Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/130126

A New Log Lindley Distribution with Applications

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Received: 16 Sep. 2022, Revised: 2 Mar. 2023, Accepted: 10 Jul. 2023.
Published online: 1 Jan. 2024.

Abstract: This paper introduces a new generalization of the Lindley distribution introduced by [1], using the basic idea of [2] and along the lines of [3]. The new distribution is a compound of the Lindley and logarithmic distributions. We refer to the new model as the logarithmic-Lindley (Log-L) distribution. This model is capable of modeling various shapes of aging and failure criteria. The properties of the Log-L model are discussed, and the maximum likelihood estimation method is used to evaluate the parameters involved. Finally, the usefulness of the new model for modeling reliability data is illustrated using a two real data sets with simulation study.

Keywords: Lindley distribution; logarithmic distribution; maximum likelihood estimation; order statistics; Rényi entropy.

1 Introduction

Lifetime distribution represents an attempt to describe, mathematically, the length of the life of systems or devices. Lifetime distributions are most frequently used in many fields as medicine, engineering ...etc. Many parametric models such as exponential, gamma and Weibull have been frequently used in statistical literature to analyze lifetime data. But there is no clear motivation for the gamma and Weibull distributions. They only have more general mathematical closed forms than the exponential distribution with one additional parameter.

Recently, the one parameter Lindley distribution has attracted the researchers for its use in modeling lifetime data. It has been observed in several papers that this distribution has performed excellently. The Lindley distribution was originally proposed by [1] in the context of Bayesian statistics, as a counter example of fiducial statistics. One can glean it as a mixture of exponential(θ) and gamma(2, θ).

Some of the advances in the literature of Lindley distribution are given by [4] who has introduced a two-parameter weighted Lindley distribution and has pointed that Lindley distribution is particularly useful in modeling biological data from mortality studies. Mahmoudi et al. [5] proposed generalized Poisson-Lindley distribution. Bakouch et al. [6] come up with extended Lindley distribution, [7] introduced a two-parameter Lindley distribution. [8] proposed a new two parameter lifetime distribution. Hassan [9] introduced a convolution of Lindley distribution. Elbatal et al. [10] proposed a new generalized Lindley distribution. Afify and Alizadeh [11] proposed the odd Dagum Lindley distribution. Al-Babtain et al. [12] introduced Weibull Marshall-Olkin power-Lindley distribution. Furthermore, Al-Babtain et al. [13] proposed the discrete version of the continuous Lindley called natural discrete Lindley (NDL) as a mixture of geometric and negative binomial distributions. Almazah et al. [14] addressed the reliability properties of the NDL distribution. Hosseini et al. [15] studied the weighted-Lindley distribution.

Definition 1 A random variable X is said to have the Lindley distribution with parameter (θ) if its probability density function (pdf) is defined as

$$g\,(x) = \frac{\theta^2}{\theta+1}(1+\,x)e^{-\theta x},\;\; x>0,\theta>0 \tag{1}$$

while the corresponding survival, or reliability, function is given by

$$\overline{G}\,(x) = \frac{\theta+1+\theta x}{\theta}e^{-\theta x},\quad x>0. \tag{2}$$

The hazard rate function (hrf) is

$$r(x) = \frac{\theta^2(1+x)}{\theta+1+\theta x}\,,\quad x>0. \tag{3}$$

In the context of reliability and survival analysis, [3] proposed a transformation of a distribution $G\,(x;\theta)$ that introduces a

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new parameter $\alpha > 0$. This transformation is defined through the cumulative distribution function (cdf).

$$f(x;\theta,\alpha) = \frac{G(x;\theta)}{G(x;\theta) + \alpha\overline{G}(x;\theta)}. \tag{4}$$

The interpretation of the parameter α is given in [3] in terms of the behavior of the ratio of hazard rates of G and F. This ratio is increasing in x for $\alpha \geq 1$ and decreasing in x for $0 < \alpha < 1$. This transformation is then proposed for the exponential and Weibull distributions in [3] in order to generate more flexible models for lifetime data. Clearly, for $\alpha = 1$, G and F coincide.

A lot of papers had been published by using Marshall-Olkin (M-O) transformation given in (4). Alice and Jose [16] introduced M-O extended semi-Pareto model and studied its geometric extreme stability. Semi-Weibull distribution and generalized Weibull distributions are considered by [17]. Ristic et al. [18] introduced and studied the M-O gamma distribution. Ghitany et al. [19] proposed the M-O extended Lomax distribution. The M-O beta distribution as an extension of the basic distribution with four parameters was presented by [20]. Gomez-Deniz [21] presented a new generalization of the geometric distribution using the M-O scheme. Garcia et al. [22] defined a generalized normal distribution by applying this transformation to a normal distribution G. Afify et al. [23] proposed the M-O additive Weibull distribution. Nassar et al. [24] proposed the M-O alpha power family. The M-O power generalized Weibull and M-O odd Burr III-G are introduced by [25] and [26], respectively.

Pappas et al. [2] introduced a new generalization which is derived along the lines of [3]. Accordingly, starting with a survival function (sf) $\overline{G}(x)$, then the usual device of adding a new parameter results in another sf $\overline{F}(x)$ defined by

$$\overline{F}(x) = \frac{\ln[\,1-(1-p)\overline{G}(x)]}{\ln(p)}, \quad x \in \mathbb{R}, p > 0, \tag{5}$$

and when $p \to 1$, the distribution reduces to the base distribution $\overline{G}(x)$. If $f(x)$ and $h(x)$ are the pdf and hrf corresponding to $\overline{F}(x)$, then

$$f(x) = \frac{(p-1)\,g(x)}{[\,1-(1-p)\overline{G}(x)]\ln(p)}, \qquad x \in \mathbb{R}, p > 0 \tag{6}$$

and

$$h(x) = \frac{(p-1)\,\overline{G}(x)\,r(x)}{[\,1-(1-p)\overline{G}(x)]\,\ln[1-(1-p)\overline{G}(x)]}, \tag{7}$$

where $h(x)$ is the hrf corresponding to $f(x)$. It is worth mentioning that [27] followed this idea to provide the Lomax-logarithmic distribution. The aim of this paper is to introduce a new generalization of Lindley distribution [1]. This generalization is called logarithmic-Lindley (Log-L) distribution and it is flexible enough to model different types of lifetime data having different forms of failure rate. The new model can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates.

The rest of this paper will cover the following topics adequately: Section 2 introduces the pdf and the sf of the Log-L distribution, then gives an interpretation of the new model. We investigate the reliability analysis of the new model via Section 3 which includes the hrf with its shapes, the cumulative hrf and the mean residual lifetime. Section 4 presents the statistical properties of the Log-L distribution. The Log-L parameters are estimated via the maximum likelihood estimation method in Section 5. Section 6 presents a simulation study. Section 7 provides two applications illustrating the performance of the new proposed model that are applied on different real data sets. Finally, Section 8 presents some conclusions.

2 A Lindley Extension Model

In the following, Lindley distribution is extended by adding a new shape parameter, $p > 0$, using Equations (5) through (7). Now, substituting (2) into (5) and doing the necessary simplifications gives the sf of the Log-L distribution as

$$\overline{F}(x) = \frac{\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]}{\ln(p)}, \qquad x > 0, \tag{8}$$

where $\theta > 0$ is a scale parameter and $p > 0$ is a shape parameter. Then the pdf corresponding to (8) is readily found to be

$$f(x)=\frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\left[\frac{(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}}\right], \qquad x>0,\theta,p>0. \tag{9}$$

Note that the Log-L distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions. The Lindley distribution is clearly a special case when p → 1.

Interpretation: For p ∈(0,1), the pdf given by (9) can be obtained as a compound of the logarithmic and the Lindley distributions. According to [28] and [29], suppose that X_1,X_2, ...,X_y are Y iid (independent and identically distributed) lifetime random variables in a series system each with pdf (1), and let Y be a random variable distributed according to the logarithimc distribution with probability mass function (pmf) defined as

$$p(Y=y)=\frac{-(1-p)^y}{ylnp},\ \ y\in\mathbb{N},p\in(0,1).$$

Now, the conditional distribution function of $(X|Y)$ is given by

$$f(x|y)=yg(x)\left[\overline{G}(x)\right]^{y-1}=\frac{y\theta^2(1+x)}{1+\theta+\theta^2}\left(\frac{1+\theta+\theta x}{1+\theta}\,e^{-\theta x}\right)^y,$$

where $g(x)$ and $\overline{G}(x)$ are the pdf and the sf corresponding to Lindley distribution and given by (1) and (2), respectively.

Then, the joint distribution of the random variables X and Y, denoted by $f\ (x,y)$, is obtained as

$$f(x,y)=f(x|y).p(Y=p)=\frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\left[(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}\right]^y.$$

Hence, it can be found the marginal pdf of x as follows

$$f(x)=\sum_{y=1}^{\infty}f(x,y)=\frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\sum_{y=1}^{\theta}\left[(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}\right]^y$$

$$=\frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\left[\frac{(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}}{1-(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}}\right]^y$$

$$=\frac{\theta^2(p-1)}{(1+\theta)\ln(p)}\left[\frac{(1+x)e^{-}\theta x}{1-(1-p)\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}}\right],$$

which is the pdf of the Log-L distribution given by (1).

Figure 1 illustrates some of the possible shapes of the pdf of the Log-L distribution for different values of the parameters θ and p chosen from the ranges specified in Equation (9).

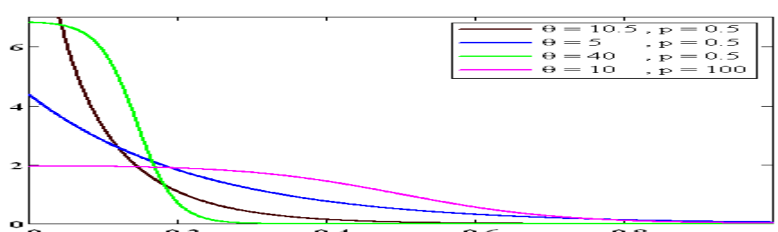


FIG. 1: The density function plots of the Log-L distribution.

3 Reliability Analysis

In this section, we present the hrf with its different shapes, the cumulative hrf (chrf) and the mean residual lifetime for the Log-L distribution.

Let X be the lifetime of a device (or a component in a system). Suppose a component follow that X has a pdf as in (9). One of the most important characteristics of X is its hrf, $h(x)$, which is defined by

$$h(x) = \lim_{\Delta x \to 0} \frac{Pr(x < X < x + \Delta x | X > x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x . R(x)} = \frac{f(x)}{R(x)},$$

which provides information about a small interval after time $x(x + \Delta x)$. Using the previous definition or by substituting (2) and (3) into (7), the hrf of a random variable $X \sim \text{Log} - \text{L}(\theta, p)$ is given by

$$h(x) = \frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]}. \quad (10)$$

By taking the limits of (10) when $x \to 0$ and when $x \to \infty$, we have

$$\lim_{x \to 0} h(x) = \frac{p-1}{p\ln(p)} \times \frac{\theta^2}{\theta+1} = \frac{p-1}{p\ln(p)} \lim_{x \to 0} r(x),$$

and

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} r(x),$$

it follows from (10) that

$$\frac{p-1}{p\ln(p)} r(x) \le h(x) \le r(x); x > 0, p \ge 1,$$

and

$$r(x) \le h(x) \le \frac{p-1}{p\ln(p)} r(x); x > 0, p \in (0,1).$$

Hence, using the ratio $\frac{h(x)}{r(x)}$, $x > 0$, it can be shown that $\frac{h(x)}{r(x)}$ is increasing for $p \ge 1$ and decreasing for $p \in (0,1)$. Figure 2 illustrates the behavior of the hrf of the Log-L distribution at different values of the parameters involved.

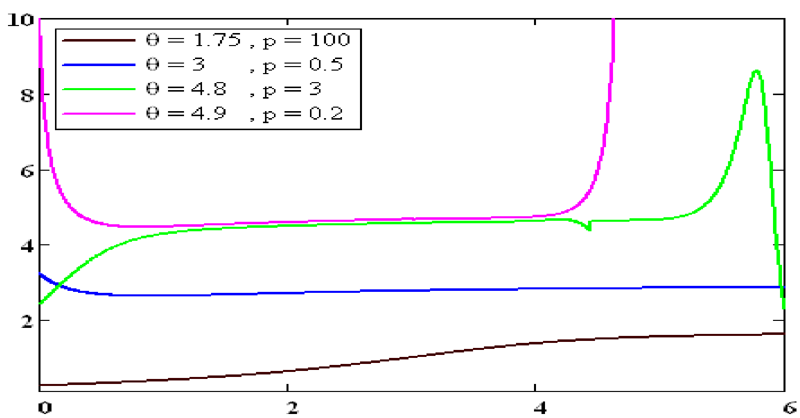


FIG. 2: Increasing, decreasing, unimodal and bathtub shapes for the hrf of the Log-L distribution.

Therefore, the new distribution can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates.

Many generalized models have been proposed in reliability literature through the relationship between the reliability function $\overline{F}(x)$ and its chrf, denoted by $H(x)$, given by $H(x) = -\ln \overline{F}(x)$. Then, the chrf of the Log-L distribution is given by

$$H(x) = \ln[\ln(p)] - \ln\left[\ln\left\{1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right\}\right]. \tag{11}$$

The additional lifetime given that the component has survived up to time x is called the residual life function of the component, then the expectation of the random variable that represent the remaining lifetime is called the mean residual lifetime (MRL) and it is given by

$$m(x) = E(X-x|X \geq x) = \left\{\frac{1}{\overline{F}(x)}\int_x^\infty t\, f(t)dt\right\} - x. \tag{12}$$

While the hrf provides information about a small interval after time x (just after x), the MRL considers information about the whole interval after x (all after x). The MRL as well as the hrf or the reliability function are very important as each of them can be used to characterize a unique corresponding life time distribution

The MRL function $m(x)$ for Log-L random variable can be derived in the following steps.

Now,

$$\int_x^\infty t\, f(t)dt = \frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\int_x^\infty \frac{(t+t^2)e^{-\theta t}}{\left[1-(1-p)\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right]}dt. \tag{13}$$

Using the expansion $(1-z)^{-1} = \sum_{j=0}^{\infty} zi$, $|z| < 1$, one has

$$\left[1-(1-p)\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right]^{-1} = \sum_{j=0}^{\infty}(1-p)^j\left(1+\frac{\theta t}{\theta+1}\right)^j e^{-j\theta t}. \tag{14}$$

Similarly, using the expansion $(1+b)^n = \sum_{l=0}^{\infty}\binom{j}{i}$, one can have

$$1+\frac{\theta+t^j}{\theta+1} = \sum_{i=0}^{\infty}\binom{j}{i}(\frac{\theta}{\theta+1})^{j-i}\ t^{j-i}. \tag{15}$$

Hence, one can rewrite (14) as

$$\left[1-(1-p)\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right]^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{j}{i}(1-p)^j\left(\frac{\theta}{\theta+1}\right)^{j-i} t^{j-i}e^{-j\theta t}. \tag{16}$$

Substitute (16) into (13) and do the necessary simplifications, one has

$$\int_x^\infty t.f(t)dt = \frac{\theta^2(p+1)}{(\theta+1)\ln(p)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{j}{i}(1-p)^{j+1}\left(\frac{\theta}{\theta+1}\right)^{j-1}\int_x^\infty (t^{j-i+1}+t^{j-i+2})e^{-(j+1)}\,\theta t. \tag{17}$$

Evaluating the integral $\int_x^\infty (t^{j-i+1}+t^{j-i+2})e^{-(j+1)\theta t}dt$ by using the substitution $u = \theta(j+1)$,

$$\int_x^\infty (t^{j-i+1}+t^{j-i+2})e^{-(j+1)\theta t}dt = \int_x^\infty t^{j-i+1}+e^{-(j+1)\theta t}\,dt + \int_x^\infty t^{j-i+2}e^{-(j+1)\theta t}\,dt.$$

Then, using

$$A = \int_x^\infty t^{j-i+1}e^{-(j+1)\theta t}dt$$

and

$$B = \int_x^\infty t^{j-i+2}e^{-(j+1)\theta t}dt.$$

Then,

$$A = \frac{1}{[\theta(j+1)]^{j-i+1}}\int_{\theta(j+1)x}^\infty u^{j-i+1}e^{-u}\,du = \frac{r(j-i+2,\theta(j+1)x)}{[\theta(j+1)]^{j-i+1}}. \tag{18}$$

Also,

$$B = \frac{1}{[\theta(j+1)]^{j-i+2}} \int_{\theta(j+1)^x}^{\infty} u^{j-i+2} e^{-u}\, du = \frac{r(j-i+3, \theta(j+1)x)}{[\theta(j+1)]^{j-i+2}}, \tag{19}$$

where $r(a,b)$ is the higher incomplete gamma function, and defined by $r(a,b) = \int_b^\infty x^{a-1} e^{-x} dx.$

Substituting (18) and (19) into (17) and doing the necessary simplifications, gives

$$\int_x^\infty t\, f(t)dt = \frac{(p-1)}{\ln(p)} \times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} iCj[\theta(j+1)\Gamma(j-i+2, \theta(j+1)x) + \Gamma(j-i+3, \theta(j+1)x)], \tag{20}$$

where $\sum_{i=0}^{j}\sum_{j=0}^{\infty} iCj$ is a constant term, and it is denoted by

$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} iCj = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{j}{i}(1-p)^j(1+\theta)^{i-j-1}}{(j+1)^{j-i+3}}.$

Finally, collecting all of the above evaluations the MRL of the Log-L distribution can be written as

$$m(x) = \frac{(p-1)}{\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]} \times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{iCj\ \ [\theta(j+1)\Gamma(j-i+2, \theta(j+1)x) + \Gamma(j-i+3, \theta(j+1)x)]\} - x. \tag{21}$$

4 Statistical Properties

This section investigates the statistical properties of the Log-L distribution such as the moments, the moment generating function, the quantiles and median, order statistics, Lorenz and Bonferroni curves and Rényi entropy.

4.1 Moments

The r^{th}non-central moment of the Log-L distribution is given by

$$E(X^r) = \acute{\mu}_r = \frac{\theta^2(p-1)}{(\theta+1)\ln p}\int_0^\infty \frac{(x^r + x^{r+1})e^{-\theta x}}{\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]} dx, \qquad r = 1,2, \dots \tag{22}$$

Using the expansion $(1- \ )^{-1} = \sum_{j=0}^{\infty} z^j$, one has

$$\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^{-1} = \sum_{j=0}^{\infty}(1-p)^j\left(1+\frac{\theta x}{\theta+1}\right)^j e^{-j\theta x}. \tag{23}$$

Similarly, using the expansion $(1+b)^n = \sum_{i=0}^{\infty}\binom{j}{i} b^i$, we can have

$$\left(1+\frac{\theta x}{1+\theta}\right)^j = \sum_{i=0}^{\infty}\binom{j}{i}\left(\frac{\theta}{1+\theta}\right)^i x^i. \tag{24}$$

Substitute (23) and (24) into (22), gives us

$$\mu_r' = \frac{\theta^2(p-1)}{(\theta+1)\ln p}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{j}{i}(1-p)^j\left(\frac{\theta}{1+\theta}\right)^i\int_0^\infty (x^{r+i} + x^{r+i+1})e^{-(j+1)\theta x} dt. \tag{25}$$

Finally, evaluating the integral

$$\int_0^\infty (x^{r+i} + x^{r+i+1})e^{-(j+1)\theta x} dt\,.$$

and doing the necessary simplifications the r^{th} non-central moment of the Log-L distribution can be written as

$$\mu_r' = \frac{(p-1)}{\theta^r \ln p} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[\frac{\binom{j}{i} (1-p)^j [\theta(j+1) + r + i + 1](r+i)!}{(1+\theta)^{i+1} (j+1)^{r+i+2}} \right]. \quad (26)$$

Depending on (26), we can conclude the basic statistical properties of the Log-L distribution as follows.

(i) The mean, $\mu_1' = \mu$, and the variance, $Var(X)$, of the Log-L random variable X are, respectively, given by

$$\mu = \frac{(p-1)}{\theta \ln p} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{j!\,(1-p)^j [\theta(j+1) + i + 2](i+1)}{(1+\theta)^{i+1} (j+1)^{i+3} (j-i)!} \right] \quad (27)$$

and

$$Var(X) = \acute{\mu}_2 - \acute{\mu}_1,$$

where μ_2' is the second non-central moment and given by

$$\mu_2' = \frac{(p-1)}{\theta^2 \ln p} \sum_{i=0}^{j} \sum_{j=0}^{\infty} \left[\frac{j!\,(1-p)^j [\theta(j+1) + i + 3](i+2)(i+1)}{(1+\theta)^{i+1} (i+1)^{i+4} (j-i)!} \right]. \quad (28)$$

(ii) The n^{th} central moments μ_n can be obtained easily from the r^{th} moments through the relation

$$\mu_n = E(x-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} \acute{\mu}_r.$$

Then the n^{th} central moment of the Log-L distribution is given by

$$\mu_x = \frac{(p-1)}{\ln p} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\binom{j}{i} t^3 (1-p)^j [\theta(j+1) + r + i + 1](r+i)}{\theta^3 (1+\theta)^{i+1} (j+1)^{r+i+2}\, r!} \right]. \quad (29)$$

4.2 The Quantiles

Let X be a random variable with cdf associated with (9). Then, the quantile function (qf), x_q, defined by $F\,(xq) = q$ is the root of the equation

$$\left(\frac{\theta + 1 + \theta x_q}{\theta + 1} \right) e^{-\theta x_q} = \left(\frac{1 - p^{1-q}}{1-p} \right), \quad 0 < q < 1. \quad (30)$$

Substituting, $y_{q\,=-1-\theta-\theta x_q,}$ we can rewrite (30) as

$$y_q e^{yq} = -(\theta+1) e^{-(\theta+1)} \left(\frac{1 - p^{1-q}}{1-p} \right), \quad 0 < q < 1. \quad (31)$$

Hence, the solution of y_q is

$$y_q = W\left[-(\theta+1) e^{-(\theta+1)} \left(\frac{1 - p^{1-q}}{1-p} \right) \right], \quad 0 < q < 1, \quad (32)$$

where $W(.)$ is the Lambert W function, see [30] for more details about the properties of the Lambert W function. Inverting (32), one has

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(-(\theta+1) e^{-(\theta+1)} \left(\frac{1 - p^{1-q}}{1-p} \right) \right), \quad 0 < q < 1. \quad (33)$$

Remark 1 A particular case of (33) at $p \to 1$ gives the qf of the Lindley distribution; see Jodra (2010), as

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(-(\theta+1) e^{-(\theta+1)} (1-q) \right). \quad (34)$$

When $q = 0.5$ in (33), one can obtain the median of the distribution as

$$x_{0.5} = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(\frac{-(\theta+1)e^{-(\theta+1)}}{1+\sqrt{p}}\right). \tag{35}$$

A series expansion for (33) around $q = 1$ can be obtained as

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta}\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}, \tag{36}$$

where $z = -(\theta+1)e^{-(\theta+1)}\left(\frac{1-p^{1-q}}{1-p}\right)$. These kind of expansions for computing $W(.)$ are widely available, for example, ProductLog[.] in Mathematica software.

4.3 Distribution of Order Statistics

Let $X_1X_2 \dots \dots X_n$ denote n independent random variables from a distribution function $F_X(x) = 1 - \overline{F}_X(x)$ with pdf $f_X(x)$, and then the pdf of $X_{(j)}$ (the *j* order sample arrangement) is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}, \qquad j = 1,2,\dots \tag{37}$$

Using (8) and (9) into (37), then the pdf of $X_{(j)}$ according to the Log-L distribution is given by

$$f_{X_{(j)}}(x) = \frac{n!\ (p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{(j-1)!\,(n-j)!\,[\ln(p)]^{n-j+1}} \frac{\left\{\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]\right\}^{n-j}}{\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1)}\right)e^{-\theta x}\right]}$$

$$\times\left\{1-\frac{\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]}{\ln(p)}\right\}^{j-1}, \tag{38}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$and the smallest order statistic $X_{(1)}$ are, respectively, given by

$$f_{X_{(n)}}(x) = \frac{n}{\ln p}\left[\frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}}\right]\left[1-\frac{\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}}{\ln(p)}\right]^{n-1} \tag{39}$$

and

$$f_{X(1)}(x) = \frac{n\,(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{[\ln(p)]^n\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1)}\right)e^{-\theta x}\right]}\left\{\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]\right\}^{n-1}. \tag{40}$$

4.4 Measures of Inequality

In this section, Lorenz and Bonferroni curves are introduced as measures of inequality.

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution. In fact, Lorenz and Bonferroni curves are depending on the length-biased distribution with pdf $f^*(x)$ defined by

$$f^*(x) = \frac{x\,f(x)}{\mu},$$

where $f(x)$ is the pdf of the base distribution with mean μ.

Accordingly, Lorenz and Bonferroni curves, denoted by $L(x)$ and $B(x)$ respectively, are defined by

$$L(x) = \frac{f^*(x)}{\mu}, \qquad \text{and} \qquad B(x) = \frac{l(x)}{f(x)}, \tag{41}$$

where $F^*(x)$ cdf of the length-biased distribution.

Now, we shall derive the expressions of $L(x)$ and $B(x)$ based on $F^*(x)$ and $F^*(x)$ for Log-L distribution. It is easily shown that the pdf of the length biased distribution according to the Log-L distribution can be obtained as follows.

$$f^*(x,\theta,p)=\frac{x\,f^*(x,\theta,p)}{\mu}=\frac{\theta^2(p-1)}{\mu(\theta+1)\text{In}(p)}\left[\frac{(x+x^2)e^{-\theta x}}{1-(1-p)\left(\frac{\theta+1+\theta zi}{\theta+1}\right)e^{-\theta x}}\right] \tag{42}$$

Which cdf $F^*(x)$ given by

$$F^*(x)\ =\frac{(p-1)}{\theta\mu\ \text{Inp}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta)j+1)x], \tag{43}$$

where $\gamma(a,b)$ is the lower incomplete gamma function defined by $\gamma(a,b)=\int_a^b u^{a-1}e^{-u}\,du$

It follow from (8), (41) and (43) that $L(x)$ and $B(x)$ are

$$L(x)=\frac{(p-1)}{\theta u^2\ \text{lnp}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta)j+1)x] \tag{44}$$

and

$$\text{B(x)}=\frac{(p-1)}{\theta u^2\left(\ln p-\ln\left\{1-\left[1-p\left(\frac{\theta+1+\theta xi}{\theta+1}\right)e^{-\theta xi}\right]\right\}\right)}$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta(j+1)x)]. \tag{45}$$

4.5 Rényi Entropy

If X is a random variable having an absolutely continuous cdf $F(x)$ and pdf $f(x)$, then the basic uncertainty measure for distribution F (called the entropy of F) is defined as $[T(x)\ =\ E[—\ln(f\ (x))]$. Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Abundant entropy and information indices, among them the Rényi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.

Rényi entropy is an extension of Shannon entropy. Rényi entropy of the Log-L distribution is defined to be

$$\gamma_v(f(x,\theta,p)=\frac{\ln\left(\int_0^\infty f^v\,(x;\theta,p)dt\right)}{1-v}, \tag{46}$$

where $v\ >\ 0$ and $v\ =\ 1$. Rényi entropy tends to Shannon entropy as $v\to 1$. Now

$$\int_0^\infty f^v\,(x,\theta,p)dr=\left[\frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\right]^v\int_0^\infty\frac{(1+x)^{ve-v\theta x}}{\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^v}dx. \tag{47}$$

Using the following expansions

$$\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^v=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{v+j-1}{j}\binom{j}{i}(1-p)^j\left(\frac{\theta}{\theta+1}\right)^i x^i e^{-j\theta x} \tag{48}$$

and

$$(1+x)^v=\sum_{k=0}^{\infty}\binom{v}{k}x^k. \tag{49}$$

Then Equation (47) cab be written follows

$$\int_0^{\infty} f^{v}\,(x,\theta,p)dr=\left[\frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\right]^{v}\sum_{j,k,i=0}^{\infty}\binom{v}{k}\binom{v+j-1}{j}\binom{j}{i}(1-p)^j\left(\frac{\theta}{\theta+1}\right)\int_0^{\infty} x^{i+k}\,e-(j+v)^{\theta x}dx. \quad (50)$$

Evaluating the integral in (50) using the gamma function. Then, collecting all of the above evaluations and substituting in (46), the Rényi entropy of the Log-L distribution can be defined as

$$r_v\big(f(x,\theta,p)\big)=v\ln\left(\frac{p-1}{\ln(p)}\right)+\ln\left[\sum_{j,k,i=0}^{\infty}\frac{\binom{v}{k}\binom{v+j+1)}{j}\binom{j}{i}\theta^{2v-k-1}(1-p)^j(i+k)^i}{(\theta+1)^{v+1}(j+v)^{i+k+1}}\right]. \quad (51)$$

5 Estimation of the Parameters

In this section, we introduce the method of likelihood to estimate the parameters involved.

Let $X_1, X_2, ..., X_n$ be a sample size n from Log-L distribution. Then the likelihood function is given by

$$\ell=\prod_{i=1}^{n} fi(x)=\left(\frac{1}{\ln p}\right)^n(p-1)^n\left(\frac{\theta^2}{\theta+1}\right)^n\frac{\prod_{i=1}^{n}(1+xi)e-\theta\sum_{i=1}^{n}x_i}{\prod_{i=1}^{n}\left[1-(1-p)\left(\frac{\theta+1+\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]}. \quad (52)$$

Hence, the log-likelihood function, $\mathcal{L}=\ln\ell$, becomes

$$\mathcal{L}=-n\ln[\ln p]+n\ln(p-1)+n\ln(\frac{\theta2}{\theta+1})+\sum_{i=1}^{n}\ln(1+x_i)-\theta\sum_{i=1}^{n}x_i$$
$$-\sum_{i=1}^{n}\ln\left[1-(1-p)\left(\frac{\theta+1+\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]. \quad (53)$$

Therefore, the maximum likelihood estimators (MLEs) of θ and p are derived from the derivatives of $\mathcal{L}$. They should satisfy the following equations

$$\frac{\partial\mathcal{L}}{\partial\theta}=\frac{n(\theta+2)}{\theta(\theta+1)}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_i[1-(\theta+1)(\theta+1+\theta x_i)]e^{-\theta x_i}}{\theta+1}=0 \quad (54)$$

and

$$\frac{\partial\mathcal{L}}{\partial p}=\frac{n}{p-1}-\frac{n}{p\ln p}-\sum_{i=1}^{n}\frac{(\theta+1+\theta x_i)e^{-\theta x_i}}{1-(1-p)\left(\frac{\theta+1+\theta x_i}{\theta+1}\right)e^{-\theta x_i}}=0. \quad (55)$$

To solve the Equations (54) and (55), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

6 Simulation Analysis

The performance of the MLEs of the Log-L parameters are addressed in this section using some numerical simulations in terms of the sample size n. The Log-L distribution is simulated using its qf (33) given in Section 4.3.

Using the software R programming language R (R Core Team, 2020), 5,000 random samples from the Log-L distribution are generated with four sample sizes $n=80, 200, 300$ and $n=450$. The true values of the Log-L parameters are selected as follows: $\theta=(0.3, 0.5, 0.75, 1.5, 2.75, 1.0, 3.5)$ and $p=(0.2, 0.55, 0.85, 1.5, 2.3, 3.4)$. Tables 1 and 2 report the averages estimates (AvEs) of the parameters along with the averages mean square errors (MSEs), $\text{MSEs}(\hat{\boldsymbol{\delta}})=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}\right)^2$, averages absolute biases (ABs), $\text{ABs}(\hat{\boldsymbol{\delta}}) = \frac{1}{N}\sum_{i=1}^{N}|\hat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}|$, and averages mean relative estimates (MREs), $\text{MREs}(\hat{\boldsymbol{\delta}}) = \frac{1}{N}\sum_{i=1}^{N}|\hat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}|/\boldsymbol{\delta}$, where $\boldsymbol{\delta} = (\theta, p)^T$. These measures are calculated for all sample sizes and all values of the parameters θ and p. Tables 1 and 2 illustrate that the MLEs of the Log-L parameters are stable and consistent. Additionally, these tables reveal that the MSEs, ABs, and MREs of the parameters decay toward zero as the sample size increases.

TABLE 1: Simulation results of the Log-L distribution for different parametric values

Parameters		n	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$
θ	p		AvEs		MSEs		ABs		MREs	
0.3	0.2	80	0.31694	0.47511	0.00484	0.56080	0.05050	0.34580	0.16833	0.72902
		200	0.30612	0.23971	0.00152	0.02257	0.03022	0.09617	0.10073	0.48087
		300	0.30477	0.22955	0.00096	0.01191	0.02432	0.07602	0.08107	0.38008
		450	0.30297	0.21704	0.00061	0.00629	0.01953	0.05779	0.06511	0.28896
0.3	0.55	80	0.32000	0.75793	0.00552	0.39156	0.05217	0.41232	0.17389	0.88604
		200	0.30631	0.69968	0.00153	0.29321	0.03011	0.31391	0.10038	0.47075
		300	0.30452	0.64739	0.00099	0.13052	0.02466	0.23753	0.08219	0.43187
		450	0.30323	0.61038	0.00065	0.06479	0.02007	0.18321	0.06692	0.33311
0.3	0.85	80	0.32222	1.01622	0.00619	0.52327	0.05531	0.55096	0.18435	0.59525
		200	0.30751	0.97259	0.00163	0.44290	0.03120	0.43305	0.10402	0.42711
		300	0.30407	0.93921	0.00099	0.34428	0.02482	0.38449	0.08272	0.35234
		450	0.30278	0.87512	0.00065	0.17860	0.02026	0.29967	0.06754	0.25256
0.5	0.2	80	0.52835	0.34414	0.01276	0.28226	0.08495	0.21685	0.16990	1.08424
		200	0.50914	0.23712	0.00389	0.01867	0.04869	0.09208	0.09739	0.46042
		300	0.50665	0.22642	0.00258	0.01043	0.03999	0.07267	0.07997	0.36337
		450	0.50552	0.21828	0.00174	0.00627	0.03289	0.05764	0.06577	0.28821
0.5	0.55	80	0.53181	0.76405	0.01749	1.48602	0.08717	1.64056	0.17434	1.80103
		200	0.51130	0.69381	0.00417	0.23556	0.05044	0.30085	0.10087	0.54701
		300	0.50767	0.64185	0.00275	0.11564	0.04094	0.22977	0.08189	0.41776
		450	0.50422	0.60405	0.00170	0.06149	0.03252	0.17530	0.06503	0.31872
0.5	0.85	80	0.53346	1.46405	0.01683	0.58199	0.08981	0.46405	0.17962	0.45770
		200	0.51238	1.13346	0.00449	0.43574	0.05173	0.33302	0.10346	0.37410
		300	0.50857	1.01583	0.00284	0.34175	0.04176	0.28401	0.08352	0.27403
		450	0.50626	0.88029	0.00182	0.17991	0.03361	0.15130	0.06722	0.13074
0.5	1.5	80	0.53570	2.71889	0.01910	2.45475	0.09244	1.71799	0.18488	1.47866
		200	0.51153	2.09601	0.00472	1.57981	0.05255	1.07495	0.10511	0.71664
		300	0.50859	1.85834	0.00312	1.54067	0.04349	0.76869	0.08698	0.51246
		450	0.50461	1.69813	0.00187	0.66356	0.03415	0.56213	0.06830	0.37476
0.75	0.2	80	0.79190	4.21120	0.02699	45.44993	0.12180	0.28820	0.16240	1.44102
		200	0.76333	0.23580	0.00913	0.01917	0.07412	0.09001	0.09882	0.45007

		300	0.75965	0.22154	0.00549	0.00891	0.05825	0.06704	0.07766	0.33521
		450	0.75505	0.21372	0.00369	0.00528	0.04821	0.05428	0.06428	0.27142
0.75	0.55	80	0.80049	1.76526	0.04028	1.55808	0.13162	1.76527	0.17549	3.20957
		200	0.76739	0.69323	0.00967	0.23350	0.07624	0.29956	0.10165	0.54465
		300	0.75985	0.63381	0.00594	0.10589	0.06108	0.22196	0.08144	0.40357
		450	0.75660	0.60247	0.00379	0.05377	0.04851	0.16890	0.06468	0.30710
0.75	0.85	80	0.80130	1.91428	0.04728	1.83225	0.13229	1.91429	0.17639	2.25210
		200	0.76797	1.10107	0.00973	0.75523	0.07653	0.49104	0.10205	0.57770
		300	0.76089	0.99511	0.00605	0.28771	0.06092	0.36124	0.08123	0.42499
		450	0.75863	0.94993	0.00399	0.16551	0.04956	0.28478	0.06608	0.33504
0.75	1.5	80	0.80482	3.26588	0.05376	8.00336	0.13935	1.26588	0.18579	8.43926
		200	0.76823	2.08548	0.01087	5.31825	0.07941	1.05180	0.10588	0.70120
		300	0.76397	1.84341	0.00686	1.34647	0.06475	0.74802	0.08633	0.49868
		450	0.75973	1.72140	0.00421	0.65376	0.05106	0.55650	0.06808	0.37100

TABLE 2: Simulation results of the Log-L distribution for different parametric values

Parameters		n	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$	$\hat{\theta}$	$\hat{p}$
θ	p		AvEs		MSEs		ABs		MREs	
1.5	0.2	80	1.58833	0.31284	0.11347	0.12525	0.25339	0.17741	0.16893	0.88706
		200	1.53281	0.23395	0.03700	0.01480	0.14979	0.08429	0.09986	0.42143
		300	1.51950	0.22105	0.02432	0.00864	0.12312	0.06623	0.08208	0.33115
		450	1.51210	0.21342	0.01499	0.00466	0.09660	0.05102	0.06440	0.25508
1.5	0.55	80	1.59117	2.01895	0.12011	4.86218	0.25555	1.66083	0.17037	3.01969
		200	1.53332	0.66852	0.03671	0.16771	0.14929	0.26742	0.09953	0.48621
		300	1.52170	0.62507	0.02411	0.08918	0.12169	0.20338	0.08113	0.36979
		450	1.51411	0.59757	0.01497	0.04837	0.09647	0.15984	0.06432	0.29061
1.5	0.85	80	1.59639	1.77159	0.13199	7.33083	0.26281	1.71735	0.17520	2.02041
		200	1.53779	1.06834	0.03812	0.50950	0.15217	0.45179	0.10145	0.53151
		300	1.52815	0.99321	0.02483	0.25813	0.12326	0.34322	0.08217	0.40379
		450	1.51355	0.93476	0.01602	0.13721	0.10019	0.26742	0.06679	0.31461
1.5	1.5	80	1.60249	5.08890	0.14791	1.23925	0.26966	1.07962	0.17978	1.38641
		200	1.53368	1.96510	0.04025	1.02425	0.15411	0.90472	0.10274	0.60315
		300	1.52343	1.77345	0.02583	1.00369	0.12519	0.66413	0.08346	0.44276
		450	1.51440	1.67079	0.01650	0.52903	0.10081	0.51629	0.06721	0.34419
1.5	2.3	80	1.60556	3.39512	0.17702	2.63373	0.27651	2.29512	0.18434	1.97877
		200	1.53401	3.14273	0.04309	1.00506	0.15971	1.56762	0.10647	0.68157
		300	1.52444	2.78803	0.02636	0.03332	0.12657	1.11692	0.08438	0.48562
		450	1.51530	2.60310	0.01709	0.01595	0.10248	0.85686	0.06832	0.37255

1.5	3.4	80	1.62842	4.47498	0.23805	1.00127	0.29189	4.47498	0.19460	1.49166
		200	1.54597	4.10715	0.04494	0.49161	0.16133	2.31825	0.10755	0.77275
		300	1.52521	3.73738	0.02732	0.66812	0.12902	1.58327	0.08601	0.52776
		450	1.52050	3.49487	0.01774	0.31720	0.10439	1.19979	0.06960	0.39993
0.3	1.5	80	0.32462	7.25657	0.00745	2.62443	0.05724	2.25656	0.19078	1.83771
		200	0.30810	2.14692	0.00182	1.88845	0.03265	1.13286	0.10882	0.75524
		300	0.30460	1.84945	0.00108	1.62774	0.02575	0.77844	0.08582	0.51896
		450	0.30332	1.72992	0.00074	0.79045	0.02139	0.59799	0.07128	0.39866
0.5	1.5	80	0.53695	2.70146	0.02044	2.21170	0.09298	2.01465	0.18597	2.34310
		200	0.51228	2.10550	0.00473	1.94844	0.05277	1.07444	0.10554	0.71629
		300	0.50848	1.85383	0.00303	1.51587	0.04279	0.76900	0.08559	0.51266
		450	0.50441	1.69997	0.00194	0.72683	0.03463	0.57168	0.06926	0.38112
0.75	1.5	80	0.80271	2.71733	0.05245	3.87108	0.13672	6.11733	0.18229	4.07822
		200	0.76786	2.03921	0.01060	2.53648	0.07969	1.00678	0.10625	0.67119
		300	0.76284	1.83426	0.00679	1.37205	0.06405	0.74251	0.08540	0.49501
		450	0.75846	1.71167	0.00428	0.71168	0.05131	0.55866	0.06841	0.37244
1.5	1.5	80	1.61364	2.46324	0.15165	1.60352	0.27552	1.36860	0.18368	1.57907
		200	1.53840	1.97694	0.04154	1.37073	0.15801	0.92044	0.10534	0.61363
		300	1.52464	1.78708	0.02493	1.07877	0.12239	0.66557	0.08159	0.44372
		450	1.51652	1.68431	0.01698	0.58691	0.10198	0.52405	0.06798	0.34936
2.75	1.5	80	2.93546	2.27728	0.56415	1.99830	0.49444	1.27728	0.17980	1.51818
		200	2.81671	1.91105	0.13457	1.96676	0.28367	0.83513	0.10315	0.55675
		300	2.79057	1.75528	0.08661	0.92530	0.22890	0.62860	0.08324	0.41907
		450	2.77315	1.64879	0.05736	0.48235	0.18863	0.48956	0.06859	0.32637

7 Applications

In this section, we use two real data sets to show that the Log-L distribution can be better than the Lindley distribution and other non-nested models such as Lindley-exponential by [31], new generalized Lindley (NGL) by [10], Weibull and weighted Lindley by [4].

7.1 Data Set 1

The first data set represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in [32]. These data are addressed by [33]. In order to compare the six distributions, we consider criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) for the two data sets. We also consider the Kolmogorov-Smirnov (KS) statistic. The better distribution corresponds to smaller AIC, AICC and BIC values:

$$\mathrm{AIC} = 2k - 2i, \qquad \mathrm{AICC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1} \quad \text{and} \quad \mathrm{BIC} = -2i + K\log(n),$$

where k is the number of parameters in the statistical model, n the sample size and i is the maximized value of the log-likelihood function under the considered model.

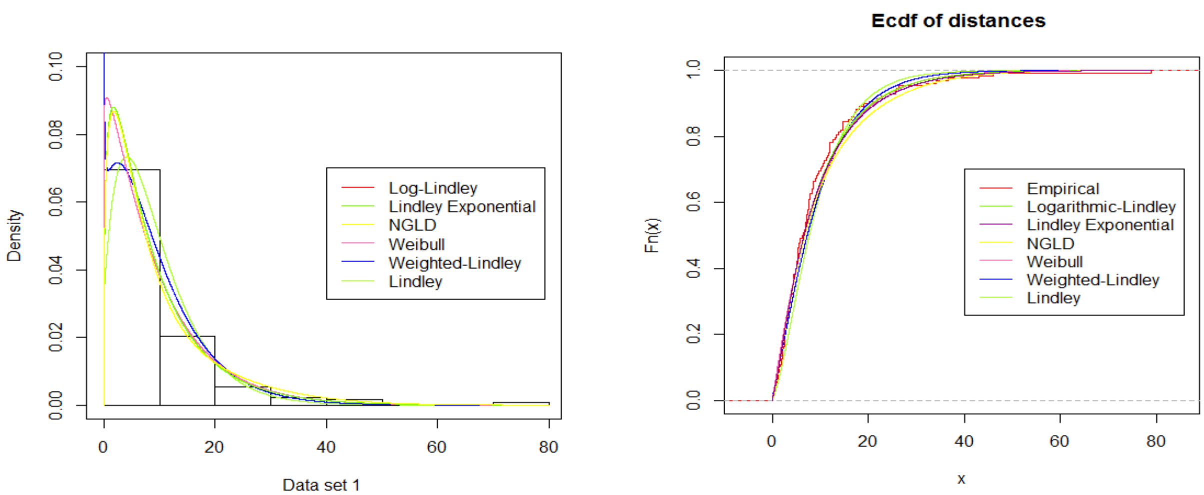
Table 3 shows MLEs of the parameters to each one of the six fitted distributions with their standard errors (S.E.) for data set with 95% confidence interval, while Table 4 represents the values of $-\mathcal{L}$, AIC, BIC, AICC and KS.

The values in Table 4 indicate that the Lo-L is a strong competitor to other distributions used here for fitting the data set. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 3). The fitted density for the Log-L model is closer to the empirical histogram than the fits of the other models.

TABLE 3. The MLEs and S.E. of the parameters with 95% CI for data set 1

Model	Parameter estimates	S.E.	95% CI
Log-L	0.1238	0.0186	[0.0872, 0.1604]
	0.0979	0.0472	[0.0053, 0.1906]
Lindley Exponential	0.1093	0.0137	[0.0824, 0.1363]
	1.5687	0.1638	[1.2476, 1.8898]

	0.1827	0.0355	[0.1130, 0.2525]
NGL	4.6807	1.3080	[2.1169, 7.2445]
	1.3243	0.1718	[0.9874, 1.6611]
Weibull	1.0478	0.0676	[0.9153,1.1803]
	0.1045	0.0093	[0.0862, 0.1229]
Weighted-Lindley	0.1594	0.0172	[0.1257, 0.1931]
	0.6827	0.1115	[0.4640, 0.9014]
Lindley	0.1960	0.0123	[0.1718, 0.2202]



(a) (b)

FIG. 3. (a) Estimated densities of data set 1. (b) Empirical, Log-Lindley, Lindley, Weibull, Lindley exponential, NGL, and Weighted-Lindley cdf of data set 1.

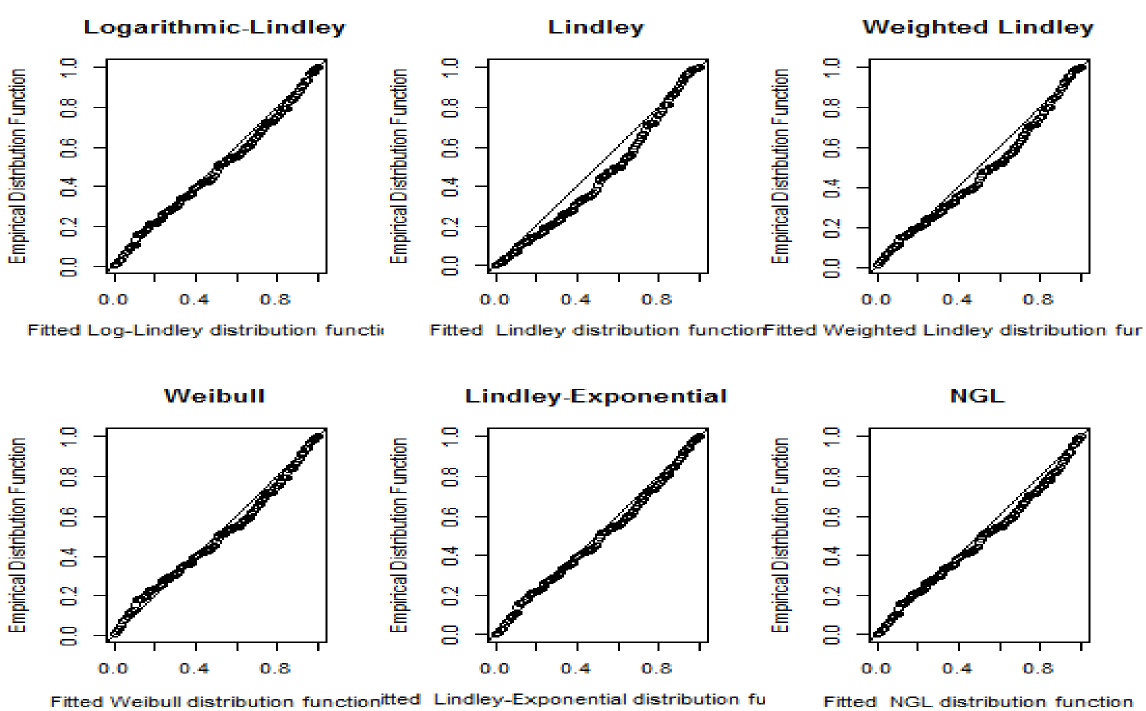


FIG4. Probability-probability plots for the Log-Lindley, Lindley, Weibull, Lindley-exponential, weighted Lindley and

NGL distributions of data set 1.

TABLE 4. The goodness of fit measures values under considered models based on data set 1

Model	-$\mathcal{L}$	AIC	AICC	BIC	KS
Log-L	411.7701	827.5403	827.6363	833.2443	0.0619
Lindley- Exponential	412.0493	828.0985	828.1945	833.8026	0.0621
NGL	412.7503	831.5006	831.6942	840.0567	0.0740
Weibull	414.0869	832.1738	832.2698	837.8778	0.0701
Weighted Lindley	416.4422	836.8845	836.9805	842.5885	0.0925

7.2 Data Set 2

The data represents 46 repair times (in hours) for an airborne communication transceiver and available in [34]. Table 5 shows parameter MLEs and S.E. to each one of the six fitted distributions for data set 2 with 95% confidence interval, while Table 6 lists the values of $-\mathcal{L}$, AIC, BIC, AICC and KS.

The values in Table 4, indicate that the Log-L is a strong competitor to other distributions used here for fitting data set.

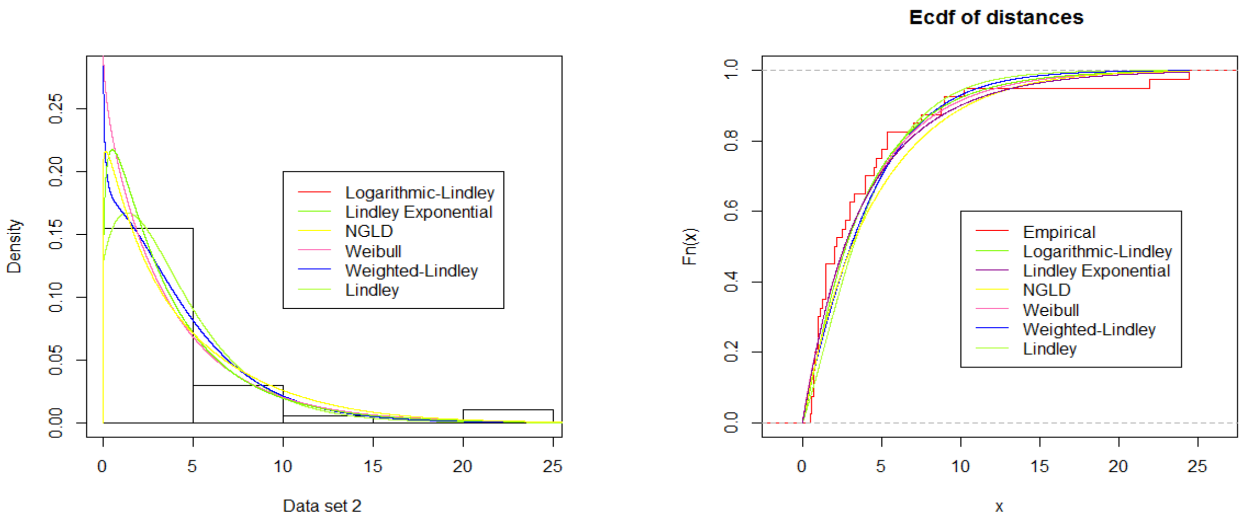
TABLE 5. The MLEs and S.E. of the parameters with 95% CI for data set 2

Model	Parameter estimates	S.E.	95% CI
Log-L	0.2473	0.0640	[0.0872, 0.1604]
	0.0694	0.0313	[0.0080, 0.1307]
Lindley-Exponential	0.242675	0.05649785	[0.1319393, 0.3534108]
	1.460206	0.2715473	[0.9279731, 1.992438]
NGL	0.3552648	0.1721713	[0.01780907, 0.6927206]
	3.089682	3.338087	[2.02514, 9.632333]
	1.058484	0.2930719	[0.4840632, 7.601135]
Weibull	0.960359	0.06812665	[0.8268308, 1.093887]
	0.2546432	0.1467211	[0.02586985, 0.4834164]
Weighted Lindley	0.3551453	0.06812665	[0.221617,0.4886735]
	0.7471963	0.1867211	[0.381223,1.11317]
Lindley	0.4242097	0.04852818	[0.3290945, 0.519325]

TABLE 6. The goodness of fit measures values under considered models based on data set 2

Model	$-\mathcal{L}$	AIC	AICC	BIC	KS
Log-L	94.39982	192.7996	193.124	196.1774	0.1212
Lindley- Exponential	94.614	193.2284	193.5527	196.6062	01488
NGL	96.13662	198.2732	198.9399	203.3399	0.1621635
Weibull	95.51136	195.0227	195.0347	198.4005	0.1224559
Weighted Lindley	98.04943	200.0989	200.4232	203.4766	0.1722701
Lindley	98.79132	199.5826	199.6879	201.2715	0.2156951

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 5). The fitted density for the Log-L model is closer to the empirical histogram than the fits of the Lindley distribution and other non-nested models.



(a) (b)

FIG. 5. (a) Estimated densities of data set 1. (b) Empirical, Logarithmic-Lindley, Lindley, Weibull, Lindley-exponential, NGL, and weighted Lindley cdf's of data set 2.

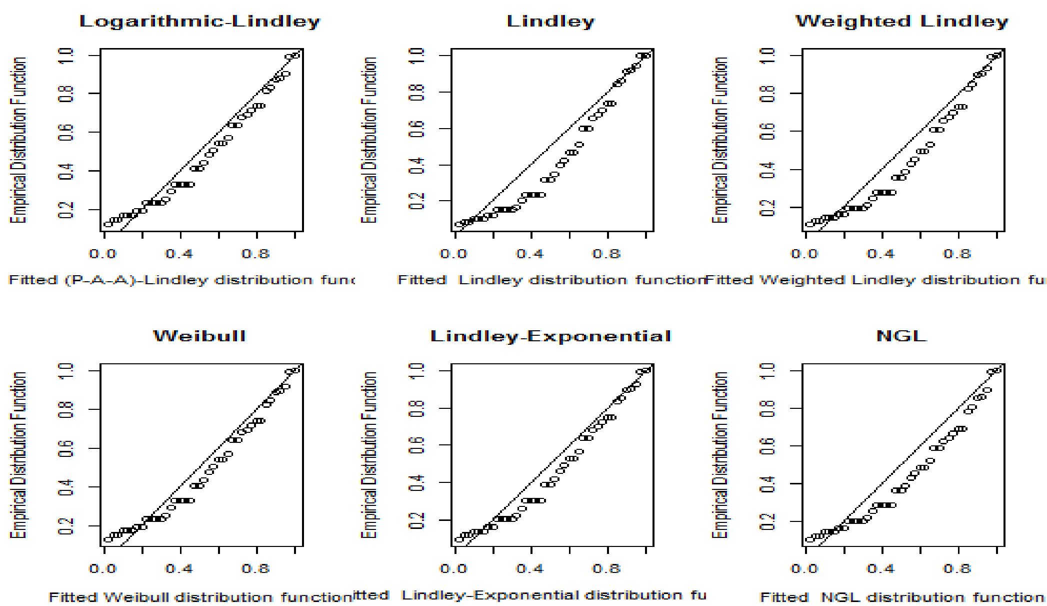


FIG. 6. Probability plots for the Logarithmic-Lindley, Lindley, Weibull, Lindley-exponential, weighted Lindley and NGL distributions of data set 2.

8 Conclusion

The logarithmic-Lindley distribution is proposed to extend the Lindley distribution. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for the moments, moment generating function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mean residual lifetime distribution, quantiles, Lorenz curves, Bonferroni curves, order statistics, and Rényi entropy. The estimation of parameters is approached by the method of maximum likelihood. A numerical simulation study is presented to explore the performance of the maximum likelihood estimates. Two applications of the logarithmic-

Lindley distribution to real data show that the new distribution can be used quite effectively to provide better fits than the Lindley distribution and other non-nested models such as Lindley-exponential, weighted Lindley, Weibull and new generalized Lindley distributions.

Acknowledgement

The authors would like to thank the Editorial Board and the Referees for their careful reading of the manuscript and for their constructive remarks and comments, which have enhanced the presentation.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this article.

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