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Optimal Control Analysis of the Dynamics of COVID-19 with Application to Ethiopian Data

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Abstract: In this paper, we proposed an optimal control of the COVID-19 transmission dynamics. First, we investigated system features such as solution boundedness, positivity, disease-free and endemic equilibrium, and the local and global stability of equilibrium points. Besides, a disease-free equilibrium point is globally asymptotically stable if the basic reproduction number is less than one, and an endemic equilibrium point exists otherwise. Secondly, we have shown the sensitivity analysis of the basic reproduction number. Also the model is then fitted using COVID-19 infected reported in Ethiopia from February 1,2023 to March 2,2023. The values of model parameters are then estimated from the data reported using the least square method together with the the MATLAB software. Moreover, the optimal corruption minimization strategies are determined using three controls strategies, namely prevention, vaccination and treatment. The existence of the optimal controls and characterization is established using Pontryagin's Maximum Principle. Finally, based on analysis of optimality system, the combination of the prevention and treatment of infected is the most optimal and least cost strategy to minimize the burden of the disease.

Keywords: COVID-19 model; Optimal control analysis; Cost effective strategy; Parameter estimation; Numerical simulation

1 Introduction

The COVID-19 is a new corona virus which leads to corona virus infection. It was found as a result of a respiratory disease epidemic in Wuhan City, Hubei, China [1]. The report was originally submitted to the WHO on December 31, 2019. On January 30, 2020, the WHO declared the epidemic a worldwide health hazard [2]. According to the most recent research, COVID-19 spreads between people by close contact with infected people, dirty surfaces, or direct touch [3]. These are discharged from an infected person's mouth or nose when they talk, cough, or sing [4]. The virus also spread to numerous African countries. Up to August 26, 2020, confirmed cases in Africa totaled 22,313, with 1,124 fatalities and 5,492 recoveries. Also the first case report of COVID-19 in Ethiopia was on March 13,2020, then on April 16,2020, the Ethiopian Public Health Institute reported a total of 92 confirmed cases, 3 deaths and 2 recovers [5]. Now up to January 19,2023 confirmed cases are 499,254 and 7,571 are deaths [6].

Several researchers investigated the transmission dynamics of COVID-19 using a mathematical model. For instance Gurmu et al. [7] formulate mathematical model of COVID-19 dynamics. The authors results suggest that reducing the contact rate between the infected individual and the susceptible individual is the best one to combat COVID-19 infection. Bugalia et al. [8] proposed a COVID-19 dynamics using mathematical model in Indian. Their studies show that if there is a partial or no closure instance, the endemic level will be quite high and India might face more than six million illnesses if precautions are not followed. The study [9] presented a the SEIAHR model for transmission dynamics of COVID-19. The authors had not taken into account vaccinated groups in their model. Abriham et al. [10] proposed SEQIHR compartmental model. The authors concluded that increasing isolation and quarantine rates will control COVID-19 disease. The author [11] proposed mathematical model for COVID-19 dynamics. The model has two equilibrium points: the disease free and the endemic equilibrium point. The result of stability analysis of the DFE equilibrium is locally asymptotically stable whenever $R_0 < 1$ and endemic equilibrium point is

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globally asymptotically whenever $R_0 > 1$. The authors conclude that it is better to minimise contact between the exposed individual and the susceptible individual. The study [12] proposed mathematical modelling and analysis of a SEIQR model to study the dynamics of COVID-19. The study is conclude this demonstrates that asymptomatic cases caused by an exposed population play an important role in increasing COVID-19 infection among the population. The author [13] proposed a model based on is an evaluation that is used to explain the dynamics of corona virus infection with early interventions. The concluded that the problem can be solved by either providing more space, equipment, and personnel or reducing the burden on health care organisations to accept more corona virus and other patients.

A number of scholars studied about optimal control of COVID-19 dynamics with cost effectiveness analysis. For instances Molla et al. [14] formulated and studied a new optimal control dynamical model for the 2019 coron avirus disease. The numerical illustrations of the various control strategies revealed that the third strategy, which encompasses all four time-dependent control functions, produces the most efficient results. Kouidere et al.[15] proposed an optimal control of mathematical modelling of COVID-19 pandemic propagate, with a concentrate on the adverse effect of quarantine on diabetics. The authors implemented three controls, which are sensitization and prevention, quarantine, and a diabetic awareness program. Duressa et.al.[16] analyzed an extension of the SEIR model to asymptomatic and hospitalized classes. The authors believe that the combined effects of health care education, personal protective measures, and hospitalisation treatment contribute to slowing the spread of the disease. Nana-Kyere et al. [17] presented SEIRW COVID-19 compartmental model with optimal control. The authors, investigated a mathematical model of COVID-19 dynamics with three-time dependent control measures. They conclude that a combination of personal protection, drug treatment, and disinfectant spraying should be considered in order to drastically decrease the number of exposed and infected people in the population. The study [18] proposed the current global issue of coron avirus pandemic containment as an optimal control problem. The researchers concluded that these control and prevention efforts should be kept perpetually and gradually relaxed as new cases of the disease become rare and the overall incidence declines. Asamoah et al.[19] proposed the COVID-19 model was used to investigate how to control the spread of the corona virus in Saudi Arabia by employing four control indicators: personal hygiene, adequate security precautions, following proper protocol, and fumigating schools. The author concludes that, in the absence of vaccination, implementing physical or social distance protocols is the most efficient and affordable control intervention. Lemecha Obsu & Feyissa Balcha [20] proposed three control measures to study the possibility of transmission of COVID-19 with the best possible control the burden of the disease. The authors conclude that medical care and intensive prevention are effective control measures for reducing the number of exposed and infected populations.

To the best of our knowledge, many of them did not mentioned optimal control strategies with cost effectiveness analysis. In this study, the corona virus transmission model [21] is extended to the optimal control problem using three control measure: prevention, vaccination and treatment with cost effective strategy. Moreover, we used real data in Ethiopia to estimate some parameter's values for numerical simulation.

The remaining portion of the paper is organised as follows. **In section 2**, we formulate a mathematical model for COVID-19. **In section 3**, explain the analysis of of the model. **Section 4**, represents the sensitivity analysis of the parameters used in the model. **In section 5**, we express an optimal control model for COVID-19 with three controls measures. **In section 6**, we present a simulation of the model to confirm the analytical results. **In section 7**, discus cost-effectiveness analysis. **In section 8**, we conclude by discussing the control measures and results.

2 Model Formulation and It's Description

In this section, we considered the COVID-19 transmission model with SEIQR model represent of the human population. The total human population at time (t), denoted by $N(t)$, is divided into five disjoint compartments. Susceptible persons (S) are those who are not affected with the disease but may be infectious. Exposed individuals (E) are those who are in the incubation phase of the disease and do not exhibit any clinical symptoms. Infected individuals (I) are those who got a sickness symptom. Regarding the diagnosed individual, we anticipate that infected individuals will be immediately transported to an appointed quarantine for isolation and treatment, and that this population group will be converted to quarantine Q. Also, individuals who have recovered from the disease via treatment or natural recovery, are referred to as recovered individuals (R). As a result, total human population is given as $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. Then the recruitment rate π of susceptible human occurs through either the flow of people or birth. The susceptible individuals contracted corona virus disease through β contact with exposed or infected individuals, which spread to the exposed compartment. Individuals who have been exposed become infectious and join the infected compartment in proportion α. The recovery rate without being quarantine by the contact rate ε and σ is the quarantine rate from the infected. The parameter γ is recovery rate of the quarantine and die due to the corona virus disease with a rate of δ. Those who have temporary

immunity and have recovered from the disease. Then the recovered individuals become again susceptible to the disease with a rate of η. Natural death rate of all populations denoted by μ. All parameters in the model are positive. Figure (1) show that the flow diagram of proposed model and the parameters are described as below.

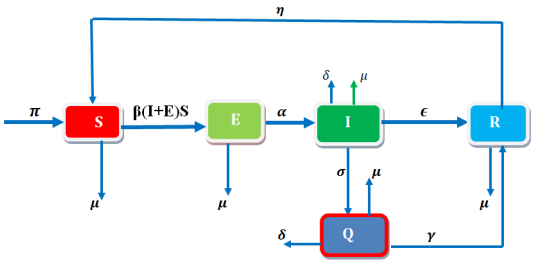


Fig. 1: Flow diagram of the model of COVID-19

The flow chart in Figure (1), the govern nonlinear ordinary differential equation of the model is expressed as follows:

$$\begin{cases} \frac{dS}{dt} = \pi - \beta(I+E)S + \eta R - \mu S, \\ \frac{dE}{dt} = \beta(I+E)S - (\alpha+\mu)E, \\ \frac{dI}{dt} = \alpha E - (\sigma+\varepsilon+\mu+\delta)I, \\ \frac{dQ}{dt} = \sigma I - (\gamma+\delta+\mu)Q, \\ \frac{dR}{dt} = \varepsilon I + \gamma Q - (\eta+\mu)R, \end{cases} \tag{1}$$

with the initial condition

$$S(0) \geq 0, \quad E(0) \geq 0, \quad I(0) \geq 0, \quad Q(0) \geq 0, \quad R(0) \geq 0.$$

3 Model Analysis

3.1 Invariant Region

In this section, we considered in which the solutions of model (1) is bounded. Then we take into account the total human population by $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. After that differentiate $N(t)$ with respect to time, we get:

$$\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dQ}{dt} + \frac{dR}{dt} = \pi - \delta Q - \delta I - \mu N. \tag{2}$$

Then it implies that Eq. (2) becomes,

$$\frac{dN}{dt} \leq \pi - \mu N. \tag{3}$$

Table 1: Parameters description used for the model

Parameters	Parameters Description
π	Rate of recruitment into vulnerable population
μ	Natural mortality rate of human
β	Individual contact rates
α	Rate of exposed become infected
ε	Rate of recovery without quarantine
σ	Proposition of infected individuals become quarantine
γ	Rate of quarantine patients who recovers
η	The proportion of recovered individuals who are vulnerable
δ	Corona virus-related deaths are on the rise

By solving the Eq.(3), the total population $N \to \frac{\pi}{\mu}$ as time tends to infinity. This shows as $N \leq \frac{\pi}{\mu}$ and the invariant region of system (1) of the human population is positive invariant is given by

$$\Omega = \left\{ (S,E,I,Q,R) \in R_+^5 : 0 < S+E+I+Q+R \leq \frac{\pi}{\mu} \right\}. \tag{4}$$

3.2 Positivity of the Solutions

In this subsection, we will show that all solutions of system with positive initial data will remain positive for all time $t \geq 0$.

Theorem 3.1. If $S(0), E(0), I(0), Q(0)$ and $R(0)$ are non-negative, then the solution $S(t), E(t), I(t), Q(t)$ and $R(t)$ of Eq. (1) all are non-negative for $t \geq 0$.

Proof. Take from the first equations from model (1) is given as

$$\frac{dS}{dt} = \pi - \beta(I+E)S + \eta R - \mu S. \tag{5}$$

Then Eq. (5) becomes,

$$\frac{dS}{dt} \geq -\beta[(I+E)+\mu]S. \tag{6}$$

Integrating Eq. (6) with respect to time and using the method of separation variable, we obtain:

$$S(t) \geq S(0)e^{-\beta[(I+E)+\mu]t} \geq 0. \tag{7}$$

It is possible to show using the same procedure for other state variables that:

$$E(t) \geq E(0)e^{-(\alpha+\mu)t} \geq 0 \ , I(t) \geq I(0)e^{-(\gamma+\varepsilon+\mu+\delta)t} \geq 0 \ ,$$
$$Q(t) \geq Q(t)e^{-(\gamma+\delta+\mu)t} \geq 0 \ , R(t) \geq R(0)e^{-(\eta+\mu)t} \geq 0 \ . \tag{8}$$

This shows that all solutions of Eq. (1) are positive for all t≥0. Therefore, the proposed COVID-19 transmission model stated in Eq. (1) is both ecologically significant and numerically well posed in an attainable region Ω.

3.3 Disease Free Equilibrium(DFE)

The disease free equilibrium points Eq. (1) is where there is no COVID-19 infection in the community. It was computed by equating all Eq. (1) to zero and denoted by E_0 as given as

$$\mathrm{E}_0 = \left(\frac{\pi}{\mu}, 0, 0, 0, 0\right). \tag{9}$$

3.4 Basic reproduction number

The basic reproduction number is denoted by $\Re_0$ and is referred to as the predicted amount of people getting another infection among the entire population at risk [22, 23]. Here, we used the next generation matrix method. Then, we start with newly infective classes and rewrite the model equations as:

$$\begin{aligned}
\frac{dE}{dt} &= \beta(I+E)S - (\alpha+\mu)E,\\
\frac{dI}{dt} &= \alpha E - (\sigma+\varepsilon+\mu+\delta)I,\\
\frac{dQ}{dt} &= \sigma I - (\gamma+\delta+\mu)Q\,.
\end{aligned} \tag{10}$$

Then, the right hand side of Eq. (10) can be written in the form $f-v$, where

$$f = \begin{pmatrix} \beta(I+E)S \\ \alpha E \\ \sigma I \end{pmatrix} \text{ and } \quad v = \begin{pmatrix} (\alpha+\mu)E \\ (\sigma+\varepsilon+\mu+\delta)I \\ (\gamma+\delta+\mu)Q \end{pmatrix} \tag{11}$$

Computing the partial derivatives of f and v at the DFE gives $\mathbf{F}$ and $\mathbf{V}$ respectively, where

$$F = \begin{pmatrix} \frac{\beta\pi}{\mu} & \frac{\beta\pi}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$V = \begin{pmatrix} (\alpha+\mu) & 0 & 0 \\ -\alpha & (\sigma+\varepsilon+\delta+\mu) & 0 \\ 0 & -\sigma & (\gamma+\delta+\mu) \end{pmatrix}. \tag{12}$$

Then the product of fv^{-1} is obtain as:

$$fv^{-1} = \begin{pmatrix} \frac{\beta\pi(\sigma+\varepsilon+\mu+\delta)+\beta\pi}{\mu(\alpha+\mu)(\sigma+\varepsilon+\mu+\delta)} & \frac{\beta\pi}{\mu(\sigma+\varepsilon+\mu+\delta)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

Therefore, the basic reproduction number is given us the largest (dominant) of the eigenvalue of Eq. (13) as

$$R_0 = \frac{\beta\pi(\sigma+\varepsilon+\mu+\delta)+\beta\pi\alpha}{\mu(\alpha+\mu)(\sigma+\varepsilon+\mu+\delta)}. \tag{14}$$

3.5 Local Stability of Disease Free Equilibrium

Theorem 3.2. The DFE of the model (1) is locally asymptotically stable if $R_0 < 1$.

Proof. The Jacobian matrix of model (1) at disease free equilibrium is given by

$$J(\mathrm{E}_0) = \begin{pmatrix} -\mu & -\frac{\beta\pi}{\mu} & -\frac{\beta\pi}{\mu} & 0 & \eta \\ 0 & \frac{\beta\pi}{\mu}-\alpha-\mu & \frac{\beta\pi}{\mu} & 0 & 0 \\ 0 & \alpha & -\sigma-\varepsilon-\mu-\delta & 0 & 0 \\ 0 & 0 & \sigma & -\gamma-\delta-\mu & 0 \\ 0 & 0 & \varepsilon & \gamma & -\eta-\mu \end{pmatrix}. \tag{15}$$

From the Eq. (15), we have the last characteristic equation as;

$$(-\mu-\lambda)(-\gamma-\delta-\mu-\lambda)(-\eta-\mu-\lambda)(\lambda^2+A\lambda+B)=0, \tag{16}$$

Where,

$$\begin{aligned}
A &= \frac{\beta\pi}{\mu}+2\mu+\alpha+\sigma+\varepsilon,\\
B &= -\frac{\beta\pi}{\mu}(\sigma+\varepsilon+\mu+\delta)-\frac{\beta\pi\alpha}{\mu}+(\alpha+\mu)(\sigma+\varepsilon+\mu+\delta).
\end{aligned} \tag{17}$$

From the of Eq. (16) we will get that

$$\lambda_1 = -\mu < 0, \quad \lambda_2 = -(\eta+\mu) < 0, \quad \lambda_3 = -(\gamma+\eta+\mu) < 0 \tag{18}$$

and again, from the last characteristic Eq. (16) we get,

$$\lambda^2 + A\lambda + B = 0. \tag{19}$$

By using the Routh-Hurwitz criteria [24] from the Eq. (19) has a real root that is negative if A $>$ 0 and $B > 0$ and $AB > 0$. Hence, we can observe that $A > 0$, since it is the sum of non-negative parameters and the value of B is given as

$$B = (\alpha+\mu)(\sigma+\varepsilon+\mu)(1-R_0).$$

However, when B is non-negative $1-R_0$ could be positive, which implies that $R_0 < 1$. Therefore, the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$.

3.6 Global stability of Disease free Equilibrium

Theorem 3.3. The DFE of the model (1) is globally asymptotically stable if $R_0 < 1$.

Proof. Let us prove the globally stability of the equilibrium point we construct the Lyapunov function as

$$V = (\sigma+\varepsilon+\mu)E+(\alpha+\mu)I. \tag{20}$$

Then differentiating Eq. (20) with respect to t it gives

$$\frac{dV}{dt} = (\sigma+\varepsilon+\mu)\frac{dE}{dt}+(\alpha+\mu)\frac{dI}{dt}. \tag{21}$$

Then depending above the Eq. (21) we substituting $\frac{dE}{dt}$ and $\frac{dI}{dt}$ from the model (1), we get:

$$\begin{aligned}\frac{dV}{dt} = (\sigma+\varepsilon+\mu)(\beta[I+E]S-(\alpha+\mu)E)+\\ (\alpha+\mu)(\alpha E-(\sigma+\varepsilon+\mu)I.\end{aligned} \tag{22}$$

When we solving Eq. (22) to obtain

$$\frac{dV}{dt} = (\beta)(\sigma+\varepsilon+\mu)S-(\alpha+\mu)(\sigma+\varepsilon+\mu)(E+I). \tag{23}$$

Then we calculate in simplifying form of the Eq. (22) with Eq. (23) to obtain

$$\frac{dV}{dt} = (\alpha+\mu)(\sigma+\varepsilon+\mu)[R_0-1](E+I). \tag{24}$$

Hence, from the Eq. (24), we seen that the $\frac{dV}{dt} = 0$ if $E = 0 = I$ and R_0 this means that the highest invariant set in $\{(S,E,I,Q,R) \in R_+^5\}$ is the singleton DFE, $\mathbb{E}_0$ and by Lasalle's invariance principle [25,26,27,28], $\mathbb{E}_0$ is globally asymptotically stable in R_+^5.

3.7 The Endemic Equilibrium point

An endemic equilibrium point shows that the disease is going to keep impacting the population in a stable state. It is obtained by putting the left side in (1) equal to zero. Let they are represented by (S^*,E^*,I^*,Q^*,R^*) at the endemic's steady state, respectively as:

$$\begin{cases} S^* = \frac{\mu(\alpha+\mu)(\sigma+\varepsilon+\mu)}{\beta[\alpha+(\sigma+\varepsilon+\mu)]}, \\ E^* = \frac{\mu(\alpha+\mu)(\eta+\mu)(\sigma+\varepsilon+\mu)^2(\gamma+\delta+\mu)}{\beta k}(R_0-1), \\ I^* = \frac{\alpha\mu(\alpha+\mu)(\eta+\mu)(\sigma+\varepsilon+\mu)(\gamma+\delta+\mu)}{\beta k}(R_0-1)), \\ Q^* = \frac{\alpha\mu\sigma(\alpha+\mu)(\eta+\mu)(\sigma+\varepsilon+\mu)}{\beta k}(R_0-1), \\ R^* = \frac{\alpha\mu(\alpha+\mu)(\eta+\mu)(\sigma+\varepsilon+\mu)[\varepsilon(\gamma+\delta+\mu)+\gamma\sigma]}{\beta k}(R_0-1). \end{cases} \tag{25}$$

4 Sensitivity Analysis

In this section, we investigated the impact of the essential parameters on the dynamics of COVID-19 using sensitivity analysis. The we performed sensitivity analysis using the definition of the standardized sensitivity index provided in [29,30]. The normalized forward sensitivity index of a variable, R_0, that depends differentiable on a parameter, Q, is defined as:

$$\Pi_Q^{R_0} = \frac{\partial R_0}{\partial Q}\times\frac{Q}{R_0} \tag{26}$$

For example, the forward sensitivity indices of R_0 with respect to parameter β.

$$\Pi_\beta^{R_0} = \frac{\partial R_0}{\partial \beta}\times\frac{\beta}{R_0} = 1.$$

Also the sensitivity indices at different values for parameters are provided **Table 2**.

Table 2: Sensitivity indices of parameters

Parameters symbol	Sensitivity index
π	1
α	- 0.5790827
β	1
ε	-0.0000246
σ	- 0.063369
μ	-0.00387685
δ	-0.000485
γ	-0.022

4.1 Interpretation of the sensitivity indices

The basic reproduction number sensitivity indices have been explained $\Re_0$ in relation to the parameters in Table (2). The results showed that the parameters with a positive sensitivity index raised more as their values risen, while the remaining parameters remained constant. Furthermore, if the values of the parameters with negative measurements are increased while the amounts of the other parameters remain constant, the value of $\Re_0$ decreases. If the values of the basic parameters with negative sensitivity indices lower while the other the parameters remain constant, the COVID-19 disease rises.

4.2 Parameter Estimations

In studies, we fit the our model using the real data from COVID-19 infection situations in Ethiopia and predict the unknown parameters for the model. The daily real data of total confirmed cases of COVID-19 infection in Ethiopia from December 20, 2022 to January 19, 2023 extracted from Ethiopian Minister of Health situation reports in Table (4). Now, we formulated the system in the

following form to solve the dynamic parameter estimation problem [31]:

$$\frac{dx}{dt} = f(t,x,u), \quad x(t_0) = x_0, \tag{27}$$

where t denotes the number of independent variables, the system's state vector is denoted by x, x_0 is the system's starting condition, and u is the vector of unidentified parameters. The error is represented by the sum of squares error, which has the form

$$\theta(u) = \sum_{i=1}^{n} (x_i - \bar{x}_i)^2 \tag{28}$$

where x_i is the real data and $\bar{x}_i$ is the solution of ordinary differential Eq. (27) for the given u. Our main goal is to determine the least square estimators u in order to minimize the errors Eq. (28) under the constraint Eq. (27) to gain the best estimated parameters using least square method.

Table 3: Daily confirmed case COVID -19 infectious from February 1, 2023 to March 2, 2023

No	Month	Total I(t) cases	No	Month	Total I(t) cases
1	1 February	17	16	16 February	10
2	2 February	32	17	17 February	16
3	3 February	31	18	18 February	10
4	4 February	20	19	19 February	6
5	5 February	13	20	20 February	7
6	6 February	16	21	21 February	18
7	7 February	17	22	22 February	13
8	8 February	25	23	23 February	0
9	9 February	24	24	24 February	35
10	10 February	0	25	25 February	11
11	11 February	11	26	26 February	0
12	12 February	16	27	27 February	15
13	13 February	17	28	28 February	12
14	14 February	12	29	1 March	1
15	15 February	29	30	2 March	2

That will see in Figure (2) the model parameters of system (1) are estimated using least-square fitting method, which results in a better fit for the model solution to the real data. The corresponding parameter values are shown in Table (4).

When we estimated the basic reproduction number is given by the expression $R_0 = 2.57571 > 1$. The previous of COVID-19 will cause an epidemic because $R_0 > 1$.

Next, we will see an optimal control of COVID-19 dyanmics to identify an optimal strategy that is most optimal to minimize the spread of COVID-19 transmission.

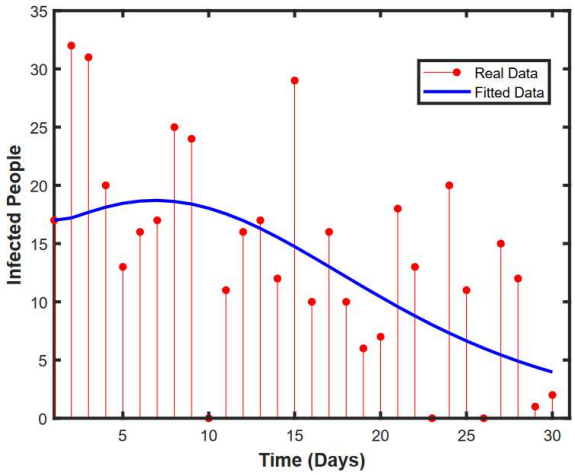


Fig. 2: SEIQR model fit with real data on the number of COVID-19 cases in Ethiopia

Table 4: Parameter descriptions and taken values for model (1)

Parameters	Parameter descriptions	Values	References
π	Covid-19 incremental contact rate	4995	Estimated
α	Rate exposed human become infected	0.0022	Fitted
β	Incremental contact rate of human	0.00012	Estimated
γ	Recover rate of hospitalized patients	0.8693	Estimated
ε	Recover rate without being hospitalized	0.0012	Fitted
σ	Infected individual become hospitalized	0.3237	Estimated
μ	Human natural death rate	0.7640	Estimated
η	Recovered individuals to be susceptible	0.1890	Estimated
δ	Death rate due to corona virus	0.00000116	[32]

5 Optimal control of COVID-19 dynamics

In this section, we extended the mathematical model dynamic COVID-19 model Eq. (1) to an optimal control problem. Using this method, we hope to identify the best illness avoidance strategy, the COVID-19 model in Eq. (1), after incorporating the controls into the model, the state equations obtained is:

$$\begin{cases} \frac{dS}{dt} = \pi - (1-u_1)\beta(I+E)S + \eta R - (\mu+u_2)S, \\ \frac{dE}{dt} = (1-u_1)\beta(I+E)S - (\alpha+\mu)E, \\ \frac{dI}{dt} = \alpha E - (\sigma+\varepsilon+\mu+\delta+u_3)I, \\ \frac{dQ}{dt} = \sigma I - (\gamma+\delta+\mu)Q, \\ \frac{dR}{dt} = (\varepsilon+u_3)I + \gamma Q - (\eta+\mu)R. \end{cases} \tag{29}$$

As the first control (u_1) represents prevention that help to lower contact rate, (u_2) denote the vaccination and (u_3) refers to the treatment of COVID-19 patients in order to reduce disease severity. The objective functional is given by [33]:

$$\begin{aligned} &\mathrm{J}(u_1,u_2,u_3) = \\ &\underbrace{\min}_{u_1,u_2,u_3} \int_0^{t_f} \left(\mathrm{A}_1 E + \mathrm{A}_2 I + \frac{1}{2}(\mathrm{B}_1 u_1^2 + \mathrm{B}_2 u_2^2 + \mathrm{B}_3 u_3^2) \right) dt \\ &\longrightarrow min \end{aligned} \tag{30}$$

Subject to Eq. (1) where A_1, A_2, B_1, B_2 and B_3 are positive. The weight constants B_1, B_2 and B_3 are the measure of relative costs of interventions associated with the controls u_1, u_2 and u_3, respectively, and also balances the units of integrand. Our goal is to reduce the number of exposed, infected, and quarantined individuals responsible for the novel virus in the population at the lowest possible cost. The main goal is to find a triple optimal control u_1, u_2, u_3 such that

$$J(u_1^*, u_2^*, u_3^*) = min J(u_1, u_2, u_3) : u_1, u_2, u_3 \in U \tag{31}$$

were U=(u_1, u_2, u_3): $u_i(t)$ such that u_1, u_2 and u_3 are lebesgue measurable on t $\in$ [0,t_f] with 0$< u_i(t) \leq 1$ as the control set.

5.1 Existence of the optimal controls

In this section, the limitation of system solutions for a finite time interval is used to show the existence of optimal controls.

Theorem 5.1. Given $J(u_1, u_2, u_3)$ subject to state system Eq. (29), then there exist optimal controls $u^* = (u_1^*, u_2^*, u_3^*)$ and corresponding to the optimal solution $(S^*, E^*, I^*, Q^* R^*)$ such that

$$J(u_1^*, u_2^*, u_3^*) = min\{J(u_1, u_2, u_3)\} : u_1, u_2, u_3 \in U \tag{32}$$

Proof: The proof of the existence of an optimal controls was similar with the prove in [34].

5.2 Hamiltonian System

By using the Pontryagin's minimum principle [35,36] the optimal control problem, which is characterised by the Hamiltonian (H) function, consists of Eq. (29) is represented as,

$$\begin{aligned} H = A_1 E + A_2 I + \frac{1}{2}\sum_{i=1}^{3} B_i u_i^2 + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dE}{dt} + \\ \lambda_3 \frac{dI}{dt} + \lambda_4 \frac{dQ}{dt} + \lambda_5 \frac{dR}{dt}. \end{aligned} \tag{33}$$

Substituting Eq. (29) and (30), into Eq. (33), with respect to controls u_1, u_2, u_3 is given by,

$$\begin{aligned} H = &[A_1 E + A_2 I + \frac{1}{2}(B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2)] \\ &+ \lambda_1(\pi - (1-u_1)\beta(I+E)S + \eta R - (\mu+u_2)S) \\ &+ \lambda_2((1-u_1)\beta(I+E)S - (\alpha+\mu)E) \\ &+ \lambda_3((\alpha E - (\sigma+\varepsilon+\mu+\delta+u_3)I) \\ &+ \lambda_4((\sigma I - (\gamma+\delta+\mu)Q) \\ &+ \lambda_5((\varepsilon+u_3)I + \gamma Q - (\eta+\mu)R) \end{aligned} \tag{34}$$

where λ_1, λ_2, λ_3, λ_4 and λ_5 are adjoint variables.

Theorem 5.2. For an optimal controls triples such as u_1^*, u_2^*, u_3^* and a solution of S^*, E^*, I^*, Q^* and R^* minimizers of the corresponding state system $\mathrm{J}(u_1, u_2, u_3)$ over U subject to Eq. (1) and adjoint variables λ_1, λ_2, λ_3, λ_4 and λ_5 are found, then the adjoint system are given

$$\begin{cases} \frac{d\lambda_1}{dt} = (1-u_1)\beta(I+E)(\lambda_1-\lambda_2) + \lambda_1(\mu+u_2), \\ \frac{d\lambda_2}{dt} = (1-u_1)\beta S(\lambda_1-\lambda_2) + \lambda_2(\alpha+\mu) - \lambda_3\alpha - A_1, \\ \frac{d\lambda_3}{dt} = (1-u_1)\beta S(\lambda_1-\lambda_2) + \lambda_3(\sigma+\varepsilon+u_3) - \\ \qquad \lambda_4\sigma - \lambda_5(\varepsilon+u_3) - A_2, \\ \frac{d\lambda_4}{dt} = \lambda_4(\gamma+\delta+\mu) - \lambda_5\gamma, \\ \frac{d\lambda_5}{dt} = -\lambda_1\eta + \lambda_5(\eta+\mu), \end{cases} \tag{35}$$

with transversality conditios, $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = 0$. Furthermore, the optimal controls u_1^*, u_2^*, u_3^* are represented by,

$$\begin{aligned} u_1^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_2-\lambda_1)(\beta(I+E)S}{\mathrm{B}_1}\right\}\right\}, \\ u_2^* &= \max\left\{0, \min\left\{1, \frac{\lambda_1 S}{\mathrm{B}_2}\right\}\right\}, \\ u_3^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_3-\lambda_5)I}{\mathrm{B}_3}\right\}\right\}. \end{aligned} \tag{36}$$

Proof: In the form of the co-state problems, determine the derivative of the hamiltonian function (H) of Eq. (35) with respect to S, E, I, Q and R respectively then adjoint or co-state equations were then achieved [37,38].

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\partial \mathscr{H}}{\partial S} = ((1-u_1)\beta(I+E)(\lambda_1-\lambda_2))+ \\ \lambda_1(\mu+u_2), \\ \frac{d\lambda_2}{dt} = -\frac{\partial \mathscr{H}}{\partial E} = \\ (1-u_1)\beta S(\lambda_1-\lambda_2)+\lambda_2(\alpha+\mu)-\lambda_3\alpha-A_1, \\ \frac{d\lambda_3}{dt} = -\frac{\partial \mathscr{H}}{\partial I} = (1-u_1)\beta S(\lambda_1-\lambda_2))+ \\ \lambda_3(\sigma+\varepsilon+u_3)-\lambda_4\sigma-\lambda_5(\varepsilon+u_3)-A_2, \\ \frac{d\lambda_4}{dt} = -\frac{\partial \mathscr{Q}}{\partial H} = \lambda_4(\gamma+\delta+\mu)-\lambda_5\gamma, \\ \frac{d\lambda_5}{dt} = -\frac{\partial \mathscr{H}}{\partial R} = -\lambda_1\eta+\lambda_5(\eta+\mu), \end{cases} \tag{37}$$

with transversality conditios $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = 0$.
Then to compute the control values, we used the partial derivatives of the Hamiltonian with respect to controls, given by $\frac{\partial H}{\partial ui} = 0$, for $i = 1,2,3$ as follows:

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= 0 = B_1u_1 - (\lambda_1+\lambda_2)\beta(I+S), \\ \frac{\partial H}{\partial u_2} &= 0 = B_2u_2 - \lambda_1 S, \\ \frac{\partial H}{\partial u_3} &= 0 = B_3u_3 - \lambda_3 I + \lambda_5 I. \end{aligned} \tag{38}$$

Solving for optimal control from the Eq. (38) we will get

$$u_1^* = \frac{(\lambda_2-\lambda_1)\beta(I+E)S}{B_1}, u_2^* = \frac{\lambda_1 S}{B_2}, u_3^* = \frac{(\lambda_3-\lambda_5)I}{B_3}. \tag{39}$$

Hence, rearranging the solution of Eq. (39) with the boundary condition of each control, we get,

$$\begin{aligned} u_1^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_2-\lambda_1)(\beta(I+E)S}{\mathrm{B}_1}\right\}\right\}, \\ u_2^* &= \max\left\{0, \min\left\{1, \frac{\lambda_1 S}{\mathrm{B}_2}\right\}\right\}, \\ u_3^* &= \max\left\{0, \min\left\{1, \frac{(\lambda_3-\lambda_5)I}{\mathrm{B}_3}\right\}\right\}. \end{aligned} \tag{40}$$

Next, we will see the simulation of the optimality system to identify an optimal strategy that is most optimal to minimize the spread of COVID-19 transmission.

6 Numerical Simulation

In this section, we solved an optimality system with one system to find the best strategy. The state equations yield five ordinary differential systems, and the adjoint equations yield five. Using forward-backward Runge Kutta to solve the state and adjoint systems. In solving state equations Eq. (29) the forward fourth order Runge-Kutta was used due to the initial value of the state variables. Because of the transversality condition, we used the backward fourth order Runge-Kutta method to solve the adjoint equations holding the state equations solution and optimal controls values. For numerical simulation of the optimality system, the initial condition that we used was: $S(0) = 121279189$, $E(0) = 20, I(0) = 17, Q(0) = 16$, $R(0) = 10$ then the parameter Table (4) were used. We used the following weight constant values for the state and controls: $A_1 = 80, A_2 = 70, B_1 = 60, B_2 = 80$, and $B_3 = 60$.

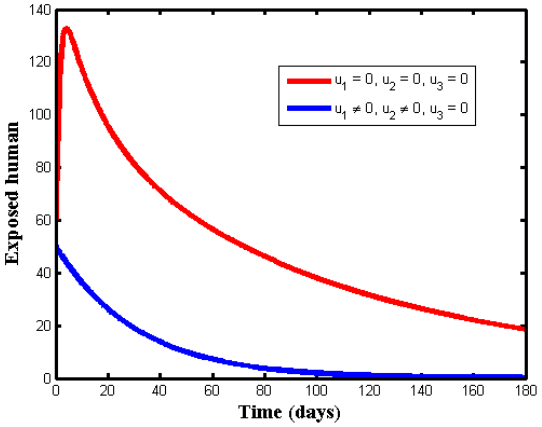
Besides, we proposed four strategies with varying combinations of more than two controls at a time to demonstrate the impact of each control on corona virus reduction.

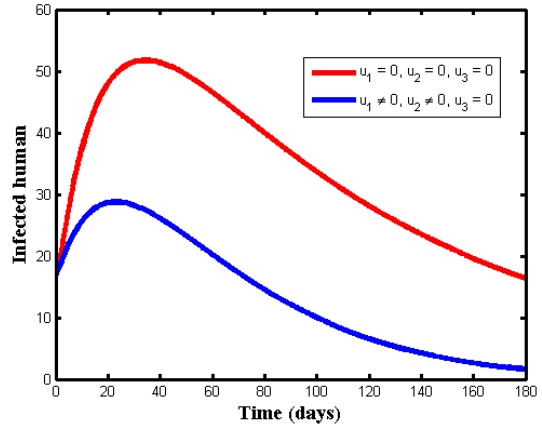
Strategy A: Combination of prevention (u_1) and vaccination (u_2)

This strategy to optimise the objective function Eq. (30) with personal protective control (u_1) and vaccination (u_2) whereas treatment of COVID-19 patients control (u_3) set to zero. The numerical simulation result shown in Figure 3(a) when we used controls, the total population of exposed humans decreased, but increased when no controls were used. Figure 3(b) infected humans use fewer controls and increase if there is no control disease, and they tend to have the lowest value at the end of the intervention. Figure 3(c) indicates that the use of the personal protective u_1 remained between zero and the highest, then gradually decreases to its lower value at the end of time whereas u_2 kept its highest bound 100% after 20 days of time of strategy.

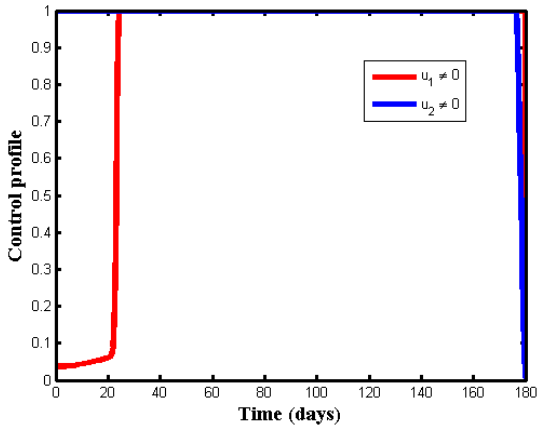
Strategy B: Combination of prevention (u_1) and treatment (u_3)

In this strategy, to minimize the objective functional Eq. (30), the combinations of two controls personal protective (u_1) and treatment of the infected (u_3) are implemented, while the control vaccination u_2 is zero. From the numerical result in Figure 5(a) the overall number of COVID-19-exposed humans increases rapidly in the absence of control and reduces in the presence of control. In Figure 4(b) if there is a control, the total number of corona virus infected humans decreases. The number of COVID-19-infected populations is increasing in the absence of controls. In Figure 4(c) the profile of the control implies that use of the personal protective u_1 is





(**a**) (**b**)

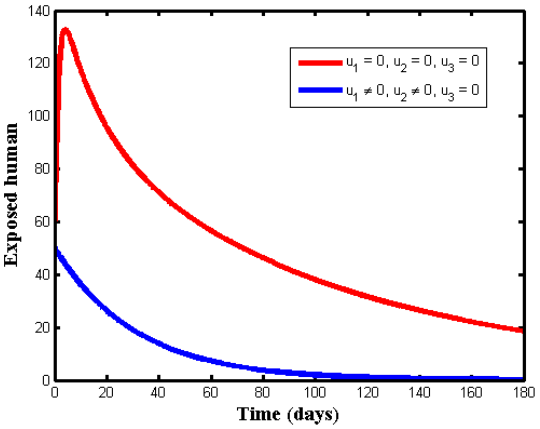


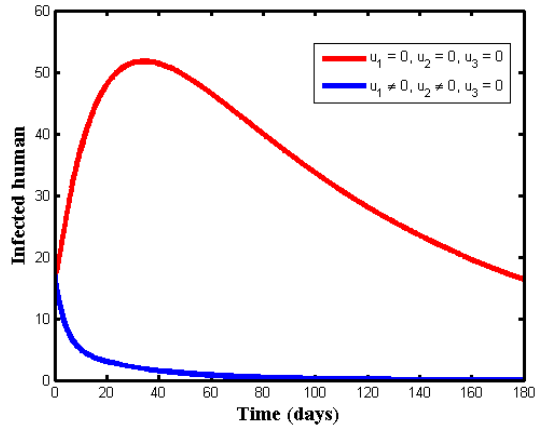
(**c**)

Fig. 3: Figure shows numerical simulations with prevention (u_1) and vaccination (u_2).

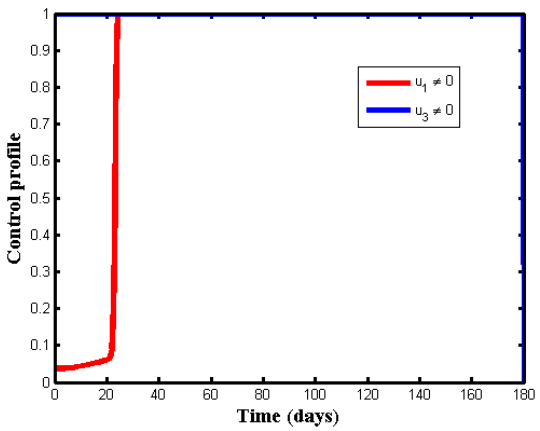
retained its high level and reduces gradually to their lower bound, while treatment use preserved its upper bound 100% for the entire duration of the strategy.

Strategy C: Combination of vaccination (u_2) and treatment (u_3).





(**a**) (**b**)



(**c**)

Fig. 4: Figure shows simulations with prevention (u_1) and treatment (u_3)

In this study, we used vaccination (u_2) and treatment of the COVID-19 patients control (u_3) to reduce total exposed and infected populations and save money. In Figure 5(**a**) we observed that using this control strategy, the number of exposed humans becomes smaller if there is use of controls than without controls and declines to its lowest value. In Figure 5(**b**) when used as a control, the total number of infected humans decreases to its lowest value and in the absence of control, the number of infected people appears to be growing. Also, the control profiles (u_2) and (u_3) are shown in Figure 5(**c**) were kept their maximum level for 178 days and is gradually increases.

Strategy D: Applying all control strategies

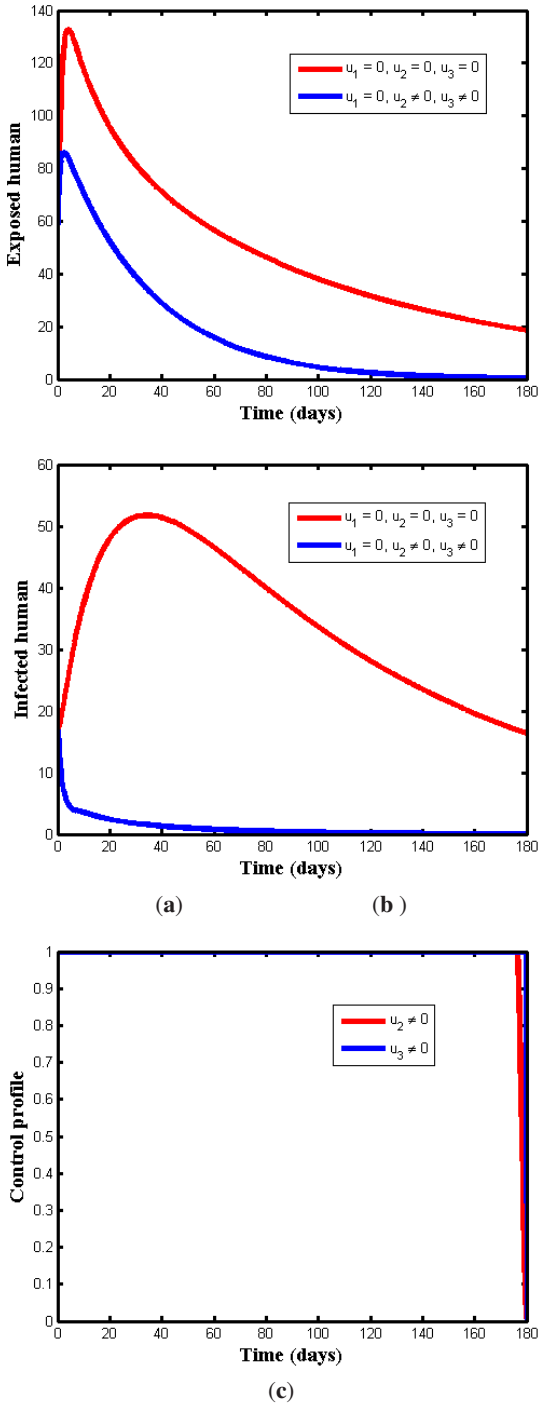
In this strategy, to reduce the objective function eq. (30), we applied the three control interventions on each compartment. In Figures 6(**a**) and 6(**b**) the total population exposed and infected population are increased in the absence of controls while there are decreased in the presence of controls. At the end of 180 days, the total number of exposed and infected humans has decreased to its lowest value. Also, the control profile depicted in figure 6(**c**) u_1 show that the control profile implies that is maintained its upper bound for (90%) for 160 days in the entire time of strategy. The control profile u_2 and u_3 are at their highest level(100%) in every days and the control u_1 is increasing for 20 days, then keep maximum level (100%) in 178 days.

7 Cost-Effectiveness Analysis

Cost-effectiveness analysis is used to identify the most optimal and least cost-effective measure for control for the combined and all execution of the three given control measures in order to effectively reduce the transmission of COVID-19. In this work, we used incremental cost effectiveness ratio [39,40]. The formula for the ICER is given by

$$\frac{\text{Difference in averted costs between two strategies}}{\text{Difference in the total number of infections averted}}.$$

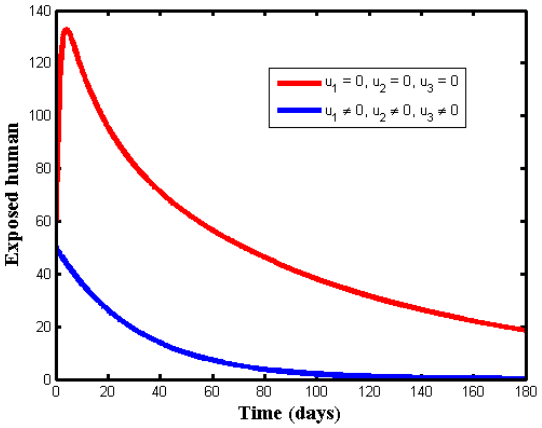
Besides, a numerical simulation of the optimal control problem was used to calculate the total cost avoided and total infections saved, and the control strategy was ordered in increasing order based on the total infections saved, as shown in Table (5). The total getting sick saved can be calculated by subtracting the total number of humans infected with COVID-19 with control from the total number of humans infected with COVID-19 without control, whereas the cost avoided by each strategy was calculated using the cost function indicated by $\frac{1}{2}B_1u_1^2, \frac{1}{2}B_2u_2^2$ and $\frac{1}{2}B_3u_3^2$ over the time [39,40,41]. The total infection saved and total cost of all strategies with their ICER are given in Table (5).

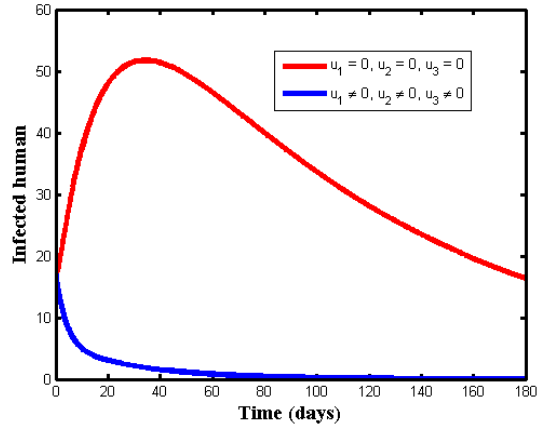


(**a**) (**b**) (**c**)

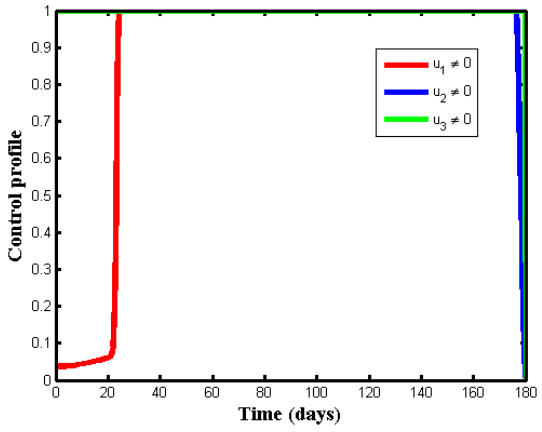
Fig. 5: Figure shows simulations with vaccination (u_2) and treatment (u_3)

The cost-effectiveness ratio (ICER) is calculated from the total number of people saved and the total cost of used in each of the strategies listed in the Table (5), which





(**a**) (**b**)



(**c**)

Fig. 6: Simulations with prevention (u_1), vaccination (u_2) and treatment (u_3)

compares the differences between two strategies obtained and provided by

Table 5: Amount of the total infections saved and total cost averted for all strategies

Strategy	Total infections averted	Total cost($)	ICER
C	14450	210	0.0145
A	11200	240	-0.00923
B	14210	260	0.0066
D	14480	320	0.222

$$\text{ICER}(\mathbf{C}) = \frac{210}{14450} = 0.0145$$
$$\text{ICER}(\mathbf{A}) = \frac{240-210}{11200-14450} = -0.00923$$
$$\text{ICER}(\mathbf{B}) = \frac{260-240}{14210-11200} = 0.0066$$
$$\text{ICER}(\mathbf{D}) = \frac{320-260}{14480-14210} = 0.222$$

With the above result the number of infections saved with ICER for four different strategies as shown in the Table (6).

Table 6: Amount of the total infections saved and total cost averted for all strategies

Strategy	Total infections averted	Total cost($)	ICER
C	14450	210	0.0145
A	11200	240	-0.00923
B	14210	260	0.0066
D	14480	320	0.222

In the above Table (6) comparing the interventions **C** and **A**. ICER(**A**) is less than ICER(**C**) as shown in the Table (6). It implies that the strategy **C** is costly and has a low an opportunity of saving lives. As a result, **A** more people are saved than **C**. Then, the competing strategies have been eliminated **C**. Then, as shown in Table (7), determining the ICER for remaining strategies **A**, **B**, and **D**.

Table 7: Amount of the total infection averted and total cost used with ICER

Strategies	Amount of infections saved	Total cost ($)	ICER
A	11200	240	-0.00923
B	14210	260	0.0066
D	14480	320	0.222

The ICER(**B**) is more than the ICER(**A**) from the list of competing intervention strategies in Table (7). It illustrates the fact that ICER(**A**) strategy outperforms ICER(**B**). As an out come, ICER(**B**) is less efficient and more costly than ICER(**A**) as a result, that was cleared the strategy **B** from the list of competing. The ICER was re-calculated strategies as shown in Table (8).

Table 8: Amount of the total infection averted and total cost used with ICER

Strategies	Amount of infections saved	Total cost ($)	ICER
A	11200	240	0.021
D	14480	320	0.222

From the above Table (8) with intervention strategies **A** and **D** shows that ICER(**A**) is less than ICER(**D**). This suggests that strategy **A** performs better strategy **D**. As a result, the **A** the most optimal strategy has the lowest total cost. Based on our findings, the combination of prevention and treatment of infected humans is the best optimal and least expensive strategy for limiting the dynamics of the disease.

8 Conclusion

In this study, we have propose and presented an optimal control analysis for the transmission model for the 2019 corona virus disease. To begin we shown that all solutions of the model are positive and bounded with initial conditions. Then by applying the next matrix generation, we have found as basic reproduction number of the system, which helps us to determine the dynamical behavior of the system. Besides, we fit the proposed model using real data of COVID-19 infection cases in Ethiopia February 1, 2023 to march 2, 2023 and we estimate the unknown model parameters. Moreover, we have extended the model to an optimal control model of corruption dynamics using three control strategies namely personal protection, vaccination and treatment to examine the dynamics of COVID-19 deadly infectious disease. In this case, the optimal control strategies are realized by minimizing the number of exposed and infected people while accounting for implementation costs. The existence of optimal controls and characterization is established with the help of Pontryagin's Maximum Principle. Also, the cost-effectiveness analysis is described. Lastly, the best optimal and least cost strategy for limiting disease spread is the combination of prevention and treatment infected humans.

Data Availability

All data were included in the manuscript.

Conflict of Interest

The authors have no particular conflicts of interest on the manuscript.

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