approximations to continuous functions over compact intervals of the real line or over the whole \mathbb{R} with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally they treat completely the related X -valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X -valued high order derivative, or X -valued fractional derivatives and given by very tight Jackson-type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by a parametrized hyperbolic tangent function, which is a sigmoid function.

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Feed-forward X -valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x , and σ is the activation function of the network. About neural networks in general read [12], [13], [14].

2 Preliminaries

We consider here the parametrized hyperbolic tangent function $\tanh \lambda x$, $x \in \mathbb{R}$, $\lambda > 0$:

$$\tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}} = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1}. \quad (1)$$

For small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to perform better than ReLu and Leaky ReLu activation functions. It has the properties $\tanh \lambda x = 0$, $-1 < \tanh \lambda x < 1$, $\forall x \in \mathbb{R}$, and $\tanh \lambda(-x) = -\tanh \lambda x$.

Furthermore $\tanh \lambda x \rightarrow 1$ as $x \rightarrow \infty$, and $\tanh \lambda x \rightarrow -1$, as $x \rightarrow -\infty$, and it is strictly increasing on \mathbb{R} , in fact it holds $\frac{d}{dx} \tanh \lambda x = \frac{\lambda}{\cosh^2 \lambda x} > 0$. This function plays the great role of an activation function in the hidden layer of neural networks.

The activation function here will be

$$\theta(x) := \frac{1}{4} (\tanh \lambda(x+1) - \tanh \lambda(x-1)) > 0, \quad \forall x \in \mathbb{R}. \quad (2)$$

We observe that

$$\begin{aligned} \theta(-x) &= \frac{1}{4} (\tanh \lambda(-x+1) - \tanh \lambda(-x-1)) \\ &= \frac{1}{4} (-\tanh \lambda(x-1) + \tanh \lambda(x+1)) \\ &= \frac{1}{4} (\tanh \lambda(x+1) - \tanh \lambda(x-1)) = \theta(x), \end{aligned} \quad (3)$$

that is $\theta(x) = \theta(-x)$, i.e. θ is even function on \mathbb{R} . Clearly $\theta(x)$ is differentiable, thus continuous.

Proposition 1. $\theta(x)$ is strictly decreasing for $x \geq 0$.

Proof. We have that

$$\begin{aligned} \frac{d\theta(x)}{dx} &= \frac{\lambda}{4} \left(\frac{1}{\cosh^2 \lambda(x+1)} - \frac{1}{\cosh^2 \lambda(x-1)} \right) \\ &= \frac{\lambda}{4} \left(\frac{\cosh^2 \lambda(x-1) - \cosh^2 \lambda(x+1)}{\cosh^2 \lambda(x+1) \cosh^2 \lambda(x-1)} \right) \\ &= \frac{\lambda}{4} \left(\frac{\cosh \lambda(x-1) + \cosh \lambda(x+1)}{\cosh^2 \lambda(x+1) \cosh^2 \lambda(x-1)} \right) (\cosh \lambda(x-1) - \cosh \lambda(x+1)). \end{aligned} \quad (4)$$

Let $x \geq 1$, then $x-1 \geq 0$ and $x-1 < x+1$, so that

$0 < \cosh \lambda(x-1) < \cosh \lambda(x+1)$ and $\cosh \lambda(x-1) - \cosh \lambda(x+1) < 0$, hence $\theta'(x) < 0$, and $\theta(x)$ is strictly decreasing for $x \geq 1$.

Let now $0 < x < 1$, then $1-x > 0$, and for $x > 0$ we have that

$$0 < \cosh \lambda(x-1) = \cosh \lambda(1-x) < \cosh \lambda(1+x) = \cosh \lambda(x+1). \quad (5)$$

Hence again $\cosh \lambda(x-1) - \cosh \lambda(x+1) < 0$, and $\theta'(x) < 0$. So that $\theta(x)$ is decreasing also over $0 < x < 1$. Furthermore θ is continuous everywhere and in particular at zero. The claim is proved.

Clearly $\theta(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{\lambda \rightarrow -\infty} \theta(x) = 0 = \lim_{\lambda \rightarrow +\infty} \theta(x)$. In fact $\theta(x)$ has the bell shape with horizontal asymptote the x -axis. So the maximum of θ is at zero,

$$\theta(0) = \frac{1}{4} (\tanh \lambda - \tanh(-\lambda)) = \frac{\tanh \lambda}{2}. \tag{6}$$

Theorem 1. We have that

$$\sum_{i=-\infty}^{\infty} \theta(x-i) = 1, \forall x \in \mathbb{R}. \tag{7}$$

Proof. We notice that

$$\begin{aligned} & \sum_{i=-\infty}^{\infty} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) = \\ & \sum_{i=0}^{\infty} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) + \sum_{i=-\infty}^{-1} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)). \end{aligned}$$

Furthermore ($\rho \in \mathbb{Z}^+$)

$$\begin{aligned} & \sum_{i=0}^{\infty} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) = \\ & \lim_{\rho \rightarrow \infty} \sum_{i=0}^{\rho} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) \text{ (telescoping sum)} \\ & = \lim_{\rho \rightarrow \infty} (\tanh \lambda(x) - \tanh \lambda(x-(\rho+1))) = 1 + \tanh \lambda x. \end{aligned}$$

Similarly,

$$\begin{aligned} & \sum_{i=-\infty}^{-1} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) = \lim_{\rho \rightarrow \infty} \sum_{i=-\rho}^{-1} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) = \\ & \lim_{\rho \rightarrow \infty} (\tanh \lambda(x+\rho) - \tanh \lambda(x)) = 1 - \tanh \lambda(x). \end{aligned}$$

By adding the last two limits we derive

$$\sum_{i=-\infty}^{\infty} (\tanh \lambda(x-i) - \tanh \lambda(x-1-i)) = 2, \forall x \in \mathbb{R}. \tag{8}$$

Consequently we get

$$\sum_{i=-\infty}^{\infty} (\tanh \lambda(x+1-i) - \tanh \lambda(x-i)) = 2, \forall x \in \mathbb{R}.$$

Therefore it holds

$$\sum_{i=-\infty}^{\infty} (\tanh \lambda(x+1-i) - \tanh \lambda(x-1-i)) = 4, \forall x \in \mathbb{R}, \tag{9}$$

proving the claim.

Thus

$$\sum_{i=-\infty}^{\infty} \theta(nx-i) = 1, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}. \tag{10}$$

Furthermore we give:

Because θ is even it holds $\sum_{i=-\infty}^{\infty} \theta(i-x) = 1, \forall x \in \mathbb{R}$.

Hence $\sum_{i=-\infty}^{\infty} \theta(i+x) = 1, \forall x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} \theta(x+i) = 1, \forall x \in \mathbb{R}$.

Theorem 2. It holds

$$\int_{-\infty}^{\infty} \theta(x) dx = 1. \tag{11}$$

Proof. We observe that

$$\int_{-\infty}^{\infty} \theta(x) dx = \sum_{j=-\infty}^{\infty} \int_j^{j+1} \theta(x) dx = \sum_{j=-\infty}^{\infty} \int_0^1 \theta(x+j) dx = \int_0^1 \left(\sum_{j=-\infty}^{\infty} \theta(x+j) \right) dx = \int_0^1 1 dx = 1.$$

So that θ is a density function on \mathbb{R} .

We need

Theorem 3. Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \theta(nx-k) < e^{4\lambda} e^{-2\lambda n^{1-\alpha}}, \quad \lambda > 0. \quad (12)$$

Proof. Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we obtain

$$\theta(x) = \frac{1}{4} \cdot 2 \cdot \frac{\lambda}{\cosh^2 \lambda \xi} = \frac{\lambda}{2 \cosh^2 \lambda \xi},$$

for some $x-1 < \xi < x+1$. We get

$$\cosh \lambda(x-1) < \cosh \lambda \xi < \cosh \lambda(x+1),$$

and

$$\cosh^2 \lambda(x-1) < \cosh^2 \lambda \xi < \cosh^2 \lambda(x+1).$$

Therefore

$$0 < \frac{\lambda}{2 \cosh^2 \lambda \xi} < \frac{\lambda}{2 \cosh^2 \lambda(x-1)},$$

and

$$\begin{aligned} \theta(x) &< \frac{\lambda}{2 \cosh^2 \lambda(x-1)} = \frac{4\lambda}{2(e^{\lambda(x-1)} + e^{\lambda(1-x)})^2} \\ &= \frac{2\lambda}{(e^{\lambda(x-1)} + e^{\lambda(1-x)})^2} =: (*). \end{aligned} \quad (13)$$

From $e^{\lambda(x-1)} + e^{\lambda(1-x)} > e^{\lambda(x-1)}$, we obtain

$$(e^{\lambda(x-1)} + e^{\lambda(1-x)})^2 > e^{2\lambda(x-1)}, \text{ and}$$

$$\frac{1}{(e^{\lambda(x-1)} + e^{\lambda(1-x)})^2} < \frac{1}{e^{2\lambda(x-1)}}.$$

So that

$$\begin{aligned} (*) &< \frac{2\lambda}{e^{2\lambda(x-1)}} = \frac{2\lambda e^{2\lambda}}{e^{2\lambda x}} \\ &= 2\lambda e^{2\lambda} e^{-2\lambda x} \end{aligned}$$

We have proved that

$$\theta(x) < 2\lambda e^{2\lambda} e^{-2\lambda x}, \quad x \geq 1.$$

Thus

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \theta(nx-k) = \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \theta(|nx-k|) <$$

$$2\lambda e^{2\lambda} \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} e^{-2\lambda|nx-k|} \leq 2\lambda e^{2\lambda}$$

$$\int_{n^{1-\alpha}-1}^{\infty} e^{-2\lambda x} dx = e^{2\lambda} \int_{n^{1-\alpha}-1}^{\infty} e^{-2\lambda x} d(2\lambda x) = e^{2\lambda} \int_{n^{1-\alpha}-1}^{\infty} e^{-y} dy = e^{2\lambda} \left\{ -e^{-y} \Big|_{n^{1-\alpha}-1}^{\infty} \right\} =$$

$$e^{2\lambda} \left\{ e^{-2\lambda x} \Big|_{\infty}^{n^{1-\alpha}-1} \right\} = e^{2\lambda} e^{-2\lambda(n^{1-\alpha}-1)}$$

$$= e^{2\lambda} e^{-2\lambda n^{1-\alpha}} e^{2\lambda} = e^{4\lambda} e^{-2\lambda n^{1-\alpha}},$$

proving the claim.

Denote by $[\cdot]$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.
We present

Theorem 4. Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)} < \frac{4}{\tanh 2\lambda} = \frac{1}{\theta(1)}. \tag{14}$$

Proof. We observe that

$$1 = \sum_{k=-\infty}^{\infty} \theta(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(|nx-k|) > \theta(|nx-k_0|), \tag{15}$$

$\forall k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx-k_0| < 1$.

Therefore it holds

$$\theta(|nx-k_0|) > \theta(1) = \frac{\tanh 2\lambda}{4}, \tag{16}$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(|nx-k|) > \frac{\tanh 2\lambda}{4} = \theta(1). \tag{17}$$

We have proved that

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(|nx-k|)} < \frac{4}{\tanh 2\lambda}, \tag{18}$$

establishing the claim.

We make

Remark. We also notice that

$$1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nb-k) = \sum_{k=-\infty}^{\lceil na \rceil-1} \theta(nb-k) + \sum_{k=\lfloor nb \rfloor+1}^{\infty} \theta(nb-k)$$

$$> \theta(nb - \lceil na \rceil - 1) \tag{19}$$

(call $\varepsilon := nb - \lceil na \rceil, 0 \leq \varepsilon < 1$)

$$= \theta(\varepsilon - 1) = \theta(1 - \varepsilon) \geq \theta(1) > 0.$$

Therefore

$$\lim_{n \rightarrow \infty} \left(1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nb-k) \right) > 0. \quad (20)$$

Similarly,

$$\begin{aligned} 1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(na-k) &= \sum_{k=-\infty}^{\lceil na \rceil - 1} \theta(na-k) + \sum_{k=\lfloor nb \rfloor + 1}^{\infty} \theta(na-k) \\ &> \theta(na - \lceil na \rceil + 1) \\ (\text{call } \eta &:= \lceil na \rceil - na, 0 \leq \eta < 1) \\ &= \theta(1 - \eta) \geq \theta(1) > 0. \end{aligned} \quad (21)$$

Therefore again

$$\lim_{n \rightarrow \infty} \left(1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(na-k) \right) > 0. \quad (22)$$

Here we find that

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \neq 1, \text{ for at least some } x \in [a, b]. \quad (23)$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 1. Let $f \in C([a, b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X -valued linear neural network operators

$$H_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \theta(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)}, \quad x \in [a, b]. \quad (24)$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. The same H_n is used for real valued functions. We study here the pointwise and uniform convergence of $H_n(f, x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \theta(nx-k), \quad (25)$$

(the same H_n^* can be defined for real valued functions) that is

$$H_n(f, x) := \frac{H_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)}. \quad (26)$$

So that

$$\begin{aligned} H_n(f, x) - f(x) &= \frac{H_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)} - f(x) = \\ &= \frac{H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)}. \end{aligned} \quad (27)$$

Consequently, we derive that

$$\begin{aligned} \|H_n(f, x) - f(x)\| &\leq \frac{4}{\tanh 2\lambda} \left\| H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) \right) \right\| = \\ &= \frac{4}{\tanh 2\lambda} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \theta(nx - k) \right\|. \end{aligned} \tag{28}$$

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a, b], X)$ the first modulus of continuity

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{29}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from \mathbb{R} into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X -valued), and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a, b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \rightarrow 0} \omega_1(f, \delta) = 0$, see [15].

We make

Definition 2. When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define

$$\overline{H}_n(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \theta(nx - k), \tag{30}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X -valued quasi-interpolation neural network operator.

We give

Remark. We have that

$$\left\| f\left(\frac{k}{n}\right) \right\| \leq \|f\|_{\infty, \mathbb{R}} < +\infty,$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| \theta(nx - k) \leq \|f\|_{\infty, \mathbb{R}} \theta(nx - k) \tag{31}$$

and

$$\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| \theta(nx - k) \leq \|f\|_{\infty, \mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} \theta(nx - k) \right),$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \theta(nx - k) \leq \|f\|_{\infty, \mathbb{R}}, \tag{32}$$

a convergent series in \mathbb{R} .

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \theta(nx - k)$ is absolutely convergent in X , hence it is convergent in X and $\overline{H}_n(f, x) \in X$. We denote by $\|f\|_{\infty} := \sup_{x \in [a, b]} \|f(x)\|$, for $f \in C([a, b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a set of X -valued neural network approximations to a function given with rates.

Theorem 5. Let $f \in C([a, b], X)$, $\lambda > 0$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$. Then

i)

$$\|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} \left[\omega_1 \left(f, \frac{1}{n^\alpha} \right) + 2 \|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}} \right] =: \tau, \quad (33)$$

and

ii)

$$\|H_n(f) - f\|_\infty \leq \tau. \quad (34)$$

We get that $\lim_{n \rightarrow \infty} H_n(f) = f$, pointwise and uniformly.

Proof. We see that

$$\begin{aligned} & \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f \left(\frac{k}{n} \right) - f(x) \right) \theta(nx - k) \right\| \leq \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f \left(\frac{k}{n} \right) - f(x) \right\| \theta(nx - k) = \\ & \sum_{\substack{k=\lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f \left(\frac{k}{n} \right) - f(x) \right\| \theta(nx - k) + \\ & \sum_{\substack{k=\lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f \left(\frac{k}{n} \right) - f(x) \right\| \theta(nx - k) \leq \\ & \sum_{\substack{k=\lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \omega_1 \left(f, \left| \frac{k}{n} - x \right| \right) \theta(nx - k) + \\ & 2 \|f\|_\infty \sum_{\substack{k=\lceil na \rceil \\ |k - nx| > n^{1-\alpha}}}^{\lfloor nb \rfloor} \theta(nx - k) \leq \\ & \omega_1 \left(f, \frac{1}{n^\alpha} \right) \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\infty} \theta(nx - k) + \\ & 2 \|f\|_\infty \sum_{\substack{k=-\infty \\ |k - nx| > n^{1-\alpha}}}^{\infty} \theta(nx - k) \stackrel{\text{(by Theorem 3)}}{\leq} \\ & \omega_1 \left(f, \frac{1}{n^\alpha} \right) + 2 \|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}} \end{aligned} \quad (35)$$

That is

$$\begin{aligned} & \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f \left(\frac{k}{n} \right) - f(x) \right) \theta(nx - k) \right\| \leq \\ & \omega_1 \left(f, \frac{1}{n^\alpha} \right) + 2 \|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}}. \end{aligned} \quad (37)$$

Using the last equality we derive (33).

Next we give

Theorem 6. Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $\lambda > 0$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then

i)

$$\|\overline{H}_n(f, x) - f(x)\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}} =: \gamma, \tag{38}$$

and

ii)

$$\|\overline{H}_n(f) - f\|_\infty \leq \gamma. \tag{39}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n \rightarrow \infty} \overline{H}_n(f) = f$, pointwise and uniformly.

Proof. We observe that

$$\begin{aligned} \|\overline{H}_n(f, x) - f(x)\| &= \left\| \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \theta(nx-k) - f(x) \sum_{k=-\infty}^{\infty} \theta(nx-k) \right\| = \\ & \left\| \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) \theta(nx-k) \right\| \leq \\ & \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \theta(nx-k) = \\ & \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \theta(nx-k) + \\ & \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \theta(nx-k) \leq \\ & \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\infty} \omega_1\left(f, \left| \frac{k}{n} - x \right|\right) \theta(nx-k) + \\ & 2\|f\|_\infty \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\infty} \theta(nx-k) \leq \\ & \omega_1\left(f, \frac{1}{n^\alpha}\right) \sum_{\substack{k=-\infty \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\infty} \theta(nx-k) + 2\|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}} \leq \\ & \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty e^{4\lambda} e^{-2\lambda n^{1-\alpha}}, \end{aligned} \tag{40}$$

$$\tag{41}$$

proving the claim.

We need the X -valued Taylor's formula in an appropriate form:

Theorem 7. ([16], [17]) Let $N \in \mathbb{N}$, and $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and X is a Banach space. Let any $x, y \in [a, b]$. Then

$$f(x) = \sum_{i=0}^N \frac{(x-y)^i}{i!} f^{(i)}(y) + \frac{1}{(N-1)!} \int_y^x (x-t)^{N-1} \left(f^{(N)}(t) - f^{(N)}(y) \right) dt. \tag{42}$$

The derivatives $f^{(i)}$, $i \in \mathbb{N}$, are defined like the numerical ones, see [18], p. 83. The integral \int_y^x in (42) is of Bochner type, see [19].

By [17], [20] we have that: if $f \in C([a, b], X)$, then $f \in L_\infty([a, b], X)$ and $f \in L_1([a, b], X)$.

In the next we discuss high order neural network X -valued approximation by using the smoothness of f .

Theorem 8. Let $f \in C^N([a, b], X)$, $n, N \in \mathbb{N}$, $\lambda > 0$, $0 < \alpha < 1$, $x \in [a, b]$ and $n^{1-\alpha} > 2$. Then

i)

$$\|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} \left\{ \sum_{j=1}^N \frac{\|f^{(j)}(x)\|}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} \right] + \right. \\ \left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} \right] \right\}, \quad (43)$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$, for some $x_0 \in [a, b]$, it holds

$$\|H_n(f, x_0) - f(x_0)\| \leq \frac{4}{\tanh 2\lambda} \cdot$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} \right\}, \quad (44)$$

and

iii)

$$\|H_n(f) - f\|_\infty \leq \frac{4}{\tanh 2\lambda} \left\{ \sum_{j=1}^N \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{(b-a)^j e^{4\lambda}}{e^{2\lambda n^{(1-\alpha)}}} \right] + \right. \\ \left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N e^{4\lambda}}{e^{2\lambda n^{(1-\alpha)}}} \right] \right\}. \quad (45)$$

Again we obtain $\lim_{n \rightarrow \infty} H_n(f) = f$, pointwise and uniformly.

Proof. Next we apply the X -valued Taylor's formula with Bochner integral remainder (42). We have (here $\frac{k}{n}, x \in [a, b]$)

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^N \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \int_x^{\frac{k}{n}} (f^{(N)}(t) - f^{(N)}(x)) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt. \quad (46)$$

Then

$$f\left(\frac{k}{n}\right) \theta(nx - k) = \sum_{j=0}^N \frac{f^{(j)}(x)}{j!} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \\ \theta(nx - k) \int_x^{\frac{k}{n}} (f^{(N)}(t) - f^{(N)}(x)) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt. \quad (47)$$

Hence

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \theta(nx - k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) = \\ \sum_{j=0}^N \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \\ \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) \int_x^{\frac{k}{n}} (f^{(N)}(t) - f^{(N)}(x)) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt. \quad (48)$$

Thus

$$H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) \right) = \sum_{j=1}^N \frac{f^{(j)}(x)}{j!} H_n^* \left((\cdot - x)^j \right) + \Lambda_n(x), \tag{49}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx - k) \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt. \tag{50}$$

We assume that $b - a > \frac{1}{n^\alpha}$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left[(b - a)^{-\frac{1}{\alpha}} \right]$.

Thus $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}$.

Let

$$\psi := \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt, \tag{51}$$

in the case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$, we find that

$$\|\psi\| \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N N!}} \tag{52}$$

for $x \leq \frac{k}{n}$ or $x \geq \frac{k}{n}$.

We prove it next.

i) Indeed, for the case of $x \leq \frac{k}{n}$, we have

$$\begin{aligned} \|\psi\| &= \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| \leq \\ &\int_x^{\frac{k}{n}} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \leq \\ &\int_x^{\frac{k}{n}} \omega_1 \left(f^{(N)}, |t - x| \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt = \\ &\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{\left(\frac{k}{n} - x\right)^N}{N!} \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N N!}}. \end{aligned} \tag{53}$$

ii) for the case of $x > \frac{k}{n}$, we have

$$\begin{aligned} \|\psi\| &= \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| = \\ &\left\| \int_{\frac{k}{n}}^x \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\| \leq \\ &\int_{\frac{k}{n}}^x \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \\ &\int_{\frac{k}{n}}^x \omega_1 \left(f^{(N)}, |t - x| \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \int_{\frac{k}{n}}^x \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt = \\ &\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{\left(x - \frac{k}{n}\right)^N}{N!} \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N N!}}. \end{aligned} \tag{54}$$

We have proved (52).

We treat again ψ , see (51), but differently:

Notice also for $x \leq \frac{k}{n}$ that

$$\begin{aligned} & \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| \leq \\ & \int_x^{\frac{k}{n}} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \leq \\ & 2 \left\| f^{(N)} \right\|_{\infty} \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left(\frac{k}{n} - x\right)^N}{N!} \\ & \leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N!}. \end{aligned} \quad (55)$$

Next assume $\frac{k}{n} \leq x$, then

$$\begin{aligned} & \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| = \\ & \left\| \int_{\frac{k}{n}}^x \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\| \leq \\ & \int_{\frac{k}{n}}^x \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \\ & 2 \left\| f^{(N)} \right\|_{\infty} \int_{\frac{k}{n}}^x \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left(x - \frac{k}{n}\right)^N}{N!} \\ & \leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N!}. \end{aligned} \quad (56)$$

Thus

$$\|\psi\| \leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N!}. \quad (57)$$

in the two cases.

Therefore

$$\Lambda_n(x) = \sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \theta(nx - k) \psi + \sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \theta(nx - k) \psi. \quad (58)$$

Hence

$$\begin{aligned} \|\Lambda_n(x)\| & \leq \sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \theta(nx - k) \left(\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{N! n^{\alpha N}} \right) + \\ & \left(\sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \theta(nx - k) \right) 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N!} \leq \\ & \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{N! n^{\alpha N}} + e^{4\lambda} e^{-2\lambda n(1-\alpha)} 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N!}. \end{aligned} \quad (59)$$

That is

$$\| \Lambda_n(x) \| \leq \frac{\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right)}{N! n^{\alpha N}} + \frac{2 \| f^{(N)} \|_\infty (b-a)^N}{N!} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}}, \tag{60}$$

$\forall x \in [a, b]$.

We further see that

$$H_n^* \left((\cdot - x)^j \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \left(\frac{k}{n} - x \right)^j, \tag{61}$$

where H_n^* is defined similarly for real valued functions.

Therefore

$$\begin{aligned} \left| H_n^* \left((\cdot - x)^j \right) \right| &\leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \left| \frac{k}{n} - x \right|^j = \\ &\begin{cases} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \left| \frac{k}{n} - x \right|^j & \left\{ \begin{array}{l} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{array} \right. \\ \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \left| \frac{k}{n} - x \right|^j & \left\{ \begin{array}{l} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{array} \right. \end{cases} \tag{12} \\ &\frac{1}{n^{\alpha j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}}. \end{aligned} \tag{62}$$

That is

$$\left| H_n^* \left((\cdot - x)^j \right) \right| \leq \frac{1}{n^{\alpha j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}}, \tag{63}$$

for $j = 1, \dots, N$.

Putting things together we have proved

$$\begin{aligned} \left\| H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k) \right) \right\| &\leq \sum_{j=1}^N \frac{\| f^{(j)}(x) \|}{j!} \\ &\left[\frac{1}{n^{\alpha j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} \right] + \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \| f^{(N)} \|_\infty (b-a)^N}{N!} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} \right], \end{aligned} \tag{64}$$

that is establishing the theorem.

All integrals from now on are of Bochner type [19].

We need

Definition 3. ([17]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a, b] \rightarrow X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo-Bochner left fractional derivative of order α :

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad \forall x \in [a, b]. \tag{65}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary X -valued derivative (defined similar to numerical one, see [18], p. 83), and also set $D_{*a}^0 f := f$.

By [17], $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^\alpha f \in L_1([a, b], X)$.

If $\| f^{(m)} \|_{L_\infty([a, b], X)} < \infty$, then by [17], $D_{*a}^\alpha f \in C([a, b], X)$, hence $\| D_{*a}^\alpha f \| \in C([a, b])$.

We mention

Lemma 1. ([15]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a, b], X)$ and $f^{(m)} \in L_\infty([a, b], X)$. Then $D_{*a}^\alpha f(a) = 0$.

We mention

Definition 4. ([16]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \rightarrow X$. We call the Caputo-Bochner right fractional derivative of order α :

$$(D_{b-}^{\alpha} f)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall x \in [a, b]. \quad (66)$$

We observe that $(D_{b-}^m f)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $(D_{b-}^0 f)(x) = f(x)$.

By [16], $(D_{b-}^{\alpha} f)(x)$ exists almost everywhere on $[a, b]$ and $(D_{b-}^{\alpha} f) \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_{\infty}([a, b], X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [16], $D_{b-}^{\alpha} f \in C([a, b], X)$, hence $\|D_{b-}^{\alpha} f\| \in C([a, b])$.

We need

Lemma 2. ([15]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.

We mention the left fractional Taylor formula

Theorem 9. ([17]) Let $m \in \mathbb{N}$ and $f \in C^m([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Then

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} (D_{*a}^{\alpha} f)(z) dz, \quad (67)$$

$\forall x \in [a, b]$.

We also mention the right fractional Taylor formula

Theorem 10. ([16]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([a, b], X)$. Then

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} (D_{b-}^{\alpha} f)(z) dz, \quad (68)$$

$\forall x \in [a, b]$.

Convention 2 We assume that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \quad (69)$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \quad (70)$$

for all $x, x_0 \in [a, b]$.

We mention

Proposition 3. ([15]) Let $f \in C^n([a, b], X)$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, b]$.

Proposition 4. ([15]) Let $f \in C^m([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a, b]$.

We also mention

Proposition 5. ([15]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad (71)$$

for all $x, x_0 \in [a, b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0 .

Proposition 6. ([15]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_{\infty}([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta-x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \quad (72)$$

for all $x, x_0 \in [a, b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0 .

Corollary 1. ([15]) Let $f \in C^m([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a, b]$. Then $D_{*x_0}^\alpha f(x)$, $D_{x_0-}^\alpha f(x)$ are jointly continuous functions in (x, x_0) from $[a, b]^2$ into X , X is a Banach space.

We need

Theorem 11. ([15]) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Consider

$$G(x) = \omega_1(f(\cdot, x), \delta, [x, b]), \tag{73}$$

$\delta > 0$, $x \in [a, b]$.

Then G is continuous on $[a, b]$.

Theorem 12. ([15]) Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Then

$$S(x) = \omega_1(f(\cdot, x), \delta, [a, x]), \tag{74}$$

$x \in [a, b]$, is continuous in $x \in [a, b]$, $\delta > 0$.

We make

Remark. ([15]) Let $f \in C^{n-1}([a, b])$, $f^{(n)} \in L_\infty([a, b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then

$$\|D_{*a}^\nu f(x)\| \leq \frac{\|f^{(n)}\|_{L_\infty([a, b], X)}}{\Gamma(n - \nu + 1)} (x - a)^{n - \nu}, \quad \forall x \in [a, b]. \tag{75}$$

Thus we observe

$$\begin{aligned} \omega_1(D_{*a}^\nu f, \delta) &= \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \|D_{*a}^\nu f(x) - D_{*a}^\nu f(y)\| \leq \\ &\sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \left(\frac{\|f^{(n)}\|_{L_\infty([a, b], X)}}{\Gamma(n - \nu + 1)} (x - a)^{n - \nu} + \frac{\|f^{(n)}\|_{L_\infty([a, b], X)}}{\Gamma(n - \nu + 1)} (y - a)^{n - \nu} \right) \\ &\leq \frac{2\|f^{(n)}\|_{L_\infty([a, b], X)}}{\Gamma(n - \nu + 1)} (b - a)^{n - \nu}. \end{aligned} \tag{76}$$

Consequently

$$\omega_1(D_{*a}^\nu f, \delta) \leq \frac{2\|f^{(n)}\|_{L_\infty([a, b], X)}}{\Gamma(n - \nu + 1)} (b - a)^{n - \nu}. \tag{77}$$

Similarly, let $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_\infty([a, b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1(D_{b-}^\alpha f, \delta) \leq \frac{2\|f^{(m)}\|_{L_\infty([a, b], X)}}{\Gamma(m - \alpha + 1)} (b - a)^{m - \alpha}. \tag{78}$$

So for $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_\infty([a, b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a, b]} \omega_1(D_{*x_0}^\alpha f, \delta)_{[x_0, b]} \leq \frac{2\|f^{(m)}\|_{L_\infty([a, b], X)}}{\Gamma(m - \alpha + 1)} (b - a)^{m - \alpha}, \tag{79}$$

and

$$\sup_{x_0 \in [a, b]} \omega_1(D_{x_0-}^\alpha f, \delta)_{[a, x_0]} \leq \frac{2\|f^{(m)}\|_{L_\infty([a, b], X)}}{\Gamma(m - \alpha + 1)} (b - a)^{m - \alpha}. \tag{80}$$

By [17] we get that $D_{*x_0}^\alpha f \in C([x_0, b], X)$, and by [16] we obtain that $D_{x_0-}^\alpha f \in C([a, x_0], X)$. We present the following X -valued fractional approximation result by neural networks.

Theorem 13. Let $\alpha, \lambda > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $0 < \beta < 1$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$\begin{aligned} & \left\| H(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n((\cdot - x)^j)(x) - f(x) \right\| \leq \\ & \frac{4}{\tanh 2\lambda} \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \\ & \left. \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\}, \end{aligned} \quad (81)$$

ii) if $f^{(j)}(x) = 0$, for $j = 1, \dots, N-1$, we have

$$\begin{aligned} & \|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} \frac{1}{\Gamma(\alpha + 1)} \\ & \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \\ & \left. e^{4\lambda} e^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\}, \end{aligned} \quad (82)$$

iii)

$$\begin{aligned} & \|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} \\ & \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}(x)\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\beta)}} \right\} + \right. \\ & \left. \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \right. \\ & \left. \left. e^{4\lambda} e^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\} \right\}, \end{aligned} \quad (83)$$

$\forall x \in [a, b]$,

and

iv)

$$\begin{aligned} & \|H_n f - f\|_\infty \leq \frac{4}{\tanh 2\lambda} \\ & \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^{4\lambda} e^{-2\lambda n^{(1-\beta)}} \right\} + \right. \\ & \left. \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \right. \end{aligned}$$

$$e^{4\lambda} e^{-2\lambda n^{(1-\beta)}} (b-a)^\alpha \left(\sup_{x \in [a,b]} \|D_{x-}^\alpha f\|_{\infty, [a,x]} + \sup_{x \in [a,b]} \|D_{*x}^\alpha f\|_{\infty, [x,b]} \right) \Bigg\} . \tag{84}$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X -valued fractionally type pointwise and uniform convergence with rates of $H_n \rightarrow I$ the unit operator, as $n \rightarrow \infty$.

Proof. Let $x \in [a, b]$. We have that $D_{x-}^\alpha f(x) = D_{*x}^\alpha f(x) = 0$.

From Theorem 9, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) dJ, \tag{85}$$

for all $x \leq \frac{k}{n} \leq b$.

Also from Theorem 10, using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^\alpha f(J) - D_{x-}^\alpha f(x)) dJ, \tag{86}$$

for all $a \leq \frac{k}{n} \leq x$.

Hence we have

$$f\left(\frac{k}{n}\right) \theta(nx - k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \frac{\theta(nx - k)}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) dJ, \tag{87}$$

for all $x \leq \frac{k}{n} \leq b$, iff $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$, and

$$f\left(\frac{k}{n}\right) \theta(nx - k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \frac{\theta(nx - k)}{\Gamma(\alpha)} \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^\alpha f(J) - D_{x-}^\alpha f(x)) dJ, \tag{88}$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

Therefore it holds

$$\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \theta(nx - k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \theta(nx - k) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) dJ, \tag{89}$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right) \theta(nx - k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \theta(nx - k) \left(\frac{k}{n} - x\right)^j + \tag{90}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rceil} \theta(nx-k) \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)) dJ.$$

Adding the last two equalities obtain

$$\begin{aligned} H_n^*(f, x) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rceil} f\left(\frac{k}{n}\right) \theta(nx-k) = \\ & \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rceil} \theta(nx-k) \left(\frac{k}{n} - x\right)^j + \\ & \frac{1}{\Gamma(\alpha)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rceil} \theta(nx-k) \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)) dJ + \right. \\ & \left. \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rceil} \theta(nx-k) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)) dJ \right\}. \end{aligned} \quad (91)$$

So we have derived

$$H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rceil} \theta(nx-k) \right) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \theta_n^* \left((\cdot - x)^j \right) + e_n(x), \quad (92)$$

where

$$\begin{aligned} e_n(x) &:= \frac{1}{\Gamma(\alpha)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rceil} \theta(nx-k) \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)) dJ \right. \\ & \left. + \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rceil} \theta(nx-k) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)) dJ \right\}. \end{aligned} \quad (93)$$

We set

$$e_{1n}(x) := \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rceil} \theta(nx-k) \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)) dJ, \quad (94)$$

and

$$e_{2n} := \frac{1}{\Gamma(\alpha)} \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rceil} \theta(nx-k) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} (D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)) dJ, \quad (95)$$

i.e.

$$e_n(x) = e_{1n}(x) + e_{2n}(x). \quad (96)$$

We assume $b - a > \frac{1}{n^\beta}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b - a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^\beta}$.

For $k = \lceil na \rceil, \dots, \lfloor nx \rceil$, we consider

$$\Omega_{1k} := \left\| \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} (D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)) dJ \right\| = \quad (97)$$

$$\left\| \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J) dJ \right\| \leq \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} \|D_{x-}^{\alpha} f(J)\| dJ \leq$$

$$\|D_{x-}^{\alpha} f(J)\|_{\infty, [a, x]} \frac{(x - \frac{k}{n})^{\alpha}}{\alpha} \leq \|D_{x-}^{\alpha} f\|_{\infty, [a, x]} \frac{(x - a)^{\alpha}}{\alpha}. \quad (98)$$

That is

$$\Omega_{1k} \leq \|D_{x-}^\alpha f\|_{\infty, [a,x]} \frac{(x-a)^\alpha}{\alpha}, \tag{99}$$

for $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$.

Also we have in case of $|\frac{k}{n} - x| \leq \frac{1}{n^\beta}$ that

$$\begin{aligned} \Omega_{1k} &\leq \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} \|D_{x-}^\alpha f(J) - D_{x-}^\alpha f(x)\| dJ \leq \\ &\int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} \omega_1(D_{x-}^\alpha f, |J-x|)_{[a,x]} dJ \leq \\ &\omega_1\left(D_{x-}^\alpha f, \left|x - \frac{k}{n}\right|\right)_{[a,x]} \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\alpha-1} dJ \leq \\ &\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} \frac{\left(x - \frac{k}{n}\right)^\alpha}{\alpha} \leq \omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} \frac{1}{\alpha n^{\alpha\beta}}. \end{aligned} \tag{100}$$

That is when $|\frac{k}{n} - x| \leq \frac{1}{n^\beta}$, then

$$\Omega_{1k} \leq \frac{\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]}}{\alpha n^{\alpha\beta}}. \tag{101}$$

Consequently we obtain

$$\|e_{1n}(x)\| \leq \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \theta(nx-k) \Omega_{1k} = \tag{102}$$

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \left\{ \sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : |\frac{k}{n} - x| \leq \frac{1}{n^\beta} \end{array} \right\}}^{\lfloor nx \rfloor} \theta(nx-k) \Omega_{1k} + \sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : |\frac{k}{n} - x| > \frac{1}{n^\beta} \end{array} \right\}}^{\lfloor nx \rfloor} \theta(nx-k) \Omega_{1k} \right\} \leq \\ &\frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : |\frac{k}{n} - x| \leq \frac{1}{n^\beta} \end{array} \right\}}^{\lfloor nx \rfloor} \theta(nx-k) \right) \frac{\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]}}{\alpha n^{\alpha\beta}} + \right. \\ &\left. \left(\sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : |\frac{k}{n} - x| > \frac{1}{n^\beta} \end{array} \right\}}^{\lfloor nx \rfloor} \theta(nx-k) \right) \|D_{x-}^\alpha f\|_{\infty, [a,x]} \frac{(x-a)^\alpha}{\alpha} \right\} \leq \tag{103} \\ &\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]}}{n^{\alpha\beta}} + \right. \\ &\left. \left(\sum_{\left\{ \begin{array}{l} k = -\infty \\ : |nx-k| > n^{1-\beta} \end{array} \right\}}^{\infty} \theta(nx-k) \|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha \right) \right\} \leq \end{aligned}$$

$$\frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]}}{n^{\alpha\beta}} + \frac{\|D_{x-}^\alpha f\|_{\infty,[a,x]} (x-a)^\alpha e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \right\}.$$

So we have proved that

$$\|e_{1n}(x)\| \leq \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]}}{n^{\alpha\beta}} + \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \|D_{x-}^\alpha f\|_{\infty,[a,x]} (x-a)^\alpha \right\}. \tag{104}$$

Next when $k = \lfloor nx \rfloor + 1, \dots, \lfloor nb \rfloor$ we consider

$$\Omega_{2k} := \left\| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} (D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)) dJ \right\| \leq \tag{105}$$

$$\int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \|D_{*x}^\alpha f(J) - D_{*x}^\alpha f(x)\| dJ =$$

$$\int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \|D_{*x}^\alpha f(J)\| dJ \leq$$

$$\|D_{*x}^\alpha f\|_{\infty,[x,b]} \frac{\left(\frac{k}{n} - x\right)^\alpha}{\alpha} \leq \|D_{*x}^\alpha f\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}. \tag{106}$$

Therefore when $k = \lfloor nx \rfloor + 1, \dots, \lfloor nb \rfloor$ we get that

That is

$$\Omega_{2k} \leq \|D_{*x}^\alpha f\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}. \tag{107}$$

In case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta}$ we have

$$\Omega_{2k} \leq \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \omega_1(D_{*x}^\alpha f, |J-x|)_{[x,b]} dJ \leq$$

$$\omega_1 \left(D_{*x}^\alpha f, \left| \frac{k}{n} - x \right| \right)_{[x,b]} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} dJ \leq \tag{108}$$

$$\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \frac{\left(\frac{k}{n} - x\right)^\alpha}{\alpha} \leq \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \frac{1}{\alpha n^{\alpha\beta}}.$$

So when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta}$ we derived that

$$\Omega_{2k} \leq \frac{\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{109}$$

Similarly we have that

$$\|e_{2n}(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \theta(nx-k) \Omega_{2k} \right) =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \sum_{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta} \end{array} \right.}}^{\lfloor nb \rfloor} \theta(nx-k) \Omega_{2k} + \sum_{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\beta} \end{array} \right.}}^{\lfloor nb \rfloor} \theta(nx-k) \Omega_{2k} \right\} \leq \tag{110}$$

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \theta(nx - k) \right) \frac{\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{\alpha n^{\alpha\beta}} + \right. \\
 & \left. \left(\sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \theta(nx - k) \right) \|D_{*x}^\alpha f\|_{\infty, [x,b]} \frac{(b-x)^\alpha}{\alpha} \right\} \leq \\
 & \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{n^{\alpha\beta}} + \right. \\
 & \left. \left(\sum_{\substack{k = -\infty \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\beta}}}^{\infty} \theta(nx - k) \right) \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right\} \leq \tag{111} \\
 & \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{n^{\alpha\beta}} + \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right\}.
 \end{aligned}$$

So we have proved that

$$\begin{aligned}
 \|e_{2n}(x)\| & \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{n^{\alpha\beta}} + \right. \\
 & \left. \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right\}. \tag{112}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|e_n(x)\| & \leq \|e_{1n}(x)\| + \|e_{2n}(x)\| \leq \\
 & \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]}}{n^{\alpha\beta}} + \right. \\
 & \left. \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\}. \tag{113}
 \end{aligned}$$

From the proof of Theorem 8 we get that

$$\left| H_n^* \left((\cdot - x)^j \right) (x) \right| \leq \frac{1}{n^{\beta j}} + (b-a)^j \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}}, \tag{114}$$

for $j = 1, \dots, N-1, \forall x \in [a, b]$.

Putting things together, we have established

$$\left\| H_n^*(f, x) - f(x) \left(\sum_{k=\lfloor na \rfloor}^{\lfloor nb \rfloor} \theta(nx - k) \right) \right\| \leq \sum_{j=1}^{N-1} \frac{\|f^{(j)}(x)\|}{j!} \tag{115}$$

$$\left[\frac{1}{n^{\beta j}} + (b-a)^j \frac{e^{4\lambda}}{e^{2\lambda n(1-\beta)}} \right] +$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{e^{4\lambda}}{e^{2\lambda n(1-\beta)}} \left(\|D_{x-}^{\alpha} f\|_{\infty, [a,x]} (x-a)^{\alpha} + \|D_{*x}^{\alpha} f\|_{\infty, [x,b]} (b-x)^{\alpha} \right) \right\} =: K_n(x). \quad (116)$$

As a result we derive

$$\|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} K_n(x), \quad \forall x \in [a, b]. \quad (117)$$

We further have that

$$\|K_n\|_{\infty} \leq \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_{\infty}}{j!} \left[\frac{1}{n^{\beta j}} + (b-a)^j \frac{e^{4\lambda}}{e^{2\lambda n(1-\beta)}} \right] +$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\sup_{x \in [a,b]} \left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \right) + \sup_{x \in [a,b]} \left(\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{e^{4\lambda}}{e^{2\lambda n(1-\beta)}} (b-a)^{\alpha} \left\{ \left(\sup_{x \in [a,b]} \left(\|D_{x-}^{\alpha} f\|_{\infty, [a,x]} \right) + \sup_{x \in [a,b]} \left(\|D_{*x}^{\alpha} f\|_{\infty, [x,b]} \right) \right\} \right\} =: E_n.$$

Hence it holds

$$\|H_n f - f\|_{\infty} \leq \frac{4}{\tanh 2\lambda} E_n. \quad (119)$$

We observe the following:

We have

$$(D_{x-}^{\alpha} f)(y) = \frac{(-1)^N}{\Gamma(N-\alpha)} \int_y^x (J-y)^{N-\alpha-1} f^{(N)}(J) dJ, \quad \forall y \in [a, x] \quad (120)$$

and

$$\|D_{x-}^{\alpha} f(y)\| \leq \frac{1}{\Gamma(N-\alpha)} \left(\int_y^x (J-y)^{N-\alpha-1} dJ \right) \|f^{(N)}\|_{\infty} =$$

$$\frac{1}{\Gamma(N-\alpha)} \frac{(x-y)^{N-\alpha}}{(N-\alpha)} \|f^{(N)}\|_{\infty} = \frac{(x-y)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty} \leq$$

$$\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}. \quad (121)$$

That is

$$\|D_{x-}^{\alpha} f\|_{\infty, [a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}, \quad (122)$$

and

$$\sup_{x \in [a,b]} \|D_{x-}^{\alpha} f\|_{\infty, [a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}. \quad (123)$$

Similarly we have

$$(D_{*x}^{\alpha} f)(y) = \frac{1}{\Gamma(N-\alpha)} \int_x^y (y-t)^{N-\alpha-1} f^{(N)}(t) dt, \quad \forall y \in [x, b]. \quad (124)$$

Thus we get

$$\|D_{*x}^{\alpha} f(y)\| \leq \frac{1}{\Gamma(N-\alpha)} \left(\int_x^y (y-t)^{N-\alpha-1} dt \right) \|f^{(N)}\|_{\infty} \leq \quad (125)$$

$$\frac{1}{\Gamma(N-\alpha)} \frac{(y-x)^{N-\alpha}}{(N-\alpha)} \|f^{(N)}\|_{\infty} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}.$$

Hence

$$\|D_{*x}^{\alpha} f\|_{\infty, [x, b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}, \tag{126}$$

and

$$\sup_{x \in [a, b]} \|D_{*x}^{\alpha} f\|_{\infty, [x, b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \|f^{(N)}\|_{\infty}. \tag{127}$$

From (79) and (80) we get

$$\sup_{x \in [a, b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a, x]} \leq \frac{2 \|f^{(N)}\|_{\infty}}{\Gamma(N-\alpha+1)} (b-a)^{N-\alpha}, \tag{128}$$

and

$$\sup_{x \in [a, b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \leq \frac{2 \|f^{(N)}\|_{\infty}}{\Gamma(N-\alpha+1)} (b-a)^{N-\alpha}. \tag{129}$$

That is $E_n < \infty$.

We finally notice that

$$\begin{aligned} H_n(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n((\cdot-x)^j)(x) - f(x) &= \\ \frac{H_n^*(f, x)}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)\right)} - \frac{1}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)\right)} &= \\ \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n^*((\cdot-x)^j)(x)\right) - f(x) &= \\ \frac{1}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)\right)} \left[H_n^*(f, x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n^*((\cdot-x)^j)(x)\right) \right. &= \\ \left. - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)\right) f(x) \right]. & \tag{130} \end{aligned}$$

Therefore we get

$$\begin{aligned} \left\| H_n(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n((\cdot-x)^j)(x) - f(x) \right\| &\leq \frac{4}{\tanh 2\lambda} \\ \left\| H_n^*(f, x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n^*((\cdot-x)^j)(x)\right) - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)\right) f(x) \right\|, & \tag{131} \end{aligned}$$

$\forall x \in [a, b]$.

The proof of the theorem is now finished.

Next we apply Theorem 13 for $N = 1$.

Theorem 14. Let $0 < \alpha, \beta < 1, \lambda > 0, f \in C^1([a, b], X), x \in [a, b], n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$\|H_n(f, x) - f(x)\| \leq \frac{4}{\tanh 2\lambda} \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a, x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\|D_{x-}^{\alpha} f\|_{\infty, [a, x]} (x-a)^{\alpha} + \|D_{*x}^{\alpha} f\|_{\infty, [x, b]} (b-x)^{\alpha} \right), \quad (132)$$

and
ii)

$$\|H_n f - f\|_{\infty} \leq \frac{4}{\tanh 2\lambda} \frac{1}{\Gamma(\alpha + 1)} \cdot \left\{ \frac{\left(\sup_{x \in [a, b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a, x]} + \sup_{x \in [a, b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \right)}{n^{\alpha\beta}} + \frac{(b-a)^{\alpha} e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\sup_{x \in [a, b]} \|D_{x-}^{\alpha} f\|_{\infty, [a, x]} + \sup_{x \in [a, b]} \|D_{*x}^{\alpha} f\|_{\infty, [x, b]} \right) \right\}. \quad (133)$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 2. Let $0 < \beta < 1$, $\lambda > 0$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$\|H_n(f, x) - f(x)\| \leq \frac{8}{\sqrt{\pi} \tanh 2\lambda} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a, x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \right)}{n^{\frac{\beta}{2}}} + \frac{e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\|D_{x-}^{\frac{1}{2}} f\|_{\infty, [a, x]} \sqrt{(x-a)} + \|D_{*x}^{\frac{1}{2}} f\|_{\infty, [x, b]} \sqrt{(b-x)} \right) \right\}, \quad (134)$$

and
ii)

$$\|H_n f - f\|_{\infty} \leq \frac{8}{\sqrt{\pi} \tanh 2\lambda} \cdot \left\{ \frac{\left(\sup_{x \in [a, b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a, x]} + \sup_{x \in [a, b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \right)}{n^{\frac{\beta}{2}}} + \frac{\sqrt{(b-a)} e^{4\lambda}}{e^{2\lambda n^{(1-\beta)}}} \left(\sup_{x \in [a, b]} \|D_{x-}^{\frac{1}{2}} f\|_{\infty, [a, x]} + \sup_{x \in [a, b]} \|D_{*x}^{\frac{1}{2}} f\|_{\infty, [x, b]} \right) \right\} < \infty. \quad (135)$$

We make

Remark. Some convergence analysis follows based on Corollary 2.

Let $0 < \beta < 1$, $\lambda > 0$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (135). Assume that

$$\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a, x]} \leq \frac{R_1}{n^{\beta}}, \quad (136)$$

and

$$\omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x, b]} \leq \frac{R_2}{n^{\beta}}, \quad (137)$$

$\forall x \in [a, b], \forall n \in \mathbb{N}$, where $R_1, R_2 > 0$.

Then it holds

$$\frac{\left[\sup_{x \in [a, b]} \omega_1 \left(D_{x^-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a, x]} + \sup_{x \in [a, b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x, b]} \right]}{n^{\frac{\beta}{2}}} \leq \frac{\frac{(R_1 + R_2)}{n^\beta}}{n^{\frac{\beta}{2}}} = \frac{(R_1 + R_2)}{n^{\frac{3\beta}{2}}} = \frac{R}{n^{\frac{3\beta}{2}}}, \tag{138}$$

where $R := R_1 + R_2 > 0$.

The other summand of the right hand side of (135), for large enough n , converges to zero at the speed $\frac{e^{4\lambda}}{e^{2\lambda n(1-\beta)}}$, so it is about $\frac{e^{4\lambda A}}{e^{2\lambda n(1-\beta)}}$, where $A > 0$ is a constant.

Then, for large enough $n \in \mathbb{N}$, by (135), (138) and the above comment, we obtain that

$$\|H_n f - f\|_\infty \leq \frac{B}{n^{\frac{3\beta}{2}}}, \tag{139}$$

where $B > 0$, converging to zero at the high speed of $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 5, for $f \in C([a, b], X)$ and for large enough $n \in \mathbb{N}$, the speed is $\frac{1}{n^\beta}$. So by (139), $\|H_n f - f\|_\infty$ converges much faster to zero. The last comes because we assumed differentiability of f . Notice that in Corollary 2 no initial condition is assumed.

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