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Domain of Attraction for Central and Intermediate Order Statistics under Exponential Normalization

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Abstract: In this article, the domain of attractions (DAs) of the central order statistics under linear normalization is compared with the DAs under exponential normalization. Moreover, The work of Barakat and Omar [1] for the intermediate order statistics under power normalization, is expanded to the exponential normalization. Furthermore, The DAs for the limit laws of intermediate order statistics under exponential normalization are derived. MSC (2000): 62G32; 62G30.

Keywords: Intermediate order statistics, Exponential normalization, Domain of attractions.

1 Introduction

Let $T_1, T_2, ..., T_n$ be independent random variables (RVs) with the same distribution function (DF) $F(t)$. The associated order statistics (OSs) are indicated by $T_{1:n} \leq T_{2:n} \leq ... \leq T_{n:n}$. Numerous researchers examined the OS $T_{k_n:n}$, when $\min(k_n, n-k_n) \to \infty$, as $n \to \infty$, including [1] [11] [3] [4] [5] [6] [7]. The term $T_{k:n}$ is called a right intermediate (or left intermediate) OS, if its rank sequence $\{k\} = \{k_n\}$ is like that $k_n \to \infty$, and $\frac{k_n}{n} \to 1$, as $n \to \infty$ or ($k_n \to \infty$ and $\frac{k_n}{n} \to 0$, as $n \to \infty$,) . Applications are numerous for intermediate OSs. for instance, In insurance and finance, intermediate quantile numbers are helpful when studying low-frequency, high-severity losses. On the other hand, tail quantiles can be estimated using intermediate OSs. Numerous authors, have also created estimators based on intermediate OSs e.g. [20] [24]. For more details about central and intermediate OSs we refer the reader to [8] [9] [10] [14] [15] [17] [19].

The nonlinear normalization was initially utilized by Weinstein [25] and Pantcheva [22], to obtain a larger class of limit laws of extremes. It can be used to resolve difficulties involving approximation. The work of Pancheva [22] was extended by the authors in [11] for the OSs under nonlinear normalization, where their work was mostly focused on the power normalization

$$G_n(t) := b_n \mid t \mid^{a_n} \mathscr{S}(t),\ a_n, b_n > 0,$$

where $\mathscr{S}(t) = -1, 0,$ and $+1$, according to $t < 0, t = 0,$ and $t > 0$, respectively. The authors in [11] revealed that the possible non-degenerate weak limits, that have more than two growth points, of any central OS with regular rank r_n (i.e., $\sqrt{n}(\frac{r_n}{n} - \lambda) \to 0, \lambda \in (0,1)$, as $n \to \infty$) under Both traditionally linear and power normalisations are the same. Namely, these limit laws are

$$\Phi_{\alpha}^{(1)}(t) = \Phi(c_1 t^{\alpha}) \mathrm{I}_{t>0},$$

$$\Phi_{\alpha}^{(2)}(t) = \Phi(-c_2 \mid t \mid^{\alpha}) \mathrm{I}_{t\leq 0} + \mathrm{I}_{t>0},$$

and

$$\Phi_{\alpha}^{(3)}(t) = \Phi(-c_2 \mid t \mid^{\alpha}) \mathrm{I}_{t\leq 0} + \Phi(c_1 t^{\alpha}) \mathrm{I}_{t>0},$$

where I_A is the usual indicator function. In this case, when the DF of a central OS $T_{r_n:n}$, based on a DF F and with the regular rank r_n, weakly converges to one of the above three limit laws $\Phi_{\alpha}^{(j)}(.),\ j \in \{1,2,3\}$, under linear or power normalization we claim that the *DF* is a member of the domin normal $\lambda-$attraction of $\Phi_{\alpha}^{(j)}(.)$ under linear or power

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normalization, respectively. Moreover, the authors in [11] derived the limit types of the intermediate OSs under power normalization (see Mohan and Ravi [21] for extreme case) and got the following 6 types for the lower intermediate OSs:

$$\tilde{L}_{1,\beta}(t)=\Phi(\beta\ln\ln t),\ t>1;\ \tilde{L}_{2,\beta}(t)=\Phi(-\beta\ln(-\ln t)),\ 0<t\leq 1;$$

$$\tilde{L}_{3,\beta}(t)=\Phi(\beta\ln(-\ln|t|)),\ -1<t\leq 0;\ \tilde{L}_{4,\beta}(t)=\Phi(-\beta\ln\ln|t|),\ t\leq -1;$$

$$\tilde{L}_{5}(t)=\Phi(\ln t),\ t>0;\ \tilde{L}_{6}(t)=\Phi(-\ln|t|),\ t\leq 0.$$

Once more, to expand the class of the limit laws in the extreme value theorem (see Galambos [17] for linear normalization), the authors in [23] introduced exponential normalization, that has the forme

$$G_n(t):=G_{u_n,v_n}(t)=\{\exp(u_n(|\ln|t||)^{v_n}\mathscr{S}(\ln|t|))\}\mathscr{S}(t);\ u_n,v_n>0.$$

which yields the following definition:

Definition 1.1. The two DFs $\mathscr{H}_1$ and $\mathscr{H}_2$ are of the same exponential type (denoted by e-type) if

$$\mathscr{H}_1(t)=\mathscr{H}_2(\{\exp(u(|\ln|t||)^{v}\mathscr{S}(\ln|t|))\}\mathscr{S}(t))=\mathscr{H}_2\circ G_{u,v}(t)=\mathscr{H}_2(G_{u,v}(t)),$$

for certain constants $u>0,\ v>0$. Moreover, we say that a DF F is belonging to the e-max-DAs of the non-degenerate DF $\mathscr{H}$ under e-normalization, denoted by $F\in D_e(\mathscr{H})$, if

$$P(G^{-1}_{u_n,v_n}(T_{n:n})\}\leq t)=P\left(\left\{\exp\left[\left(\frac{|\ln|T_{n;n}||}{u_n}\right)^{1/v_n}\mathscr{S}(\ln|(T_{n:n})|)\right]\right\}\mathscr{S}(T_{n:n})\leq t\right)$$

$$=P(T_{n:n}\leq G_{u_n,v_n}(t))=F_{n:n}(G_{u_n,v_n}(t))\xrightarrow[n]{w}\mathscr{H}(t). \tag{1.1}$$

Authors in [11] studied the asymptotic DFs for OSs with variable ranks under the exponential transformation. They showed that under this normalization, both log-normal and negative log-normal DFs are possible limits of the central OSs with regular ranks, Despite the fact that the normal distribution is not. The limit laws that have more growth points than two of the central OSs under exponential normalization are

$$\xi^{(1)}(t)=\Phi(c(\ln t)^{\beta})\mathrm{I}_{\{t>1\}},$$

$$\xi^{(2)}(t)=\Phi(-c|\ln|t||^{\beta}\mathscr{S}(\ln|t|))\mathrm{I}_{\{t\leq 0\}}+\mathrm{I}_{\{t>0\}},$$

and

$$\xi^{(3)}(t)=\Phi(-c'_1|\ln t|^{\beta})\mathrm{I}_{\{0<t\leq 1\}}+\Phi(c'_2(\ln t)^{\beta})\mathrm{I}_{\{t>1\}},\ c,c'_1,c'_2,\beta>0.$$

Moreover, authors in [11] derived all limit types of the intermediate term under exponential normalization, and got the following types for the lower intermediate

$$L_{1,\beta}(t)=\Phi(\beta\ln(\ln\ln t))\mathrm{I}_{\{t>e\}};\ L_{2,\beta}(t)=\Phi(-\beta\ln(-\ln\ln t))\mathrm{I}_{\{1<t\leq e\}}+\mathrm{I}_{\{t>e\}};$$

$$L_{3,\beta}(t)=\Phi(\beta\ln(-\ln(-\ln t)))\mathrm{I}_{\{\frac{1}{e}<t\leq 1\}}+\mathrm{I}_{\{t>1\}};$$

$$L_{4,\beta}(t)=\Phi(-\beta\ln(\ln(-\ln t)))\mathrm{I}_{\{0<t\leq\frac{1}{e}\}}+\mathrm{I}_{\{t>\frac{1}{e}\}};$$

$$L_{5,\beta}(t)=\Phi(\beta\ln(\ln(-\ln(-t))))\mathrm{I}_{\{\frac{1}{e}<t\leq 0\}}+\mathrm{I}_{\{t>0\}};$$

$$L_{6,\beta}(t)=\Phi(-\beta\ln(-\ln(-\ln(-t))))\mathrm{I}_{\{-1<t\leq-\frac{1}{e}\}}+\mathrm{I}_{\{t>-\frac{1}{e}\}};$$

$$L_{7,\beta}(t)=\Phi(\beta\ln(-\ln\ln(-t)))\mathrm{I}_{\{-e<t\leq-1\}}+\mathrm{I}_{\{t>-1\}};$$

$$L_{8,\beta}(t)=\Phi(-\beta\ln(\ln\ln(-t)))\mathrm{I}_{\{t\leq-e\}}+\mathrm{I}_{\{t>-e\}};\ L_{9,\beta}(t)=\Phi(\ln\ln(t))\mathrm{I}_{\{t>1\}};$$

$$L_{10,\beta}(t)=\Phi(-\ln(-\ln t)\mathrm{I}_{\{0\leq t\leq 1\}}+\mathrm{I}_{\{t>1\}};\ L_{11,\beta}(t)=\Phi(\ln(-\ln(-t))\mathrm{I}_{\{-1<t\leq 0\}}+\mathrm{I}_{\{t>0\}}$$

and

$$L_{12,\beta}(t)=\Phi(-\ln(-\ln(-\frac{1}{t}))\mathrm{I}_{\{t>-1\}}.$$

For more details about the asymptotic distribution of OSs under exponential normalization and some of their applications see Barakat et al. [13] [12] and Khaled et al. [18].

In Section 2, we compare the DAs for central OSs under linear and exponential normalization. In Section 3, we obtain the DAs for the limit types of intermediate OS under exponential normalization.

2 A comparison between DAs for limit laws of central OSs under linear and exponential normalization

The following theorem reveals an interesting fact: if a *DF* belongs to the central DAs of $\Phi_{\alpha}^{(3)}(t)$, under a linear normalization, then this DF can not belong to any DAs with growth points more than two under exponential normalization.

Theorem 2.1. Let a DF F belongs to the domain of normal $\lambda-$attraction of $\Phi_{\alpha}^{(3)}(t)$ type, like that $F^{-1}(\lambda) \neq 0$, Under linear normalization. Then, the domains of normal $\lambda-$attraction of $\xi^{(1)}(t)$, $\xi^{(2)}(t)$, and $\xi^{(3)}(t)$ types do not attract the DF F under the exponential normalization.

*Proof.*For a DF F and the real constants $\beta_n, u_n, v_n > 0$ and $\alpha_n \in \mathbb{R}$ we have

$$\left[\sqrt{n}\frac{F(\beta_n t+\alpha_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}\right] = \left[\frac{F(\beta_n t+\alpha_n)-\lambda}{F(\exp[(u_n(|\ln|t||)^{v_n})\mathscr{S}(\ln|t|)]\mathscr{S}(t))-\lambda}\right]$$
$$\left[\sqrt{n}\frac{F(\exp[(u_n(|\ln|t||)^{v_n})\mathscr{S}(\ln|t|)]\mathscr{S}(t))-\lambda}{\sqrt{\lambda(1-\lambda)}}\right]. \tag{2.1}$$

Let $\beta_n > 0$, α_n be suitable linear normalizing constants where the DF of $\frac{1}{\beta_n}(T_{r:n}-\alpha_n)$ weakly converges to $\Phi_{\alpha}^{(3)}(t)$. Consequently, by Theorem 8c in [5], $\exists$ a point of continuity, $t=t_0$, of the DF F like that $F(t_0)=\lambda$, where $F^{-1}(\lambda)=t_0$, now, we can take $\alpha_n = t_0$ and $\beta_n = F^{-1}(\lambda+\frac{1}{\sqrt{n}})-t_0$, which implies $\beta_n \to 0$, (as $n\to\infty$). Assuming that exponential normalizing constants $u_n, v_n > 0$ can be found such that the DF of $\left(\left\{\exp\left[\left(\frac{|\ln|T_{r:n}||}{u_n}\right)^{1/v_n}\mathscr{S}(\ln|T_{r:n}|)\right]\right\}\mathscr{S}(T_{r:n})\right)$ weakly converges to $\xi^{(3)}(t)$. Distinctly, we can suppose that the exponential normalizing constants can be chosen such that

$$\xi^{(3)}(t) = \begin{cases} \Phi(-c_1' \mid \ln t\mid^{\beta}), & 0<t\leq 1, \\ \Phi(c_2'(\ln t)^{\beta}), & t>1, \\ 0, & t\leq 0, \end{cases}$$

In view of (2.1), we get $\forall t>0$,

$$\frac{F(\exp(u_n|\ln|t||^{v_n}\mathscr{S}(\ln|t|))\mathscr{S}(t))-\lambda}{F(\beta_n t+\alpha_n)-\lambda} \to 1,$$

as $n\to\infty$. This implies that $F(\exp(u_n|\ln|t||^{v_n}\mathscr{S}(\ln|t|))\mathscr{S}(t))-\lambda \to 0$ as $n\to\infty$. Thus, $\exp(u_n|\ln|t||^{v_n}\mathscr{S}(\ln|t|) \to F^{-1}(\lambda) = t_0$. By putting $t=e$ we get $u_n \to \ln t_0$. On the other hand, using (2.1) we get

$$\frac{F(\exp(u_n|\ln|t||^{v_n}\mathscr{S}(\ln|t|))\mathscr{S}(t))-\lambda}{F(\beta_n t+\alpha_n)-\lambda} \to \frac{\acute{c}_1}{c_2} \neq 0,$$

as $n\to\infty$, for all $t<0$. Which leads $u_n \to -\ln t_0$, when $t=-e$. which contradicts that u_n does not depend on t. Consequently, the assumption $\exists$ exponential normalizing constants $u_n, v_n > 0$ where the DF of $\left(\left\{\exp\left[\left(\frac{|\ln|T_{r:n}||}{u_n}\right)^{1/v_n}\mathscr{S}(\ln|T_{r:n}|)\right]\right\}\mathscr{S}(T_{r:n})\right)$ weekly convergent to $\xi^{(3)}(t)$ is not true. Then the assumption that, the DF of $\left(\left\{\exp\left[\left(\frac{|\ln|T_{r:n}||}{u_n}\right)^{1/v_n}\mathscr{S}(\ln|T_{r:n}|)\right]\right\}\mathscr{S}(T_{r:n})\right)$ weekly converges to $\xi^{(1)}(t)$ or $\xi^{(2)}(t)$ for some suitable exponential normalizing constants $u_n, v_n > 0$ leads to a contradiction and merely has the evident differences. The evidence is now complete.

3 Criteria for DFs to belong to the DAs of the lower intermediate exponential types

TO show that a DF F is a member in the DAs of the laws G, under power normalization ($G_n(t) = \alpha_n|t|^{b_n}\mathscr{S}(t)$), and L, under exponential normalization, respectively we will write $F\in D_p(G)$ and $F\in D_e(L)$. In addition, the left and right end points of F, are denoted by $\ell_F = \inf\{t: F(t)>0\}$ and $r_F = \sup\{t: F(t)<1\}$ respectively. Also,for any nondecreasing function F define $F^{-}(y) = \inf\{t: F(t)>y\}$. Finally, the normalizing constants are given at the end of each theorem.

Theorem 3.1. A DF $F\in D_e(L_{1,\beta})$ if and only if (iff)

1. $\exists\, t_0$ like that $F(e^{e^{t_0}})=0$, and $F(e^{e^{t_0}}+\varepsilon)>0,\ \forall\varepsilon>0$ and
2. $\forall\tau>0$,

$$\lim_{t\downarrow 0}\frac{F(\exp(\exp(t_0+t\tau)))-F(\exp(\exp(t_0+t)))}{[F(\exp(\exp(t_0+t)))]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

If so, we could decide to set $u_n=e^{t_0}$ and $v_n=\ln(F^{-}(\frac{k}{n}))-t_0$.

*Proof.*Suppose that 1 and 2 hold. Define the DF

$$G(y)=\begin{cases}0, & y\le t_0,\\ F(e^y), & y>t_0.\end{cases} \tag{3.1}$$

Clearly, $\ell_G=t_0$. The relation 2 implies

$$\lim_{t\downarrow 0}\frac{G(\exp(t_0+\tau t))-G(\exp(t_0+t))}{[G(\exp(t_0+t))]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

Hence, from Theorem 3.1 in [1], will get $G\in D_p(\tilde{L}_{1,\beta})$, with $\alpha_n=e^{t_0}$ and $\beta_n+t_0=\ln G^{-}(\frac{k}{n})$, i.e.,

$$\frac{nG(\alpha_n t^{\beta_n})-k}{\sqrt{k}}\to\begin{cases}-\infty, & t\le 1,\\ \beta\ln\ln t, & t>1,\end{cases}$$

as $n\to\infty$.Therefore, we get

$$\frac{nF(e^{u_n(t)^{v_n}})-k}{\sqrt{k}}\to\beta\ln\ln t,\ \forall t>1, \tag{3.2}$$

as $n\to\infty$,where $u_n=\alpha_n,\ \ \beta_n=v_n$. Thus, putting $t=\ln y$, we get

$$\frac{nF(e^{u_n(\ln y)^{v_n}})-k}{\sqrt{k}}\to\beta\ln\ln\ln y,\ \forall y>e. \tag{3.3}$$

Now, for any $0<t\le 1$, $(i.e\ \ 1<y\le e)$ we get

$$\frac{nF(e^{u_n(\ln y)^{v_n}})-k}{\sqrt{k}}\to-\infty,\ \forall 1<y\le e. \tag{3.4}$$

For $t\le 0$, $(i.e. y\le 1)$, we get

$$\frac{nF(e^{u_n(\ln y)^{v_n}})-k}{\sqrt{k}}\to-\infty,\ \forall y\le 1. \tag{3.5}$$

Combining (3.3), (3.4), and (3.5) we get

$$\frac{nF(\exp(u_n(\ln y)^{v_n}\mathscr{S}(\ln y))\mathscr{S}(y))-k}{\sqrt{k}}\to\begin{cases}-\infty, & y\le e,\\ \beta\ln\ln\ln y, & y>e,\end{cases}$$

which implies $F\in D_e(L_{1,\beta})$.

Conversely, let $F\in D_e(L_{1,\beta})$, with normalizing constants $u_n=B>0$ and $v_n=\ln(F^{-}(\frac{k}{n}))-B>0$. First, we note that $\ell_F\ge 0$. Indeed, put $t=0$ in the relation

$$\frac{nF(\exp(u_n(\ln t)^{v_n}\mathscr{S}(\ln t))\mathscr{S}(t))-k}{\sqrt{k}}\to-\infty,\ \forall t\le 1,$$

we get $\frac{nF(0)-k}{\sqrt{k}}\to-\infty$. The last relation yields $F(0)<\frac{k}{n}\to 0$, which implies $F(0)=0$, and consequently $\ell_F\ge 0$. On the other hand, if $u_n=B>0$, we get $\frac{nF(e^B)-k}{\sqrt{k}}\to-\infty$, which implies that $F(e^B)=0$, and consequently $\ell_F\ge e^B>0$. This proves the relation 1. Now, the modified Khinchin's types theorem permits us to take $u_n=\ell_F=e^{t_0}$ (where $t_0=\ln\ell_F$). If we again adopt the definition (3.1), we get $\frac{nG(\alpha_n t^{\beta_n})-k}{\sqrt{k}} = \frac{-k}{\sqrt{k}} \to -\infty, \quad \forall t < 0,$ where $\beta_n = G^{-}(\frac{k}{n}) - t_0 = \ln F^{-}(\frac{k}{n}) - t_0 = v_n$. On the other hand, since $F \in D_e(L_{1,\beta})$, then $\forall t > 0$, let $t = \ln y$ we get

$$\frac{nF(\exp(u_n(\ln y)^{v_n}) - k}{\sqrt{k}} \to \beta \ln\ln\ln y, \ \forall y > e.$$

Hence,

$$\frac{nG(\alpha_n(t)^{\beta_n}) - k}{\sqrt{k}} \to \beta \ln\ln t, \ \forall t > 1.$$

Thus, $G \in D_p(\tilde{L}_{1,\beta})$, with α_n and β_n, which in view of Theorem 3.1 in [1] yields

$$\lim_{t\downarrow 0} \frac{G(\exp(t_0 + t\tau)) - G(\exp(t_0 + t))}{[G(\exp(t_0 + t))]^{\frac{2-\alpha}{2-2\alpha}}} = \ell^{\frac{-1}{1-\alpha}} \beta \ln \tau, \forall \tau > 0.$$

Therefore, since $t_0 + t, t_0 + \tau t > t_0$, we get

$$\lim_{t\downarrow 0} \frac{F(\exp(\exp(t_0 + t\tau))) - F(\exp(\exp(t_0 + t)))}{[F(\exp(\exp(t_0 + t)))]^{\frac{2-\alpha}{2-2\alpha}}} = \ell^{\frac{-1}{1-\alpha}} \beta \ln \tau, \forall \tau > 0,$$

which proves the relation 2, as well as the theorem.

Theorem 3.2. A DF $F \in D_e(L_{2,\beta})$ iff

1. $\ell_F = 0$, and
2. $\forall \tau > 0$,

$$\lim_{t\to -\infty} \frac{F(\exp(\exp(\tau t))) - F(\exp(\exp(t)))}{[F(\exp(\exp(t)))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}} \beta \ln \tau.$$

We may set $u_n = 1$ and $v_n = -\ln(F^{-}(\frac{k}{n}))$.

Proof. Suppose that the relations 1 and 2 hold. Define $G(y) = F(e^y), \forall y$. Thus, from 2, we get

$$\lim_{t\to -\infty} \frac{G(\exp(\tau t)) - G(\exp(t))}{[G(\exp(t))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}} \beta \ln \tau, \forall \tau > 0.$$

Consequently, in light of Theorem 3.2, in [1], we have $G \in D_p(\tilde{L}_{2,\beta})$, with $\alpha_n = 1$ and $\beta_n = -\ln(F^{-}(\frac{k}{n}))$. Thus, we get

$$\frac{nG(\alpha_n t^{\beta_n} s(t)) - k}{\sqrt{k}} \to \begin{cases} -\infty, & t \le 0, \\ -\beta \ln(-\ln t), & 0 < t < 1, \\ \infty, & t \ge 1. \end{cases}$$

Hence,

$$\frac{nF(e^{\alpha_n t^{\beta_n} s(t)}) - k}{\sqrt{k}} \to \begin{cases} -\infty, & t \le 0, \\ -\beta \ln(-\ln t), & 0 < t < 1, \\ \infty, & t \ge 1. \end{cases}$$

On the other hand, by putting $e^t = y > 0$, with $u_n = \alpha_n, v_n = \beta_n$ we get

$$\frac{nF(e^{u_n(\ln y)^{v_n} s(\ln y)}) - k}{\sqrt{k}} \to \begin{cases} -\infty, & y \le 1, \\ -\beta \ln(-\ln\ln y), & 1 < y < e, \\ \infty, & y \ge e, \end{cases}$$

which proves $F \in D_e(L_{2,\beta}(t))$, with $u_n = \alpha_n$ and $v_n = \beta_n$.

Conversely, let $F \in D_e(L_{2,\beta}(t))$. Then,

$$\frac{nF(e^{u_n(\ln y)^{v_n} s(\ln y)}) - k}{\sqrt{k}} \to -\infty, \ \forall y \le 1.$$

Put $y=0$, i.e., $nF(0)<k$, then for large $n, F(0)=0$, which implies $\ell_F=0$. This proves 1. Now, define again $G(y)=F(e^y), \forall y$. Hence,

$$\frac{nG(\alpha_n(\ln y)^{\beta_n}\mathscr{S}(\ln y))-k}{\sqrt{k}}=\frac{nF(e^{u_n(\ln y)^{v_n}\mathscr{S}(\ln y)})-k}{\sqrt{k}}$$
$$\to\begin{cases}-\infty, & y<1,\\ -\beta\ln(-\ln\ln y), & 1<y<e\\ \infty, & y>e.\end{cases}$$

Let $\ln y=t$. Hence,

$$\frac{nG(\alpha_n(t)^{\beta_n}\mathscr{S}(t))-k}{\sqrt{k}}\to\begin{cases}-\infty, & t<0,\\ -\beta\ln(-\ln t), & 0<t\le 1\\ \infty, & t>1,\end{cases}$$

i.e, $G\in D_p(\widetilde{L}_{2,\beta}(t))$, then from [1] we get

$$\lim_{t\to-\infty}\frac{G(\exp(\tau t))-G(\exp(t))}{[G(\exp(t))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

Hence the result.

Theorem 3.3. A DF $F\in D_e(L_{3,\beta})$ iff

1. $\exists t_0$ like that $F(-e^{-t_0})=0$, and $F(-e^{-t_0}+\varepsilon)>0$, $\forall\varepsilon>0$, and
2. $\forall\tau>0$,

$$\lim_{t\downarrow 0}\frac{F(-\exp(\exp(t_0+t\tau)))-F(-\exp(\exp(t_0+t)))}{[F(-\exp(\exp(t_0+t)))]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

We may take $u_n=e^{-t_0}$ and $v_n=-(\ln(-F^{-}(\frac{k}{n}))+t_0)$.

Proof. Suppose that 1 and 2 are true, we can define the DF $G(y)=F(-e^{-y}), \forall y$. Hence, 2 implies that

$$\lim_{t\downarrow 0}\frac{G(-\exp(t_0+\tau t))-G(-\exp(t_0+t))}{[G(-\exp(t_0+t))]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau, \forall\tau>0.$$

Thus, by using Theorem 3.3 in [1], we have $G\in D_p(\tilde{L}_{3,\beta})$, with power constants $\alpha_n=e^{t_0}$ and $\beta_n=-(\ln(-F^{-}(\frac{k}{n}))+t_0)$. This yields

$$\frac{nG(-\alpha_n t^{\beta_n})-k}{\sqrt{k}}\to\begin{cases}-\infty, & t\le -1,\\ \beta\ln(-\ln|t|), & -1<t\le 0,\\ \infty, & t>0.\end{cases}$$

This implies that

$$\frac{nF(-e^{\alpha_n t^{\beta_n}})-k}{\sqrt{k}}=\frac{nF(-e^{\alpha_n(\ln y)^{\beta_n}})-k}{\sqrt{k}}\to\begin{cases}-\infty, & y\le e^{-1},\\ \beta\ln(-\ln|\ln y|), & e^{-1}<y\le 1,\\ \infty, & y>1,\end{cases}$$

which means that $F\in D_e(L_{3,\beta})$, with u_n and v_n as was indicated. This proves the sufficiency.

Conversely, assume that $F\in D_e(L_{3,\beta})$. Since, $F^{-}(\frac{k}{n})\to\ell_F$, as $n\to\infty$, we should have $\ell_F<0$, otherwise v_n is not real. Also, if $\ell_F=-\infty$, we get $v_n\to-\infty$. This proves the necessity of 1. To prove 2, put $G(y)=F(-e^{-y}), \forall y$. Thus,

$$\frac{nG(\alpha_n(\ln y)^{\beta_n})-k}{\sqrt{k}}=\frac{nF(-e^{-\alpha_n(\ln y)^{\beta_n}})-k}{\sqrt{k}}=\frac{nF(-e^{-\alpha_n|-\ln y|^{\beta_n}\mathscr{S}(-\ln y)}-k}{\sqrt{k}}$$
$$\to\begin{cases}-\infty, & y\le e^{-1},\\ \beta\ln(-\ln|\ln y|) & e^{-1}<y\le 1,\\ \infty, & y>1.\end{cases}$$

Put $\ln y = t$ then we get

$$\frac{nG(\alpha_n(t)^{\beta_n}) - k}{\sqrt{k}} \to \begin{cases} -\infty, & t \le -1, \\ \beta \ln(-\ln|t|) & -1 < t \le 0, \\ \infty, & t > 0. \end{cases}$$

Hence, $G \in D_p(\tilde{L}_{3,\beta})$. Then from [1] we get

$$\lim_{t \downarrow 0} \frac{G(-\exp(t_0 + \tau t)) - G(-\exp(t_0 + t))}{[G(-\exp(t_0 + t))]^{\frac{2-\alpha}{2-2\alpha}}} = \ell^{\frac{-1}{1-\alpha}} \beta \ln \tau, \forall \tau > 0.$$

Thus, the necessity of 2, as well as the theorem is proved.

Theorem 3.4. A DF $F \in D_p(L_{4,\beta})$ iff

1.$\ell_F = -\infty$and
2.$\forall \tau > 0$,

$$\lim_{t \to -\infty} \frac{F(-\exp(\exp(-\tau t)) - F(-\exp(\exp(-t)))}{[F(-\exp(\exp(-t)))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}} \beta \ln \tau.$$

Assuming this, we could set $u_n = 1,\ v_n = \ln(-F^-(\frac{k}{n}))$.

*Proof.*Assume that 1 and 2 are satisfied. Define

$$G(y) = \begin{cases} F(-e^{-y}), & y \le 0, \\ 1, & y > 0. \end{cases} \quad (3.6)$$

By using 2, we get

$$\lim_{t \to -\infty} \frac{G(-\exp(-\tau t)) - G(-\exp(-t))}{[G(-\exp(-t)]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}} \beta \ln \tau,\ \forall \tau > 0.$$

Thus, from Theorem 3.4 in [1], we have $G \in D_p(\tilde{L}_{4,\beta})$, with $\alpha_n = 1$ and $\beta_n = \ln(-G^-(\frac{k}{n}))$. Therefore,

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}(t)) - k}{\sqrt{k}} = \begin{cases} -\beta \ln\ln|t|, & t \le -1, \\ \infty, & t > -1. \end{cases}$$

Then,

$$\frac{nF(-e^{-\alpha_n|t|^{\beta_n}\mathscr{S}(t)}) - k}{\sqrt{k}} = \begin{cases} -\beta \ln\ln|t|, & t \le -1, \\ \infty, & t > -1. \end{cases}$$

Put $t = \ln y$, then we get

$$\frac{nF(-e^{-\alpha_n|\ln y|^{\beta_n}\mathscr{S}(\ln y)}) - k}{\sqrt{k}} = \begin{cases} 0, & y < 0, \\ -\beta \ln\ln|\ln y|, & 0 < y \le e^{-1}, \\ \infty, & y > e^{-1}, \end{cases}$$

which implies $F \in D_e(L_{4,\beta})$, with $u_n = 1$ and $v_n = \ln(-F^-(\frac{k}{n}))$.

Conversely, let $F \in D_e(L_{4,\beta})$, with $u_n = 1$ and $v_n = \ln(-F^-(\frac{k}{n}))$. Now, by using the definition (3.6), we get

$$\frac{nG(\alpha_n|y|^{\beta_n}\mathscr{S}(y)) - k}{\sqrt{k}} = \frac{nF(-e^{-\alpha_n|y|^{\beta_n}\mathscr{S}(y)}) - k}{\sqrt{k}} \to -\beta \ln\ln(-\ln y),\ 0 < y < e^{-1}.$$

By putting $t = \ln y$, then we get

$$\frac{nG(\alpha_n|e^t|^{\beta_n}) - k}{\sqrt{k}} = \to -\beta \ln\ln|t|,\ t < -1.$$

If $t > -1$, $\frac{nG(\alpha_n|e^t|^{\beta_n}) - k}{\sqrt{k}} = \frac{n-k}{k} \to \infty$, which implies that $G \in D_p(\tilde{L_{4,\beta}})$. Then,

$$\lim_{t \to -\infty} \frac{G(-\exp(-\tau t) - G(-\exp(-t))}{[G(-\exp(-t))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}} \beta \ln \tau.$$

Since $F^{-1}(\frac{k}{n}) \to \ell_F$, then from [1], $v_n = \ln(-F^{-1}(\frac{k}{n})) \to \ln(-\ell_F$. Therefore, $\ell_F = -\infty$. This completes the proof.

Theorem 3.5. A DF $F \in D_e(L_{5,\beta})$ iff

1. $\ell_F > 0$ and
2. $\forall \tau > 0$,

$$\lim_{t\to-\infty}\frac{F(-\exp(-\exp(t_0+\tau t))-F(-\exp(-\exp(t_0+\tau t)))}{[F(-\exp(-\exp(t_0+\tau t)))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

We may set in this case $u_n = e^{t_0}$ and $v_n = \ln(F^{-1}(\frac{k}{n})) - t_0$.

Proof. Suppose 1 and 2are satisfied, define

$$G(y)=\begin{cases}0, & y\le t_0,\\ F(-e^{-y}), & y>t_0.\end{cases}$$

Therefore,

$$\lim_{t\to-\infty}\frac{G(\exp(t_0+\tau t))-G(\exp(t_0+t))}{[G(t_0+t)]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau,\ \forall\tau>0.$$

Hence, $G \in D_p(\tilde{L}_{1,\beta})$ with $\alpha_n = e^{t_0}$ and $\beta_n = \ln(F^{-1}(\frac{k}{n})) - t_0$.

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}(t))-k}{\sqrt{k}}=\begin{cases}-\infty, & t\le 1,\\ \beta\ln\ln t, & t>1.\end{cases}$$

Therefore, if $t < 1$,

$$\frac{nF(-e^{-u_n|t|^{v_n}\mathscr{S}(t)})-k}{\sqrt{k}}\to-\infty\,.$$

By putting $t = -\ln|-y|$,

$$\frac{nF(-e^{-u_n|-\ln|-y||^{v}\mathscr{S}(-\ln|-y|))})-k}{\sqrt{k}}\to-\infty\ \ ,\ y<-e^{-1}.$$

If $t > 1$, i.e $y > -e^{-1}$, we get

$$\frac{nF(-e^{-u_n|-\ln(-y)|^{v_n}\mathscr{S}(-\ln(-y))})-k}{\sqrt{k}}\to\beta\ln\ln(-\ln(-y)),\ \ -e^{-1}<y<0.$$

IF $y > 0$,

$$\frac{nF(-e^{-u_n|-\ln|-y||^{v_n}\mathscr{S}(-\ln|-y|))})-k}{\sqrt{k}}\to\infty.$$

Hence, $F \in D_e(L_{5,\beta}(t))$.

Conversely, let $F \in D_e(L_{5,\beta}(t))$, with $u_n = -e^{t_0}$ and $v_n = \ln(F^{-1}(\frac{k}{n})) - t_0$, then

$$\frac{nF(e^{-u_n|-\ln(-y)|^{v_n}\mathscr{S}(-\ln|-y|)})-k}{\sqrt{k}}=\begin{cases}-\infty, & y<-e^{-1},\\ \beta\ln\ln(-\ln(-y)), & -e^{-1}<y<0,\\ \infty & y>0.\end{cases}$$

Put $y = 0$, we get $\frac{nF(0)-k}{\sqrt{k}} \to -\infty$. For large $n, F(0) < \frac{k}{n} \to 0$. Hence, $F(0) = 0$ and $\ell_F > 0$, this proves 1. Define

$$G(y)=\begin{cases}0, & y\le t_0,\\ F(-e^{-y}), & y>t_0.\end{cases}$$

Hence,

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}(t))-k}{\sqrt{k}}=\frac{nF(-e^{-\alpha_n|t|^{\beta_n}\mathscr{S}(t)})-k}{\sqrt{k}}=\frac{nF(-e^{-u_n|-\ln|-y||^{v_n}\mathscr{S}(\ln|-y|)}-k}{\sqrt{k}}.$$

Then,

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}(t))-k}{\sqrt{k}}=\begin{cases}-\infty, & t\le 1,\\ \beta\ln\ln(t), & t>1,\end{cases}$$

i.e., $G \in D_p(\tilde{L}_{1,\beta}(t))$. Hence,

$$\lim_{t\to -\infty} \frac{G(\exp(-\tau t+t_0))-G(\exp(t+t_0))}{[G(t+t_0)]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau,$$

i.e.,

$$\lim_{t\to -\infty} \frac{F(-\exp(-\exp(\tau t+t_0)))-F(-\exp(-\exp(t+t_0)))}{[F(-\exp(-\exp(t+t_0))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

This completes the proof.

Theorem 3.6. A DF $F \in D_e(L_{6,\beta})$ iff

1.$\ell_F \le -e^{-t}$ and

2. $\lim\limits_{t\to -\infty} \frac{F(\exp(\exp(\tau t))-F(\exp(\exp(t)))}{[F(\exp(\exp(t)))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$

Assuming this, we could set $u_n = 1$ and $v_n = -\ln F^{-1}(\frac{k}{n})$.

*Proof.*Assume that 1 and 2 are satisfied. Define,

$$G(y) = \begin{cases} 0, & y \le t_0, \\ F(-e^{-y}), & y > t_0. \end{cases} \tag{3.10}$$

From 2 we get

$$\lim_{t\to -\infty} \frac{G(\exp(\tau t)-G(\exp(t))}{[G(\exp(t))]^{\frac{2-\alpha}{2-2\alpha}}} = -\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

Then from [1], $\exists \alpha_n, \beta_n$, like that

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}t)-k}{\sqrt{k}} \to \begin{cases} -\infty, & t\le 0, \\ -\beta\ln(-\ln t), & 0<t<1, \\ \infty, & t\ge 1, \end{cases}$$

i.e,

$$\frac{nF(-e^{-\alpha_n(t)^{\beta_n}\mathscr{S}t})-k}{\sqrt{k}} \to \begin{cases} -\infty, & t\le 0, \\ -\beta\ln(-\ln t), & 0<t<1 \\ \infty, & t\ge 1. \end{cases}$$

Let $t = -\ln(-y)$, we get

$$\frac{nF(-e^{-\alpha_n(-\ln(-y))^{\beta_n}\mathscr{S}(-\ln(-y))})-k}{\sqrt{k}} \to \begin{cases} -\infty, & y\le -1, \\ -\beta\ln(-\ln t), & -1<y<-e^{-1} \\ \infty, & y\ge -e^{-1}. \end{cases}$$

Hence, $F \in D_e(L_{6,\beta}(t))$. For the necessity part we define $G(y)$ as in (3.10), then $G(y_0) = 0 = F(-e^{-y})$ for $y \le t_0$. Hence 1 is true. To prove 2, write

$$\frac{nF(-e^{-\alpha_n|-\ln(-y)|^{\beta_n}\mathscr{S}(-\ln(-y))})-k}{\sqrt{k}} \to \begin{cases} -\infty, & y\le -1, \\ -\beta\ln(-\ln-\ln(-y)), & -1<y<-e^{-1} \\ \infty, & y\ge -e^{-1}. \end{cases}$$

From (3.10) we get

$$\frac{nG(\alpha_n|-\ln(-y)|^{\beta_n}\mathscr{S}(-\ln(-y)))-k}{\sqrt{k}} \to \begin{cases} -\infty, & y\le -1, \\ -\beta\ln(-\ln-\ln(-y)), & -1<y<-e^{-1} \\ \infty, & y\ge -e^{-1}. \end{cases}$$

Let $t = -\ln(-y)$. Then

$$\frac{nG(\alpha_n|t|^{\beta_n}\mathscr{S}(t))-k}{\sqrt{k}} \to \begin{cases} -\infty, & t\le 0, \\ -\beta\ln(-\ln t), & 0<t<1 \\ \infty, & t\ge 1. \end{cases}$$

Then $G \in D_p(\tilde{L}_{2,\beta}(t))$, i.e.,

$$\lim_{t\to-\infty}\frac{G(\exp(\tau t)-G(\exp(t))}{[G(\exp(t))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

$$\lim_{t\to-\infty}\frac{F(-\exp(-\exp(\tau t))-F(-\exp(-\exp(t)))}{[F(-\exp(-\exp(t)))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

the proof is followed.

The following theorems are followed in a similar way.
Theorem 3.7. A DF $F \in D_e(L_{7,\beta})$ iff

1. $\exists\, t_0$ like that $F(-e^{e^{t_0}})=0$, $F(-e^{e^{t_0}}+\varepsilon)>0$, $\forall\varepsilon>0$, and
2. $\lim_{t\to-\infty}\frac{F(-\exp(-\exp(\tau t+t_0))-F(-\exp(-\exp(t+t_0)))}{[F(-\exp(-\exp(t_0)))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$

Assuming this, we could set $u_n=\exp(-t_0)$ and $v_n=-\ln(-F^{-1}(\frac{k}{n}+t_0)$.
Theorem 3.8. A DF $F \in D_e(L_{8,\beta})$ iff

1. $\ell_F=-\infty$, and
2. $\forall\tau>0$,

$$\lim_{t\to-\infty}\frac{F(\exp(\exp(\tau t)))-F(\exp(\exp(t)))}{[F(\exp(\exp(t)))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\ln\tau.$$

We may set $u_n=1$ and $v_n=-\ln(F^{-}(\frac{k}{n}))$.

Theorem 3.9. A DF $F \in D_e(L_{9,\beta})$ iff

1. $\ell_F\geq 0$, and
2. Define the sequence $\{d_n\}$, as the smallest number for which

$$F(e^{e^{d_n}})\leq\frac{k}{n}\leq F(e^{e^{d_n}}+0),$$

satisfies the condition $\lim_{n\to\infty}\frac{d_n+z_n(\nu)-d_n}{d_n+z_n(\mu)-d_n}=\frac{\nu}{\mu}$, $\forall$ sequences $\{z_n(t)\},t\in R$, satisfying $\frac{z_n(t)}{n^{1-\frac{\alpha}{2}}}\to t$, as $n\to\infty$.

Theorem 3.10. A DF $F \in D_e(L_{10,\beta})$ iff

1. $\ell_F<0$, and
2. Define the sequence $\{d_n\}$, as the smallest number for which

$$F(-e^{e^{-d_n}})\leq\frac{k}{n}\leq F(-e^{e^{-d_n}}+0),$$

satisfies the condition $\lim_{n\to\infty}\frac{d_n+z_n(\nu)-d_n}{d_n+z_n(\mu)-d_n}=\frac{\nu}{\mu}$, $\forall$ sequences $\{z_n(t)\},t\in R$, satisfying $\frac{z_n(t)}{n^{1-\frac{\alpha}{2}}}\to t$, as $n\to\infty$.

Conclusion:The exponential normalization widens the class of the limit laws in OSs compared with both linear and power normalization. In this work, the DAs for the limit laws of intermediate-OSs under exponential normalization were derived. Moreover, a useful criterion for the DAs of the central OSs was revealed.

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