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Quantization of Fractional Constrained Systems with WKB Approximation

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Abstract: In this paper the constrained systems with two primary first class constraints are studied using fractional Lagrangian, after that we find the fractional Hamiltonian and the corresponding Hamilton Jacobi equation. Using separation of variables technique, we can find the action function S this function helps us to formulate the wave function which describe the behavior of our systems also from the action function S we can find the equations of motion and the corresponding momenta in fractional form. This work is illustrated using one example.

Keywords: constrained systems, hamilton jacobi equation, constraints, wave function, lagrangian, action function.

1 Introduction

The quantization of constrained systems has been started by Dirac [1,2], converting first class constraints into second class constraints using gauge constraints has been presented by [3,4,5,6]. Canonical formalism has been presented for studying singular systems [7,8,9,10,11].

The Hamilton Jacobi equation play a good role in quantum mechanics; it says that quantum mechanics reduce to classical mechanics in the limit $\hbar \to 0$. Studying Hamilton Jacobi Equation and calculating the Hamilton Jacobi function to formulate the wave function have been presented by [12,13,14,15]. The quantization of constrained systems has been studied using the WKB approximation [16,17], where the WKB approximation is semiclassical approximation and is a basic technique for obtaining an approximate solution to Schrodinger's equation.

The quantization method has been investigated to explain the dissipative systems by [18] in this field the separation of variables method was used, and the equations of motion are obtained using the given Lagrangian, then the Hamilton Jacobi equation was found to formulate the action integral and the conjugate momentum which help us to find the corresponding wave function for the dissipative system.

Recently, Hamilton Jacobi equation and WKB approximation have been developed for fractional systems using the canonical technique [19,20]. More recently, quantization of damped systems using fractional WKB approximation has been investigated by [21]. In this paper we wish to quantize the constrained systems using WKB approximation but in fractional model.

This paper is organized as follows: In section 2, Hamilton Jacobi formalism and fractional WKB approximation are discussed. In section 3, one illustrative example is studied in detail. The work closes with some concluding remarks in section 4.

2 Fractional Derivatives

The left Riemann–Liouville fractional derivative written as [22, 23]:

$${}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)d\tau \tag{1}$$

which is defined as the LRLFD,

and the right Riemann–Liouville fractional derivative written as:

$${}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)d\tau \tag{2}$$

which is defined as the RRLFD.

where Γ represents the Euler's gamma function and α is the order of the derivative such that $n-1 \le \alpha < n$, and is not equal to zero. . If α is an integer, these derivatives are written as:

$${}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x) \tag{3}$$

and

$${}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(x) \tag{4}$$

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The fractional operator, ${}_aD_x^\alpha f(x)$ can be written as [24].

$$ {}_aD_x^\alpha = \frac{d^n}{dx^n} {}_aD_x^{\alpha-n} \tag{5} $$

Where,

$$ \alpha = 1,2,... $$

Thus, the generalized coordinate q in fractional form is defined as:

$$ q_\alpha = {}_aD_t^{\alpha-1} q $$

and

$$ q_\beta = {}_tD_b^{\beta-1} q $$

Using that

$$ D_t^1 = \frac{d}{dt} \tag{6} $$

$$ D_t^0 = 1 \tag{7} $$

Thus, if $\alpha = \beta = 1$, we find that:

$$ {}_tD_b^\alpha = -\frac{d}{dt} \tag{8} $$

and

$$ {}_aD_t^\alpha = \frac{d}{dt} \tag{9} $$

3 Fractional WKB Approximation Formulation:

Starting from a Lagrangian containing a fractional derivative which takes the following form:

$$ L = L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \tag{10} $$

And recalling that, action function for all $x \in [a,b]$ can be defined as follows:

$$ S = \int_a^b L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t)dt \tag{11} $$

Where the generalized momenta can be obtained from:

$$ p_\alpha = \frac{\partial L}{\partial {}_aD_t^\alpha q} \tag{12} $$

and

$$ p_\beta = \frac{\partial L}{\partial {}_tD_b^\beta q} \tag{13} $$

The Hamiltonian depending on the fractional time derivatives is written as:

$$ H_0({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, p_\alpha, p_\beta, t) = $$
$$ p_\alpha\, {}_aD_t^\alpha q + p_\beta\, {}_tD_b^\beta q - L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \tag{14} $$

The Hamilton Jacobi equation is given as:

$$ H' = p_0 + H_0 \tag{15} $$

And by using

$$ p_0 = \frac{\partial S}{\partial t} \tag{16} $$

Where **S** is the Hamilton Jacobi function which can be written in fractional form as follows:

$$ S = S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t) \tag{17} $$

Now by using equation (14) and equation (16) the Hamilton Jacobi equation (15) will be:

$$ H' = \frac{\partial S}{\partial t} + H_0 = 0 \tag{18} $$

Thus, the solution of equation (18) takes the following form:

$$ S = S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t) = f(t) + W_1(E_1, {}_aD_t^{\alpha-1}q) $$
$$ + W_2(E_2, {}_tD_b^{\beta-1}q) + A \tag{19} $$

Where E_1, E_2 are constants of integration, and A is some other constant.

Also, equations of motion can be obtained from the Hamilton Jacobi function as follows:

$$ \eta_1 = \frac{\partial S}{\partial E_1} = {}_aD_t^{\alpha-1} Q $$

(20)

and

$$ \eta_2 = \frac{\partial S}{\partial E_2} = {}_tD_b^{\beta-1} Q \tag{21} $$

Now, the generalized momenta can be determined from the Hamilton Jacobi function as follows:

$$p_\alpha = \frac{\partial S}{\partial {}_aD_t^{\alpha-1}q} = \frac{\partial W}{\partial {}_aD_t^{\alpha-1}q} \quad (22)$$

and

$$p_\beta = \frac{\partial S}{\partial {}_tD_b^{\beta-1}q} = \frac{\partial W}{\partial {}_tD_b^{\beta-1}q} \quad (23)$$

Using Fractional WKB approximation and by using the relation between the wave function $\psi({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t)$ and the Hamilton's principle function $S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t)$, we obtain this formula:

$$\psi({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t) = \prod_{i=1}^{N} \psi_{0i}(q_i)$$
$$\exp\left[\frac{i}{\hbar} S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t)\right] \quad (24)$$

Where,

$$\psi_{0i}(q_i) = \frac{1}{\sqrt{p_i}} \quad (25)$$

The wave function satisfies the condition:

$$\hat{H}'\psi = 0 \quad (26)$$

Thus, we construct the fractional wave function as:

$$\psi({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t) =$$
$$\frac{1}{\sqrt{p_i}} \exp\left[\frac{i}{\hbar} S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t)\right] \quad (27)$$

The momenta are defined as operators in this form:

$$\hat{p}_\alpha = \frac{\hbar}{i} \frac{\partial}{\partial {}_aD_t^{\alpha-1}q} \quad (28)$$

Then,

$$\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial {}_tD_b^{\beta-1}q} \quad (29)$$

also

$$\hat{p}_0 = \frac{\hbar}{i} \frac{\partial}{\partial t} \quad (30)$$

4 Example:

To illustrate our work two primary first class constraints example is studied in detail [13]:

$$L = \frac{1}{2}(\dot{q}_1{}^2 + \dot{q}_3{}^2) + \dot{q}_1\dot{q}_3 + q_2\dot{q}_2 - q_1 - q_3 \quad (31)$$

Our Lagrangian in fractional form using equation (10) is written as:

$$L = \frac{1}{2}({}_0D_t^\alpha q_1)^2 + \frac{1}{2}({}_0D_t^\alpha q_3)^2 + ({}_0D_t^\alpha q_1)({}_0D_t^\alpha q_3)$$
$$+({}_0D_t^{\alpha-1} q_2)({}_0D_t^\alpha q_2) - ({}_0D_t^{\alpha-1} q_1) - ({}_0D_t^{\alpha-1} q_3) \quad (32)$$

The momentum using equation (12) is:

$$p_1 = \frac{\partial L}{\partial_0 D_t^\alpha q_1} = {}_0D_t^\alpha q_1 + {}_0D_t^\alpha q_3 \quad (33)$$

And

$$p_2 = \frac{\partial L}{\partial_0 D_t^\alpha q_2} = {}_0D_t^{\alpha-1} q_2 = -H_2 \quad (34)$$

Also,

$$p_3 = \frac{\partial L}{\partial_0 D_t^\alpha q_3} = {}_0D_t^\alpha q_3 + {}_0D_t^\alpha q_1 = p_1 = -H_3 \quad (35)$$

The canonical Hamiltonian has this form using equation (14)

$$H_0({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, p_\alpha, p_\beta, t) =$$
$$p_1\,{}_0D_t^\alpha q_1 + p_2\,{}_0D_t^\alpha q_2 + p_3\,{}_0D_t^\alpha q_3$$
$$- L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \quad (36)$$

Equation (36) can readily be solved to give:

$$H_0 = \frac{1}{2}(p_1)^2 + ({}_0D_t^{\alpha-1}q_1) + ({}_0D_t^{\alpha-1}q_3) \quad (37)$$

The corresponding Hamilton Jacobi partial differential equations are:

$$H_0' = p_0 + H_0 = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial_0 D_t^{\alpha-1} q_1}\right)^2$$
$$+({}_0D_t^{\alpha-1}q_1) + ({}_0D_t^{\alpha-1}q_3) = 0 \quad (38)$$

$$H_2' = p_2 + H_2 = \left(\frac{\partial S}{\partial_0 D_t^{\alpha-1} q_2}\right) - ({}_0D_t^{\alpha-1}q_2) = 0 \quad (39)$$

$$H_3' = p_3 + H_3 = \left(\frac{\partial S}{\partial {}_0D_t^{\alpha-1} q_3}\right) - \left(\frac{\partial S}{\partial {}_0D_t^{\alpha-1} q_1}\right) = 0 \quad (40)$$

We shall now use a change of variables to solve equation (38), (39) and equation (40) which is

$${}_0D_t^{\alpha-1} u = ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3) \quad (41)$$

Then equation (37) becomes

$$H_0 = \frac{1}{2}(p_u)^2 + {}_0D_t^{\alpha-1} u \quad (42)$$

Thus, the new Hamilton Jacobi partial differential equations read

$$H_0' = p_0 + \frac{1}{2}(p_u)^2 + {}_0D_t^{\alpha-1} u = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial {}_0D_t^{\alpha-1} u}\right)^2 + {}_0D_t^{\alpha-1} u = 0 \quad (43)$$

$$H_2' = p_2 + H_2 = p_2 - q_2 = \left(\frac{\partial S}{\partial {}_0D_t^{\alpha-1} q_2}\right) - ({}_0D_t^{\alpha-1} q_2) = 0 \quad (44)$$

It is possible to propose that:

$$S({}_0D_t^{\alpha-1} u, {}_0D_t^{\alpha-1} q_2, t) = f(t) + W({}_0D_t^{\alpha-1} u, E) + f_2({}_0D_t^{\alpha-1} q_2) + A \quad (45)$$

That is,

$$f(t) = -Et \quad (46)$$

Remembering that $f(t) = -Et$ and ${}_0D_t^{\alpha-1} q_2$ being treated as independent variables, using equation (22), (43), (45) and equation (46), W function can be written as

$$W = \int \sqrt{2(E - {}_0D_t^{\alpha-1} u)}\, d\, {}_0D_t^{\alpha-1} u \quad (47)$$

And the function f_2 equals to:

$$f_2 = \int ({}_0D_t^{\alpha-1} q_2)\, d\, {}_0D_t^{\alpha-1} q_2 \quad (48)$$

Putting equation (46), (47) and equation (48) into equation (45) we get:

$$S = -Et + \int \sqrt{2(E - {}_0D_t^{\alpha-1} u)}\, d\, {}_0D_t^{\alpha-1} u + \int ({}_0D_t^{\alpha-1} q_2)\, d\, {}_0D_t^{\alpha-1} q_2 + A \quad (49)$$

The equation of motion is:

$$\eta_1 = \frac{\partial S}{\partial E} = -t + \int \frac{d\, {}_0D_t^{\alpha-1} u}{\sqrt{2(E - {}_0D_t^{\alpha-1} u)}} \quad (50)$$

Using equation (22) and equation (23), it follows that

$$p_1 = \frac{\partial S}{\partial {}_0D_t^{\alpha-1} u} = \sqrt{2(E - {}_0D_t^{\alpha-1} u)} = p_3 = p_u \quad (51)$$

And

$$p_2 = \frac{\partial S}{\partial {}_0D_t^{\alpha-1} q_2} = {}_0D_t^{\alpha-1} q_2 \quad (52)$$

The quantization procedure in Schrodinger's assumes the form

$$\hat{H}_0'\psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial ({}_0D_t^{\alpha-1} u)^2} + {}_0D_t^{\alpha-1} u\right]\psi \quad (53)$$

And

$$\hat{H}_2'\psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial ({}_0D_t^{\alpha-1} q_2)} - {}_0D_t^{\alpha-1} q_2\right]\psi \quad (54)$$

Recalling that;

$$\psi({}_0D_t^{\alpha-1} u, q_2) = \psi_0({}_0D_t^{\alpha-1} u) \exp\left[\frac{i}{\hbar} S({}_0D_t^{\alpha-1} u, {}_0D_t^{\alpha-1} q_2, t)\right] \quad (55)$$

And

$$\psi_0({}_0D_t^{\alpha-1} u) = \left[2(E - {}_0D_t^{\alpha-1} u)\right]^{-1/4} \quad (56)$$

Using equation (28), (29) and equation (30) we have

$$\frac{\partial}{\partial t}\psi = -\frac{i}{\hbar} E\psi \quad (57)$$

$$\frac{\partial^2 \psi}{\partial ({}_0D_t^{\alpha-1} u)^2} = \frac{-1}{\hbar^2}\left[2(E - {}_0D_t^{\alpha-1} u)\right]\psi + \frac{5}{4}\left[2(E - {}_0D_t^{\alpha-1} u)\right]^{-2}\psi \quad (58)$$

$$\frac{\partial}{\partial_0 D_t^{\alpha-1} q_2}\psi = \frac{i}{\hbar}({}_0D_t^{\alpha-1} q_2)\psi \tag{59}$$

Substituting equation (57), (58) into equation (53) we get

$$\hat{H}_0'\psi = \begin{bmatrix} -E + \frac{1}{2}\left[2(E - {}_0D_t^{\alpha-1}u)\right] - \\ \frac{5\hbar^2}{8}\left[2(E - {}_0D_t^{\alpha-1}u)\right]^{-2} + {}_0D_t^{\alpha-1}u \end{bmatrix}\psi \tag{60}$$

Now we can show that in the limit $\hbar \to 0$,

$$H_0'\psi = 0 \tag{61}$$

Also, substituting equation (59) into equation (54) we obtain

$$\hat{H}_2'\psi = \left[{}_0D_t^{\alpha-1} q_2 - {}_0D_t^{\alpha-1} q_2\right]\psi = 0 \tag{62}$$

Which means;

$$H_2'\psi = 0 \tag{63}$$

5 Conclusion:

In this work constrained systems using fractional calculus have been studied using fractional Lagrangian, the Hamiltonian is obtained in fractional form using the given fractional Lagrangian. From the resulting Hamiltonian we find the Hamilton Jacobi equation which help us to find the corresponding action function S. Finally, from the action function one can obtain the conjugate momenta, equations of motion and the wave function in fractional form at a certain condition; which is $\alpha \to 1$; the results of fractional technique reduce to those obtained from classical technique. In order to test our proposed method and to get a somewhat deeper understanding, we have examined an example with two primary first class constraints.

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