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New Results by Using Advanced Definition of Fractional Derivative Without Singular Kernel

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Abstract: In this paper, an advanced definition of arbitrary order derivative i.e. Fractional derivative without singular kernel has been discussed. Also, verify the new results by using Laplace transform for time variable and Fourier transform for spatial variable by a non-local arbitrary order derivative, for which it is more convenient to work with the Fourier Transform.

Keywords: Caputo fractional derivative,Laplace transform, Fourier transform, singular kernel.

1 Introduction

The calculus of arbitrary order which commonly known as fractional calculus is a differentiation and integration of non-integer order. The calculus of arbitrary order grabbed the attention of other known mathematicians, many of them directly or indirectly increased to its development. In the last three centuries the calculus of half derivative had a distinguished development as reveal by the many mathematical literature committed to it (e.g. Baleanu et al. [1], Caponetto [2], Caputo [3], Diethelm [4], Hilfer [5], Jiao et al [6], Kilbas et al. [7], Kyriakova [8], Mainardi [9], McBride [10], Miller and Ross [11], Petras [12], Samko et al [13], Podlubny [14], Sabatier et al. [15], Torres and Malinowska [16], Ying and Chen [17]) and by the distinguished diffusion as shown by the many meetings dedicated to it and the superfluous of articles appeared in mathematical (e.g. Kilbas and Marzan [18], Heinsalu et al [19], Luchko and Gorenflo [20]).
The use of derivative of arbitrary order has also spread into many other fields of science as well mathematics and physics (e.g.Laskin [21], Naber [22], Baleanu et al. [23], Zavada [24], Baleanu et al. [25], Caputo and Fabrizio [26],[27]) such as biology (e.g. Cesarone et al. [28], Caputo and Cametti [29]), economy (e.g. Caputo [30]), demography (e.g. Jumarie [31]), geophysics (e.g. Iaffaldano [32]), medicine (e.g El Sahed [33]) and bioengineering (e.g. Magin [34]). The significant contribution in the arbitrary derivative which carried out by the researchers One can see also [35], [36], [37], [38], [39], [40] and [41]. However, some condemnation has been made for a little cumbersome mathematical expression of its definition and the resultant obstacle in the solutions of the arbitrary order differential equations.
We deal with the most advanced definition of arbitrary order derivative, which assumes two different depiction for the temporal and spatial variable. The first efforts proposed on the time variable, where the real powers come into sight and obtain the solutions of the usual arbitrary order derivative will turn into integer powers, with some compendiums in these formulae and the computations. In this framework, it is appropriately used the Laplace transform. The second definition is related to the spatial variables, thus for this non-local arbitrary order derivative it is more suitable to work with the Fourier transform [42].

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2 An Advanced Fractional Time Derivative

We know that the usual Caputo fractional time derivative (UFD_t) of order α, given by

$$D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)} \int_a^t \frac{f'(\tau)}{f(\tau)} d\tau \qquad (1)$$

with $\alpha \in [0,1]$ and $a \in (-\infty, t], f \in H^1(a,b), \quad b > a$.
Caputo and Fabrizo [42] replace the kernel $(t-\tau)^{-\alpha}$ with the function $exp[-\frac{\alpha(t-\tau)}{(1-\alpha)}]$ and $[-\frac{1}{(1-\alpha)}]$ with $\frac{M(\alpha)}{(1-\alpha)}$ and obtained the new definition of fractional time derivative NFD without singular kernel for detailed information one may go through [42].
An Advanced definition of fractional time derivative

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) exp[-\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau, \qquad (2)$$

where $M(\alpha)$ is a normalization function $M(0) = M(1) = 1$. Here, $exp[-\frac{\alpha(t-\tau)}{(1-\alpha)}]$ is replaced by $exp[\frac{\alpha(t-\tau)}{(1-\alpha)}]$ by us, then obtained as follows

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau. \qquad (3)$$

3 Some Important Results

Case I: Adding the equation (2) and (3) then we obtain the new result namely Advanced fractional derivative Type-1 which is given as under

$$D_t^{\alpha} f(t) = \frac{2M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) cosh[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau. \qquad (4)$$

Case II: Subtracting the equation (2) and (3) we get the result namely Advanced fractional derivative Type-2 which is given as under

$$D_t^{\alpha} f(t) = \frac{2M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) sinh[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau. \qquad (5)$$

Theorem 3.1: For $_{NF}D_t$ if the function $f(t)$ is such that

$$f^{(p)}(\alpha) = 0, p = 1, 2, ... \qquad (6)$$

Then we have $D_t^{(n)}\left(D_t^{(\alpha)} f(t)\right) = D_t^{(\alpha)}\left(D_t^{(n)} f(t)\right)$
Proof: If $n \geq 1$ and $\alpha \in [0,1]$ the fractional time derivative $D_t^{(\alpha+n)} f(t)$ of order $(n+\alpha)$ is defined by

$$D_t^{\alpha+n} f(t) = D_t^{(\alpha)}\left(D_t^{(n)} f(t)\right), \qquad (7)$$

we consider the $n = 1$, then the definition of (7) $D_t^{(\alpha+1)} f(t)$, we get

$$D_t^{(\alpha)}\left(D_t^{(1)} f(t)\right) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau.$$

After that integration by parts and suppose that $f'(a) = 0$, we have

$$D_t^{(\alpha)}\left(D_t^{(1)} f(t)\right) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \frac{d}{d\tau} f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau.$$

$$= \frac{M(\alpha)}{(1-\alpha)} \int_a^t \frac{d}{d\tau} f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau + \frac{\alpha}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau$$

$$= \frac{M(\alpha)}{(1-\alpha)} \left[f(\tau) + \frac{\alpha}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau \right].$$

Otherwise,

$$D_t^{(1)}\left(D_t^{(\alpha)} f(t)\right) = \frac{d}{dt} \left[\frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau \right],$$

$$= \frac{M(\alpha)}{(1-\alpha)} \left[f(t) + \frac{\alpha}{(1-\alpha)} \int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau \right].$$

It is easy to generalize it for any $n > 1$

4. The Laplace transform of the advanced definition of fractional time derivative NFD_t, [4]

In this section, we have studied the properties of NFD_t defined in equation

$$D_t^{(\alpha)} f(t) = \frac{N(\sigma)}{\sigma} \left[\int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau \right], \tag{8}$$

where $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty), \alpha = \frac{-1}{1+\alpha} \in [0,1]$ with $a = 0$ has priority the computation of its Laplace transform is given by

$$LT[D_t^{(\alpha)} f(t)] = \frac{1}{1-\alpha} \int_0^\infty exp(-pt) \left[\int_a^t f(\tau) exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] d\tau \right]. \tag{9}$$

Using the convolution theorem of Laplace transform, we have

$$LT[D_t^{(\alpha)} f(t)] = \frac{1}{1-\alpha} LT \left[exp[\frac{\alpha(t-\tau)}{(1-\alpha)}] \right],$$

$$= \frac{p[Lf(t)] - f(0)}{p - \alpha(p+1)},$$

$$LT[D_t^{(\alpha+1)} f(t)] = \frac{1}{1-\alpha} LT[f(t)] LT \left(exp \left[\frac{\alpha(t-\tau)}{(1-\alpha)} \right] \right),$$

$$= \frac{p^2[Lf(t)] - pf(0) - f'(0)}{p - \alpha(p+1)},$$

$$LT[D_t^{(\alpha+n)} f(t)] = \frac{p^{n+1}[Lf(t)] - p^n f(0) - \cdots - f'(0)}{p - \alpha(p+1)}. \tag{10}$$

5. Fractional gradient operator [42]:

In this section, we discuss a new notion of fractional gradient able to describe non-local dependence in constitutive equations [42]

Let us consider a set $\Omega \in R^3$ and a scalar function $u(\cdot) : \Omega \to R$, we have introduced the Fractional gradient of order

$\alpha \in [0,1]$ is given by

$$\nabla^{(\alpha)}u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla u(y) exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy \tag{11}$$

with $x, y \in \Omega$.

We have the property $lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)\sqrt{\pi}} exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] = \delta(x-y)$.

Then $\nabla^{(1)}u(x) = \nabla u(x)$ and $\nabla^{(0)}u(x) = \int_{\Omega} \nabla u(y)dy$.

Fractional divergence is defined for the smooth function $u(\cdot) : \Omega \to R$ is given by

$$\nabla^{(\alpha)}u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla u(y) exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy. \tag{12}$$

Theorem 5.1: The definition is given in equation (11) and (12) and we have used the function $u(x) : \Omega \to R$, such that

$$\nabla u(x).n|_{\delta\Omega} = 0. \tag{13}$$

We have the following identity

$$\nabla\nabla^{(\alpha)}u(x) = \nabla^{(\alpha)}\nabla u(x). \tag{14}$$

Proof: Using equation (11) and (14), we get

$$\nabla\nabla^{(\alpha)}u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla u(y) exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy,$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla u(y) \cdot \nabla exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy,$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega} \nabla \cdot \nabla u(y) \cdot \nabla exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy - \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\delta\Omega} \nabla u(y) \cdot n \cdot exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy.$$

Hence, using the boundary condition (13) and we have proved the identity (14).

$$\nabla\nabla^{(\alpha)}u(x) = \nabla^{(\alpha)}\nabla u(x).$$

6. Fourier transform of fractional gradient and divergence [42].

In this section, we use the smooth function $u(x) : R^3 \to R$ and Fourier transform of the Fractional gradient is defined by

$$FT\left(\nabla^{(\alpha)}u(x)\right)(\zeta) = \int_{R^3} \nabla^{(\alpha)}u(x) exp[2\pi i\zeta x]dx. \tag{15}$$

The Fourier transform

$$FT(\nabla^{(\alpha)}u)(\zeta) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT\left(\int_{R^3} \nabla u(y) exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy\right)(\zeta),$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} FT(\nabla u)(\zeta) FT\left(exp\left[\frac{\alpha^2(x)^2}{(1-\alpha)^2}\right]\right)(\zeta).$$

Here, $FT\left(exp\left[\frac{\alpha^2(x)^2}{(1-\alpha)^2}\right]\right)(\zeta) = \frac{(1-\alpha)}{\alpha}\sqrt{\pi} exp\left[\frac{\pi^2(1-\alpha)^2(\zeta)^2}{(\alpha)^2}\right]$.

Hence, $FT(\nabla^{(\alpha)}u)(\zeta) = \sqrt{\pi 1 - \alpha} FT(\nabla u)(\zeta) FT\left[exp\left(\frac{\alpha^2(x)^2}{(1-\alpha)^2}\right)(\zeta)\right]$.

Fourier transform of the fractional divergence is given by

$$FT(\nabla^{(\alpha)}u)(\zeta)=\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}FT\left(\int_{R^3}\nabla u(y)exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy\right)(\zeta). \tag{16}$$

Hence, $FT(\nabla^{(\alpha)}u)(\zeta)=\sqrt{\pi^{1-\alpha}}FT(\nabla\cdot u)(\zeta)exp\left[\frac{\pi^2(1-\alpha)^2(\zeta)^2}{(\alpha)^2}\right]$.

7. The fractional Laplacian: [42]

In this section, we use the definition of Fractional gradient and divergence, we represent the Fractional Laplacian for a smooth function $f(x):\Omega\to R^3$, such that

$$\nabla f(x).n|_{\delta\Omega}=0, \tag{17}$$

$$(\nabla^2)^{\alpha}f(x)=\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\left(\int_{\Omega}\nabla\cdot\nabla f(y)exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy\right). \tag{18}$$

By using the theorem 5.1, we have
$(\nabla^2)^{\alpha}f(x)=\nabla\cdot\nabla^{\alpha}f(x)=\nabla^{\alpha}\cdot\nabla f(x)$.
And we assume that $f(x)=0$ om $\delta\Omega$.
Hence, we can extend the function $f(x)=0$ on $R^3-\Omega$ and we can consider the Fourier transform

$$\begin{aligned}FT\left((\nabla^2)^{\alpha}f(x)\right)&=\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}FT\left(\int_{R^3}(\nabla)^2f(y)exp\left[\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy\right)(\zeta),\\ &=\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}FT(\nabla\cdot\nabla f(x))(\zeta)FT\left(exp\left[\frac{\alpha^2(x)^2}{(1-\alpha)^2}\right]\right)(\zeta),\\ &=4\pi|\zeta|^2FT(f(x))(\zeta)\sqrt{\pi^{1-\alpha}}exp\left[\frac{(1-\alpha^2)(\zeta)^2}{(\alpha)^2}\right].\end{aligned} \tag{19}$$

If $\alpha=1$, we get the solution

$$\begin{aligned}FT(\nabla^2f(x))&=-lim_{\alpha\to1}4\pi|\zeta|^2FT(f(x))(\zeta)\sqrt{\pi^{1-\alpha}}exp\left[\frac{(1-\alpha^2)(\zeta)^2}{(\alpha)^2}\right].\\ &=-4\pi|\zeta|^2FT(f(x))(\zeta).\end{aligned} \tag{20}$$

4 Conclusion

In this work, we have derived the new results by using advanced definition of fractional derivative without singular kernel. The results of the advanced definition of fractional derivative without singular kernel are same as the ones of Michele Caputo and Mauro Fabrizio [42].

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References:

[1] D. Baleanu, Z. B. Guvenc and J. A. T. Machado (Eds.), *New trends in nanotechnology and fractional calculus applications*, XI, 531, ISBN Springer, 1st Edition, 2010.
[2] R.Caponetto, *Fractional order systems (modelling and control applications)*, World Scientific, 2010.
[3] M. Caputo, *Elasticit'a e dissipazione*, Zanichelli, Bologna, 1965.
[4] K. Diethelm, *The analysis of fractional differential equations, an application oriented, exposition using differential operators of Caputo type*, Lecture Notes in Mathematics nr. 2004, Springer, Heidelbereg, 2010.

[5] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co, Singapore, 2003.
[6] Z. Jiao, Y. Chen and I. Podlubny, *Distributed order dynamic systems, modeling, analysis and simulation*, Springer, 2001.
[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, Amsterdam, 2006.
[8] V. Kiryakova, *Generalized fractional calculus and applications*, Pitman Research Notes in Mathematics, 301, Longman, Harlow, 1994.
[9] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*, World Scientific Publishing, 2010.
[10] A.C. McBride, *Fractional calculus and integral transforms of generalized functions*, Pitman Res. Notes in Mathem., nr. 31, Pitman, London, 1979.
[11] K. S. Miller and B. Ross, *An introduction to fractional calculus and fractional differential equations*, Wiley, New York, 1993.
[12] I. Petras, *Fractional calculus non-linear systems, modelling, analysis and simulation*, Springer, 2011.
[13] S. G. Samko, A. A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Sc. Publ, 1993.
[14] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 2009.
[15] J. Sabatier, O. P. Agrawal and J.A. Tenreiro Machado, *Advances in fractional calculus*, Springer, 2009.
[16] D. F. M. Torres and A. B. Malinowska, *Introduction to the Fractional Calculus of Variations*, Imperial College Press, 2012.
[17]V. L. Ying and Y. Chen, *Fractional order motion controls*, Wiley, 2012.
[18] A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with Caputo fractional derivative in the space of continuously differentiable functions, *Differ. Equ.* **41**, 84-95 (2005).
[19] E. Heinsalu, M. Patriarca, I. Goychuk, G. Schmid and P. H'anggi , Fractional Fokker Plank dynamics: numerical algorithm and simulation, *Phys. Rev. E* **73**, 046133 (2006).
[20] Y. Luchko and R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivative, *Acta Math. Vietnamica* **24**, 207-233 (1999).
[21] N. Laskin, Fractional Schr'odinger equation, *Phys. Rev. E* **66**, 056108 (2002).
[22] M. Naber Time fractional Schrodinger equation, *J. Math. Phys.* **45**(8), 3339-3352 (2004).
[23] D. Baleanu, A. K. Golmankhaneh and A. K. Golmankhaneh, The dual action and fractional multi time Hamilton equations, *Int. J.Theor. Phys.* **48**, 2558-2569 (2009). [24] P. Zavada, Relativistic wave equations with fractional derivatives and pseudo-differential operators, *J. Appl. Math.* **2**(4), 163-197 (2002).
[25] D. Baleanu, A. K. Golmankhaneh,A. K. Golmankhaneh and R. R. Nigmatullin, Newtonian law with memory, *Nonlin. Dyn.* **60**, 81-86 (2010).
[26] M.Caputo and M. Fabrizio M., Damage and fatigue described by a fractional derivative model, *J. Comput. Phys.*,162-178 (2014).
[27] M. Caputo, C. Cametti and V. Ruggiero, Time and spatial concentration profile inside a membrane by means of a memory formalism, *Phys. A* **387**, 2010–2018 (2007).
[28] F. Cesarone, M. Caputo and C. Cametti, Memory formalism in the passive diffusion across a biological membrane, *J. Membr. Sci.* **250**, 79-84 (2004).
[29] M. Caputo and C. Cametti, Memory diffusion in two cases of biological interest, *J. Theor. Biol.* **254**,697-703 (2008).
[30] M. Caputo, *Modeling social and economic cycles, in alternative public economics*, Forte F., Navarra P., MudambiR.(Eds.), Elgar, Cheltenham, 2014.
[31] G. Jumarie, New stochastic fractional models of the Malthusian growth, the Poissonian birth process and optimal management of populations, *Math. Comput. Modell.* **44**, 231-254 (2006).
[32] G. Iaffaldano, M. Caputo and S. Martino, Experimental and theoretical memory diffusion of water in sand, *Hydrol. Earth Sys. Sci.* **10**, 93–100 (2006).
[33] M. El Shaed, *A fractional calculus model of semilunar heart valve vibrations*, 2003 International Mathematica Symposium, London, 2003.
[34] R. L. Magin, *Fractional calculus in bioengineering*, Begell House Inc. Publishers, 2006.
[35] I. M. Gel'fand and G. E. Shilov, *Generalized functions 1–5*, Academic Press, 1968.
[36] E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, 1963.
[37] G. Z.Voyiadjis and R. K. Abu Al-Rub, Nonlocal gradient-dependent thermodynamics for modeling scale-dependent plasticity, *Int. J. Multis. Comput.* **5**, 295-323 (2007).
[38] G. Z. Voyiadjis and A. H. Almasri, Variable material length scale associated with nanoindentation experiments, *J. Eng. Mech. ASCE* **135**(3), 139-148 (2009).
[39] M. Fabrizio, Fractional rheological models for thermomechanical systems. Dissipation and free energies, *Fract. Calc. Appl. Anal.* **17**, 206-223 (2014).

[40] M. Fabrizio and A. Morro, Dissipativity and irreversibility of electromagnetic systems, *Math. Mod. Meth. Appl. Sci.* **10**, 217-246 (2000).
[41] M. Caputo, Mean fractional-order-derivatives differential equations and filters, *AnnaliUniversit'a di Ferrara, Sez. VII, Sci. Mat.* **41**, 73–83 (1995).
[42] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**(2), 73-85 (2015).