A Probabilistic SIRI Epidemic Model Incorporating Incidence Capping and Logistic Population Expansion

Soulaimane Aznague, Mourad El Idrissi*, Abdeladim Nait Brahim, Bilal Harchaoui, Saloua Boutouil, Adel Settati, Aadil Lahrouz, Mhamed El Merzguioui and Jamal El Amrani

Laboratory of Mathematics and Applications, FSTT, Abdelmalek Essaadi University, Tetouan, Morocco

Abstract: This study presents a newly developed stochastic SIRI epidemic model, which combines logistic growth with a saturation incidence rate. This research mainly examines the presence and uniqueness of positive solutions within the formulated model. Furthermore, we aim to analyze the long-term performance of the system and provide valuable insights into disease extinction in a population. Our investigation delves into the conditions required for disease extinction, which are crucial in predicting and controlling the spread of deadly diseases. To substantiate our assertions, we have devised a stochastic Lyapunov function, which serves as a robust mathematical framework for demonstrating the presence of a discernible stationary ergodic distribution. This mathematical foundation significantly contributes to the understanding of model behavior. To complement our analytical findings, we conduct numerical simulations, which reinforce our results and provide a comprehensive understanding of the behavior of our proposed model, and open new avenues for future research in this area.

Keywords: Stochastic SIRI model, Logistic growth, Extinction of disease, Lyapunov function, Stationary distribution

1 Introduction

Epidemics and pandemics have been problems for humans for a long time. They kill many people, disrupt society, and hurt the economy. Accurately predicting the trajectory of an outbreak is crucial to mitigating its impact, making epidemiologic modeling an essential field of study. By developing models that accurately capture the underlying dynamics of infectious disease spread, we can better understand how an outbreak will progress and devise effective strategies to control its spread. Recent advances in computational and statistical techniques have led to sophisticated models incorporating factors such as stochasticity, saturation incidence, and logistic growth. These models can provide realistic estimates of disease spread and predict the potential impact of various control measures. As a result, public health officials and policymakers increasingly rely on epidemiologic modeling to guide decision-making and resource allocation during outbreaks.

Mathematical modeling plays a vital role in epidemiology and has been extensively utilized to investigate the propagation of contagious diseases. Mathematical models help us understand the complicated things that happen when a disease spreads, predict the spread of epidemics, and evaluate potential control measures. As such, in the fight against viral infections, mathematical modeling is an essential tool, enabling us to develop effective strategies to protect public health. Kermack and McKendrick [1] pioneered epidemiological modeling and proposed the deterministic model for infectious diseases. According to the SIR model [2], individuals can be classified into discrete groups depending on their health status: susceptible S individuals vulnerable to the infectious agent, infectious I individuals who currently carry the infection, and recovered R individuals who have developed permanent immunity against the disease. However, it is crucial to recognize that some individuals who have previously recovered from a specific condition or infection may undergo a waning of their acquired immunity, rendering them susceptible once again. The development of this concept has led to the emergence of an adapted variant of the model recognized as the SIRS model, which holds significant importance in scientific circles. Epidemiological studies have extensively employed mathematical models to gain a comprehensive

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* Corresponding author e-mail: mourad.elidrissi@etu.uae.ac.ma

understanding of the transmission dynamics of infectious diseases. These models serve as invaluable tools in assessing the efficacy of potential control measures to mitigate the spread of such diseases across populations. By employing sophisticated disease propagation models, scientists can conduct simulations to study the spread of diseases effectively. This empirically driven research approach empowers scientists to estimate crucial parameters; these include the number of people infected, the duration of the epidemic, and the effectiveness of various intervention strategies. Ultimately, the SIR and SIRS models serve as valuable frameworks for forecasting and managing the propagation of infectious diseases and are indispensable instruments in the battle against epidemics. Some diseases can reactivate latent infections, leading to relapses in previously recovered individuals and a return to an infectious state. To accurately model these diseases, SIRI models are employed. These models account for the cyclical nature of the infection by incorporating a temporary immune state between the infectious and susceptible states. The persistence of infection is a crucial feature of certain diseases affecting animals and humans, including tuberculosis in both bovine and human populations and herpes (see, e.g., [3,4,5,6,7,8]). Tudor [9] investigated a model incorporating a bilinear incidence function for herpes infections and provided qualitative analysis. Moreira and Wang [10] then extended this model to incorporate nonlinear impact functions. Blower [4] created a compartmental model to study genital herpes, where disease transmission is assumed to follow a standard incidence and the recruitment rate remains constant. Vargas-De-León [11] conducted a scientific investigation involving an epidemic model utilizing the Susceptible-Infectious-Recovered-Immune (SIRI) framework, with a bilinear incidence rate. The study maintained a constant total population throughout the research process. Meanwhile, Michaelis-Menten [12] introduced a novel concept in epidemic modeling by introducing a saturated incidence rate $g(S)I$, where $g(S) = \beta S(1 + \alpha S)^{-1}$. Multiple studies [13,14,15,5,7,2, 16,17,18,19,20] have suggested that implementing the saturated incidence rate, as proposed by Michaelis-Menten, is a more suitable approach in modeling epidemics when compared to the utilization of bilinear incidence rates. Scientists have developed the logistic growth model as an improved approach for modeling population growth. The model demonstrates exponential growth within a population, which is limited by the carrying capacity of its environment. At this juncture, the population growth rate decelerates and eventually halts as the carrying capacity is reached. The following equation mathematically represents the logistic growth model:

$$dN(t) = rN\left(1 - \frac{N}{K}\right)dt.$$

The provided equation describes the temporal evolution of a population, represented by the variable N, wherein its growth rate, denoted by r, plays a pivotal role in determining the magnitude of population changes over time. The size of a population is directly related to its carrying capacity, denoted as K. The significance of this equation is that it accounts for the negative feedback effect that occurs when larger populations compete for the same resources as smaller populations. As populations near their carrying capacity, the birth of new offspring exceeds the available resources. Exceeding the carrying capacity of an environment leads to an unsustainable population overshoot, which can have dire consequences. More resources are needed for the entire population, resulting in a struggle for survival. Consequently, some individuals face severe hardship and even perish. According to several studies [21,14,22,23,24,25,26], logistic growth as a model for the influx of susceptible individuals is better suited for diseases with a high mortality rate. In light of this, our working hypothesis is based on the assumption that the vulnerable population of any given nation follows the logistic growth model. To validate this hypothesis, we offer a SIRI epidemic model with logistic growth that is deterministic, represented by the following system:

$$\begin{cases} dS = \left[rS\left(1 - \frac{S}{K}\right) - \mu S - \frac{\beta SI}{1 + \alpha S}\right]dt, \\ \\ dI = \left[\frac{\beta SI}{1 + \alpha S} - (\mu + \lambda)I + \gamma R\right]dt, \\ \\ dR = \left[\lambda I - (\mu + \gamma)R\right]dt. \end{cases} \quad (1.1)$$

The parameter denoted as r represents the inherent growth rate of the susceptible population, which quantifies the population growth rate unaffected by external factors. In contrast, the variable denoted as K signifies the country carrying capacity, representing the maximum population size that the environment can sustain, excluding the infected and recovered individuals. Through the systematic integration of essential variables and parameters, this sophisticated model is a valuable tool for gaining profound insights into the complex dynamics of infectious diseases, particularly their dissemination patterns within a specific population. It is important to note that the model assumes a closed population system, which means that births, deaths, and migration are not considered. The incidence rate is expressed as $\frac{\beta S}{1 + \alpha S}$, where β represents the rate at which a particular phenomenon is transmitted or propagated, while α denotes the concentration or intensity at which it achieves half its maximum effect or response. In the context of infectious diseases, the parameter λ denotes the recovery rate, indicating the speed at which infected individuals regain their health. On the other hand, the parameter γ represents the transition rate of previously non-infected individuals into a state of infectiousness. This parameter

is crucial in modeling infectious diseases and understanding their spread, as it influences how individuals can become reinfected and continue to spread the disease within a population. System (1.1) consists of positive constant parameters, where $r > \mu$. The basic reproductive number for the system described by equation (1.1) is defined as follows:

$$\mathscr{R}_0 = \frac{\beta S^0}{(1 + \alpha S^0)\left[\lambda + \mu - \frac{\lambda\gamma}{(\mu+\gamma)}\right]}.$$

Variability or random fluctuations frequently impact mathematical models of epidemics or other ecological systems. To account for this variability, several researchers, such as Peng et al. [27], Daqing et al. [13], Lahrouz et al. [28,29], Rajas et al. [23], Zhong et al. [24], and El Idrissi et al. [5,7] have proposed methods to examine the influence of ambient noise on population dynamics. After reviewing the prior discourse, we propose to present a stochastic SIRI model featuring logistic growth that considers the influence of ambient noise on the susceptible population, infected individuals, recovered individuals, and immune segments of the population. We aim to provide a comprehensive and scientifically sound approach to modeling the spread of infectious diseases while also considering the role of external factors such as environmental noise. The stochastic SIRI model, featuring logistic growth, can be expressed in the following manner:

$$\begin{cases} dS = \left[rS\left(1 - \frac{S}{K}\right) - \mu S - \frac{\beta SI}{1 + \alpha S}\right]dt + \sigma_1 SdB_1, \\[2mm] dI = \left[-(\mu+\lambda)I + \frac{\beta SI}{1 + \alpha S} + \gamma R\right]dt + \sigma_2 IdB_2, \\[2mm] dR = [-(\mu+\gamma)R + \lambda I]dt + \sigma_3 RdB_3. \end{cases} \quad (1.2)$$

The stochastic SIRI model with logistic growth employs a set of mutually independent standard Brownian motions denoted as $B_i(t)$, and the white noise intensities are indicated by $\sigma_i > 0$, where $i = 1, 2, 3$. Unless otherwise specified, the article assumes the existence of a complete probability space denoted as $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, which incorporates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ that adheres to the conventional conditions. These prerequisites necessitate the filtration to be both right-continuous and increasing, with $\mathscr{F}_0$ encompassing all $\mathbb{P}$-null sets. Within this space, the standard Brownian motions $B_i(t)$ are established, and the structure

$$\mathbb{R}_+^d = \{(X_1, ..., X_d) \in \mathbb{R}^d | X_i > 0, i = 1, ..., d\}.$$

In a broader context, we investigate the SDE in d-dimensions for any $t \geq t_0$

$$dX = a(X)dt + b(X)dB. \quad (1.3)$$

The provided stochastic differential equation is governed by the initial value condition $X_0 \in \mathbb{R}^d$. The differential operator $\mathscr{L}$ operates on the Lyapunov function $V(X,t) \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}^+)$ within this framework exhibits two continuous derivatives for the state variable X and one continuous derivative to time t. The stochastic differential equation and the associated Lyapunov function are relevant in various scientific disciplines, such as control theory and stochastic processes, aiding in the analysis and understanding of complex dynamic systems. Then, the resulting expression is:

$$\mathscr{L}V(X,t) = \frac{\partial V(X,t)}{\partial t} + \sum_{i=1}^{d} a_i(X,t)\frac{\partial V(X,t)}{\partial X_i}$$
$$+ \frac{1}{2}\sum_{i,j=1}^{d}\left[b^T(X,t)b(X,t)\right]_{ij}\frac{\partial^2 V(X,t)}{\partial X_i \partial X_j}.$$

Or

$$\mathscr{L}V = V_t + V_x a + \frac{1}{2}Tr\left[b^T V_{xx} b\right],$$

where

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_d}\right),$$

and

$$V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d \times d}.$$

By Itô formula, we get

$$dV = \mathscr{L}V.dt + V_x b.dB.$$

The manuscript follows a structured approach, beginning with Section 2, which presents a unique global positive solution for the system described in equation (1.2). In Section 3, we establish sufficient conditions for disease extinction. Following that, in Section 4, we conduct an in-depth analysis of the stochastic system (1.2), aiming to investigate the presence of an ergodic stationary distribution. Subsequently, Section 5 provides an in-depth discussion of our theoretical discoveries and includes numerical simulations to illustrate them. Finally, a concise conclusion succinctly outlines the primary contributions made in this study.

2 The Existence and Uniqueness

In this specialized research section, our primary objective is to conduct a rigorous investigation into the existence and uniqueness of a globally positive solution for the system described in equation (1.2). We will accomplish this by considering all positive initial values within the context. This involves investigating the existence and distinctiveness of such a solution across all possible scenarios.

Theorem 1. *For each $X(0) \in \mathbb{R}_+^3$, the system (1.2) admits a unique and well-defined solution $X(t)$ for all $t \geq 0$ and this solution remains strictly confined within the positive orthant $\mathbb{R}_+^3$ with a probability of 1, i.e., with absolute certainty.*

*Proof.*The local Lipschitz condition is verified by the system coefficients in (1.2), and for each $X(0) \in \mathbb{R}^3_+$, there is a unique local solution for $t \in [0, \tau^*]$ *a.s.*, where τ^* stands for the time of the explosion (see, e.g., [11] for further information). Our current objective is to demonstrate the global nature of this solution, meaning we must establish the continuity and boundedness of the solution beyond the interval $[0, \tau^*]$, i.e.,

$$\tau^* = \infty \quad a.s..$$

In pursuit of our objective, we consider a sufficiently large value for $n_0 \geq 1$ to ensure that S_0, I_0, and R_0 are all within the specified range $\left[\frac{1}{n_0}, n_0\right]$. We define the stopping time as per [11] for every integer $n \geq n_0$,

$$\tau_k = \inf\left\{t \in [0, \tau^*) \mid \min\{S_t, I_t, R_t\} \leq \frac{1}{n} \text{ or }\right.$$
$$\left.\max\{S_t, I_t, R_t\} \geq n\right\}.$$

In the context of this study, we consistently define $\inf \emptyset = \infty$, with $\emptyset$ representing the empty set as conventionally understood. It is evident that τ_n increases as $n \to \infty$. Define

$$\tau_\infty = \lim_{n \to \infty} \tau_n,$$

from which we deduce

$$\tau^* \geq \tau_\infty \quad a.s..$$

To demonstrate that $\tau^* = \infty$ and $(S, I, R) \in \mathbb{R}^3_+$ almost surely (*a.s.*) for each $t \geq 0$, verifying the condition $\tau_\infty = \infty$ *a.s.* is of utmost importance. In case this statement is incorrect, there must exist a pair of constants, $T > 0$ and $\varepsilon \in (0, 1)$, such that

$$\mathbb{P}\{\tau_\infty \leq T\} \geq \varepsilon.$$

Hence, there exists a certain value $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \varepsilon \quad \text{for all } n \geq n_1. \tag{2.1}$$

Consider $\hat{V} \in C^2(\mathbb{R}^3_+; \mathbb{R}_+)$, where

$$\hat{V}(X) = \left[S - a\ln\left(\frac{S}{a}\right) - a\right] + [I - \ln(I) - 1]$$
$$+ [R - \ln(R) - 1].$$

In this particular context, we assign a as a positive constant, the precise value of which will be chosen later. Then, using the Itô formula, we obtain

$$d\hat{V}(X) = \mathscr{L}\hat{V}dt + \sigma_1(S - a)dB_1 + \sigma_2(I - 1)dB_2$$
$$+ \sigma_3(R - 1)dB_3,$$

or

$$\mathscr{L}\hat{V} = \left(1 - \frac{a}{S}\right)\left[rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{1 + \alpha S} - \mu S\right] + \frac{a\sigma_1^2}{2}$$
$$+ \left(1 - \frac{1}{I}\right)\left[\frac{\beta SI}{1 + \alpha S} - (\lambda + \mu)I + \gamma R\right] + \frac{\sigma_2^2}{2}$$
$$+ \left(1 - \frac{1}{R}\right)[\lambda I - (\mu + \gamma)R] + \frac{\sigma_3^2}{2},$$

$$\leq \frac{-rS^2}{K} + \frac{(K + a)r}{K}S + \frac{a\beta I}{1 + \alpha S} + a\mu - \mu I + \lambda$$
$$+ 2\mu + \gamma + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2},$$

$$\leq \sup_{S \in (0, \infty)}\left\{\frac{S}{K}(-rS + (K + a)r)\right\} + (a\beta - \mu)I$$
$$+ a\mu + \lambda + 2\mu + \gamma + \frac{1}{2}\left(a\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right).$$

By choosing $a = \frac{\mu}{\beta}$, we obtain

$$\mathscr{L}\hat{V} \leq \sup_{S \in (0, \infty)}\left\{\frac{S}{K}(-rS + (K + a)r)\right\} + a\mu + \lambda$$
$$+ 2\mu + \gamma + \frac{1}{2}\left(a\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right),$$
$$\cong \hat{K}.$$

Here, $\hat{K}$ represents a positive constant. Hence

$$d\hat{V}(X) \leq \hat{K}dt + \sigma_1(S - a)dB_1(t) + \sigma_2(I - 1)dB_2(t)$$
$$+ \sigma_3(R - 1)dB_3(t). \tag{2.2}$$

By integrating both sides of inequality (2.2) over the interval $(0, \tau_n \wedge T)$ and then taking the expected value, we get

$$\mathbb{E}\hat{V}(X(\tau_n \wedge T)) \leq \hat{V}(X(0)) + \hat{K}\mathbb{E}(\tau_n \wedge T).$$

Hence

$$\mathbb{E}\hat{V}(X(\tau_n \wedge T)) \leq \hat{V}(X(0)) + \hat{K}T. \tag{2.3}$$

Consider $\Omega_n = \{\tau_n \leq T\}$ for each $n \geq n_1$. Based on equation (2.1), it follows that $P(\Omega_n) \geq \varepsilon$. For each $\omega \in \Omega_n$, either $S(\tau_n, \omega)$ or $I(\tau_n, \omega)$ or $R(\tau_n, \omega)$ is equal to either n or $\frac{1}{n}$. Hence

$$\hat{V}[X(\tau_n, \omega)] \geq \left[n - a - a\ln\left(\frac{n}{a}\right)\right] \wedge \left[\frac{1}{n} - a + a\ln(an)\right]$$
$$\wedge [n - 1 - \ln(n)] \wedge \left[\frac{1}{n} - 1 + \ln(n)\right].$$

Using (2.3), we get

$$\hat{V}(X_0) + \hat{K}T \geq \mathbb{E}\left[\mathbb{I}_{\Omega_n(\omega)}\hat{V}(X(\tau_n, \omega))\right],$$
$$\geq \varepsilon\left[n - a - a\ln\frac{n}{a}\right] \wedge \left[\frac{1}{n} - a + a\ln(an)\right]$$
$$\wedge [n - 1 - \ln(n)] \wedge \left[\frac{1}{n} - 1 + \ln(n)\right],$$

where $\mathbb{I}_{\Omega_n}$ represents the characteristic function of set Ω_n. By letting $n \to \infty$, we get

$$\infty > \hat{V}(X(0)) + \widehat{K}T = \infty.$$

This is a contradiction. Hene

$$\tau_\infty = \infty \quad a.s..$$

The proof is completed.

3 Extinction of Disease

This section of epidemic modeling research focuses on exploring the fascinating concept of extinction. It aims to establish the necessary criteria to eradicate the disease being investigated thoroughly.

Theorem 2. *For each $X(0) \in \mathbb{R}_+^3$. If $(r-\mu) > \dfrac{\sigma_1^2}{2}$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \left(\frac{w_1 I(t)}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}} + \frac{w_2 R(t)}{\mu+\gamma} \right) \leq v \quad a.s.,$$

where

$$v = \min\left\{ \mu+\gamma; (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0} \right\}$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}\leq 1\}}$$

$$+ \max\left\{ \mu+\gamma; (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0} \right\}$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}>1\}}$$

$$+ \frac{\beta K \sigma_1}{r(1+\alpha S^0)} \sqrt{\frac{r-\mu}{2}} - \left[2\left(\sigma_2^{-2} + \sigma_3^{-2} \right) \right]^{-1}.$$

Especially if $v < 0$, the populations of I and R are exponentially extinguished with probability 1, that is

$$\lim_{t\to\infty} I(t) = 0 \quad and \quad \lim_{t\to\infty} R(t) = 0 \quad a.s..$$

Furthermore, the distribution denoted as $S(t)$ exhibits weak convergence to the measure characterized by its highest density

$$\pi(x) = Qx^{\frac{2(r-\mu)}{\sigma_1^2} - 2} \exp\left(\frac{-2rx}{\sigma_1^2 K} \right), \quad x \in [0,\infty),$$

where

$$Q = \left[\sigma_1^{-2} \left(\frac{K\sigma_1^2}{2r} \right)^{\frac{2(r-\mu)}{\sigma_1^2} - 1} \Gamma\left(\frac{2(r-\mu)}{\sigma_1^2} - 1 \right) \right]^{-1},$$

as a constant that satisfies $\displaystyle\int_0^\infty \pi(x)dx = 1$.

Proof. Since the system (1.2) has a positive solution for all $X(0) \in \mathbb{R}_+^3$, we get

$$dS \leq \left[rS\left(1 - \frac{S}{K} \right) - \mu S \right] dt + \sigma_1 S dB_1. \tag{3.1}$$

Assume the auxiliary logistic equation with the random perturbations shown below:

$$dX = \left[rX\left(1 - \frac{X}{K} \right) - \mu X \right] dt + \sigma_1 X dB_1(t). \tag{3.2}$$

Let for all $x \in [0,\infty)$

$$b(x) = rx\left(1 - \frac{x}{K} \right) - \mu x \quad and \quad \sigma(x) = x\sigma_1(x).$$

We have

$$\int \frac{b(u)}{\sigma^2(u)} du = \frac{1}{\sigma_1^2} \int \left[\frac{r}{u}\left(1 - \frac{u}{K} \right) - \frac{\mu}{u} \right] du,$$

$$= \frac{1}{\sigma_1^2} \left[(r-\mu)\ln(x) - \frac{rx}{K} \right] + Q.$$

Hence

$$\exp\left(\int \frac{b(u)}{\sigma^2(u)} du \right) = x^{\frac{r-\mu}{\sigma_1^2}} \exp\left(Q - \frac{r}{\sigma_1^2 K} x \right).$$

Clearly, we have

$$\int_0^\infty \frac{1}{\sigma^2(x)} \exp\left(\int_1^x \frac{2b(u)}{\sigma^2(u)} du \right) dx$$

$$= C \int_0^\infty x^{-2 + \frac{2(r-\mu)}{\sigma_1^2}} \exp\left(\frac{-2r}{\sigma_1^2 K} x \right) dx < \infty, \tag{3.3}$$

where $C = \sigma_1^{-2} \exp(2Q)$ is a constant.
Thus, the requirement of Theorem 1.16 in [30] derives from (3.3). Hence, the stationary system (3.2) solution has the density provided by

$$\pi(x) = Q\sigma_1^{-2} x^{2(r-\mu)\sigma_1^{-2} - 2} \exp\left(\frac{-2r}{\sigma_1^2 K} x \right), \quad x \in [0,\infty),$$

where Q is a constant satisfying $\displaystyle\int_0^\infty \pi(x)dx = 1$ such that

$$Q = \left[\sigma_1^{-2} \left(\frac{\sigma_1^2 K}{2r} \right)^{\frac{2(r-\mu)}{\sigma_1^2} - 1} \Gamma\left(\frac{2(r-\mu)}{\sigma_1^2} - 1 \right) \right]^{-1}.$$

Using the ergodic theorem, we can conclude

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds = \int_0^\infty x\pi(x)\,dx \quad a.s.. \tag{3.4}$$

By using the stochastic differential equations (SDE) comparison theorem, as outlined in the work of Peng [27], it becomes possible to establish the characteristics of the solution $X(t)$ for equation (3.2), where $S(0) > 0$. For each $t \geq 0$

$$X(t) \geq S(t) \quad a.s.. \tag{3.5}$$

Also, we have

$$J_1 = \int_0^\infty x\pi(x)\,dx,$$

$$= Q\sigma_1^{-2}\int_0^\infty x^{-1+\frac{2(r-\mu)}{\sigma_1^2}}\exp\left(\frac{-2r}{\sigma_1^2 K}x\right)dx,$$

$$= Q\sigma_1^{-2}\int_0^\infty \left(\frac{\sigma_1^2 K}{2r}t\right)^{-1+2\sigma_1^{-2}(r-\mu)}$$

$$\times \exp\left(\frac{-\sigma_1^2 K}{2r}t\right)dt,$$

$$= Q\sigma_1^{-2}\left(\frac{\sigma_1^2 K}{2r}\right)^{\frac{2(r-\mu)}{\sigma_1^2}}\Gamma\left(\frac{2(r-\mu)}{\sigma_1^2}\right),$$

$$= \frac{\sigma_1^2 K}{2r}\frac{\Gamma\left(\frac{2(r-\mu)}{\sigma_1^2}\right)}{\Gamma\left(-1+\frac{2(r-\mu)}{\sigma_1^2}\right)},$$

$$= \frac{\sigma_1^2 K}{2r}\left[-1+\frac{2(r-\mu)}{\sigma_1^2}\right],$$

$$= \frac{K}{r}\left[(r-\mu)-\frac{\sigma_1^2}{2}\right].$$

And

$$J_2 = \int_0^\infty x^2\pi(x)\,dx,$$

$$= Q\sigma_1^{-2}\int_0^\infty x^{\frac{2(r-\mu)}{\sigma_1^2}}\exp\left(\frac{-2r}{\sigma_1^2 K}x\right)dx,$$

$$= Q\sigma_1^{-2}\int_0^\infty \left(\frac{\sigma_1^2 K}{2r}\right)^{\frac{2(r-\mu)}{\sigma_1^2}}t^{\frac{2(r-\mu)}{\sigma_1^2}}$$

$$\times \exp\left(\frac{-\sigma_1^2 K}{2r}t\right)dt,$$

thus

$$J_2 = Q\sigma_1^{-2}\left(\frac{\sigma_1^2 K}{2r}\right)^{\frac{2(r-\mu)}{\sigma_1^2}+1}\Gamma\left(\frac{2(r-\mu)}{\sigma_1^2}+1\right),$$

$$= \left(\frac{\sigma_1^2 K}{2r}\right)^2\frac{\Gamma\left(\frac{2(r-\mu)}{\sigma_1^2}+1\right)}{\Gamma\left(\frac{2(r-\mu)}{\sigma_1^2}-1\right)},$$

$$= \left(\frac{\sigma_1^2 K}{2r}\right)^2\frac{2(r-\mu)}{\sigma_1^2}\left[-1+\frac{2(r-\mu)}{\sigma_1^2}\right],$$

$$= \frac{K^2(r-\mu)}{r^2}\left[(r-\mu)-\frac{\sigma_1^2}{2}\right].$$

Then

$$\int_0^\infty \left(x-S^0\right)^2\pi(x)\,dx$$

$$= \int_0^\infty \left[x^2 - 2S^0 x + (S^0)^2\right]\pi(x)\,dx,$$

$$= J_2 - 2S^0 J_1 + \left(S^0\right)^2,$$

$$= \frac{K^2\sigma_1^2}{2r^2}(r-\mu).$$

Also, let

$$M_0 = \begin{pmatrix} 0 & \mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0} \\[2ex] \dfrac{\lambda}{(\mu+\gamma)} & 0 \end{pmatrix},$$

there exists a left eigenvector corresponding to

$$\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{(\mu+\lambda)-\dfrac{\beta S^0}{1+\alpha S^0}}},$$

which is represented by

$$(w_1, w_2) = \left(\lambda(\mu+\gamma)^{-1}; \sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}}\right),$$

thus,

$$\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}}(w_1, w_2) = (w_1, w_2)M_0.$$

Let $P \in \mathscr{C}^2\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ such that

$$P(R,I) = \alpha_1 I + \alpha_2 R,$$

where

$$\alpha_1 = \frac{w_1}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}, \quad \alpha_2 = \frac{w_2}{\mu+\gamma}.$$

Using the formula of Itô for the function $\ln(P)$, we get

$$
d(\ln P) = \mathscr{L}(\ln P).dt + \frac{1}{P}\left[\alpha_1\sigma_2 I dB_2(t) \right.
$$
$$
\left. + \alpha_2\sigma_3 R dB_3(t)\right], \tag{3.6}
$$

where

$$
\mathscr{L}(\ln P) = \frac{\alpha_1}{P}\left[\frac{\beta S I}{1+\alpha S} - (\lambda+\mu)I + \gamma R\right]
$$
$$
+ \frac{\alpha_2}{P}\left[\lambda I - (\mu+\gamma)R\right] - \frac{\alpha_1^2\sigma_2^2 I^2}{2P^2} - \frac{\alpha_2^2\sigma_3^2 R^2}{2P^2}.
$$

Moreover, we have

$$
P^2 = \left(\alpha_1\sigma_2 I \frac{1}{\sigma_2} + \alpha_2\sigma_3 R\frac{1}{\sigma_3}\right)^2,
$$
$$
\leq \left(\alpha_1^2\sigma_2^2 I^2 + \alpha_2^2\sigma_3^2 R^2\right)\left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}\right). \tag{3.7}
$$

And

$$
\frac{1}{P}\left\{\alpha_1\left[\frac{\beta SI}{1+\alpha S} + \gamma R - (\lambda+\mu)I\right]\right.
$$
$$
\left. + \alpha_2\left[-(\mu+\gamma)R + \lambda I\right]\right\}
$$
$$
= \frac{\alpha_1 I}{P}\left(\frac{\beta S}{1+\alpha S} - \frac{\beta S^0}{1+\alpha S^0}\right)
$$
$$
+ \frac{1}{P}\left\{\alpha_1\left[\frac{\beta S^0}{1+\alpha S}I + \gamma R - (\lambda+\mu)I\right]\right.
$$
$$
\left. + \alpha_2\left[-(\mu+\gamma)R + \lambda I\right]\right\},
$$
$$
= \frac{\alpha_1\beta I}{P}\frac{S - S^0}{(1+\alpha S)(1+\alpha S^0)}
$$
$$
+ \frac{1}{P}\left\{\alpha_1\left[\frac{\beta S^0}{1+\alpha S}I + \gamma R - (\lambda+\mu)I\right]\right.
$$
$$
\left. + \alpha_2\left[-(\mu+\gamma)R + \lambda I\right]\right\},
$$
$$
\leq \frac{\alpha_1\beta I}{P}\frac{X - S^0}{(1+\alpha S)(1+\alpha S^0)}
$$
$$
+ \frac{1}{P}\left\{\frac{w_1}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}\left[\frac{\beta S^0}{1+\alpha S}I + \gamma R - (\lambda+\mu)I\right]\right.
$$

$$
\left. + \frac{w_2}{\mu+\gamma}\left[-(\mu+\gamma)R + \lambda I\right]\right\},
$$
$$
\leq \frac{\alpha_1\beta I}{P}\frac{X - S^0}{(1+\alpha S)(1+\alpha S^0)}
$$
$$
+ \frac{1}{P}\left\{\frac{w_1}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}\left[\gamma R - \left(\lambda+\mu - \frac{\beta S^0}{1+\alpha S}\right)I\right]\right.
$$
$$
\left. + \frac{w_2}{\mu+\gamma}\left[\lambda I - (\mu+\gamma)R\right]\right\},
$$
$$
\leq \frac{\beta}{1+\alpha S^0}|X - S^0| + \frac{1}{P}(w_1, w_2)\left[M_0(I,R)^T - (I,R)^T\right],
$$
$$
\cong \frac{\beta}{1+\alpha S^0}|X - S^0| + \frac{1}{P}\left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1\right)
$$
$$
\times (w_1 I + w_2 R),
$$

hence
$$
\frac{1}{P}\left\{\alpha_1\left[\frac{\beta SI}{1+\alpha S} + \gamma R - (\lambda+\mu)I\right]\right.
$$
$$
\left. + \alpha_2\left[\lambda I - (\mu+\gamma)R\right]\right\}
$$
$$
\cong \frac{\beta}{1+\alpha S^0}|X - S^0| + \frac{1}{P}\left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1\right)
$$
$$
\times \left[\left(\mu+\lambda - \frac{\beta S^0}{1+\alpha S^0}\right)\alpha_1 I + (\mu+\gamma)\alpha_2 R\right],
$$
$$
\leq \frac{\beta}{1+\alpha S^0}|X - S^0| + \min\left\{\mu+\lambda - \frac{\beta S^0}{1+\alpha S^0}; \mu+\gamma\right\}
$$
$$
\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1\right)\mathbb{I}_{\{\sqrt{R_0}\leq 1\}}
$$
$$
+ \max\left\{(\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; (\mu+\gamma)\right\}
$$
$$
\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda - \dfrac{\beta S^0}{1+\alpha S^0}}} - 1\right)\mathbb{I}_{\{\sqrt{R_0}>1\}}. \tag{3.8}
$$

In view of (3.7) and (3.8), we get
$$
\mathscr{L}(\ln P) \leq \frac{\beta}{1+\alpha S^0}|X - S^0|
$$
$$
+ \min\left\{(\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; (\mu+\gamma)\right\}
$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}\leq 1\}}$$

$$+ \max\left\{ (\mu+\gamma); (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0} \right\}$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}> 1\}}$$

$$- \left[2\left(\sigma_2^{-2}+\sigma_3^{-2}\right)\right]^{-1}. \tag{3.9}$$

According to (3.6), we have

$$d(\ln P) \leq \min\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; \mu+\gamma \right\} \tag{3.10}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}\leq 1\}}$$

$$+ \max\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; (\mu+\gamma) \right\}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}> 1\}}$$

$$+ \frac{\beta}{1+\alpha S^0}|X - S^0| - \left[2(\sigma_2^{-2}+\sigma_3^{-2})\right]^{-1} dt$$

$$+ \frac{1}{P}\left(\alpha_1\sigma_2 I dB_2(t) + \alpha_2\sigma_3 R dB_3 \right). \tag{3.11}$$

Integrating (3.10) between $(0,t)$ and multiplying both sides by $\dfrac{1}{t}$ yields

$$\frac{1}{t}\ln P(t) - \frac{1}{t}\ln P(0)$$

$$\leq \min\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; \mu+\gamma \right\}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}\leq 1\}}$$

$$+ \max\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; (\mu+\gamma) \right\}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}> 1\}}$$

$$+ \frac{\beta}{(1+\alpha S^0)t}\int_0^t |X(s) - S^0|ds$$

$$- \left[2(\sigma_2^{-2}+\sigma_3^{-2})\right]^{-1} + \frac{1}{t}\int_0^t \frac{\alpha_1\sigma_2 I(s)}{P(s)}dB_2(s)$$

$$+ \frac{1}{t}\int_0^t \frac{\alpha_2\sigma_3 R(s)}{P(s)}dB_3(s),$$

so

$$\frac{1}{t}\ln P(t) - \frac{1}{t}\ln P(0)$$

$$\leq \min\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; \mu+\gamma \right\}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}\leq 1\}}$$

$$+ \max\left\{ (\mu+\lambda) - \frac{\beta S^0}{1+\alpha S^0}; (\mu+\gamma) \right\}$$

$$\times \left(\sqrt{\frac{\dfrac{\lambda\gamma}{\mu+\gamma}}{\mu+\lambda-\dfrac{\beta S^0}{1+\alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0}> 1\}}$$

$$+ \frac{\beta}{(1+\alpha S^0)t}\int_0^t |X(s) - S^0| ds$$

$$- \left[2(\sigma_2^{-2}+\sigma_3^{-2})\right]^{-1} + \frac{M_1}{t} + \frac{M_2}{t}, \tag{3.12}$$

where M_1, M_2 represent the local martingales such that

$$M_1(t) = \frac{\alpha_1\sigma_2}{t}\int_0^t \frac{I(s)}{P(s)}dB_2(s),$$

and

$$M_2(t) = \frac{\alpha_2\sigma_3}{t}\int_0^t \frac{R(s)}{P(s)}dB_3(s).$$

And their quadratic variations are the following:

$$<M_1,M_1>_t = \sigma_2^2\int_0^t \left(\frac{\alpha_1 I(s)}{P(s)} \right)^2 ds,$$

$$\leq \sigma_2^2 t,$$

and

$$<M_2,M_2>_t = \sigma_3^2\int_0^t \left(\frac{\alpha_2 R(s)}{P(s)} \right)^2 ds,$$

$$\leq \sigma_3^2 t.$$

Using the strong law of large numbers for local martingales [11], we obtain

$$\lim_{t \to \infty} \frac{1}{t} M_i(t) = 0 \quad a.s., \quad i = 1, 2. \tag{3.13}$$

Or $\int_0^t x\pi(x)\,dx < \infty$ and $X(t)$ is ergodic, we derive

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t |X(s) - S^0|ds = \int_0^\infty |x - S^0|\pi(x)dx, \tag{3.14}$$

$$\leq \left(\int_0^\infty (x - S^0)^2 \pi(x)dx \right)^{\frac{1}{2}}.$$

Using the upper limit on both sides of (3.12) and combining it with (3.13) and (3.14), we obtain

$$\limsup_{t \to \infty} \frac{1}{t} \ln(P(t))$$

$$\leq \min\left\{ \mu + \gamma; (\mu + \lambda) - \frac{\beta S^0}{1 + \alpha S^0} \right\}$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu + \lambda - \frac{\beta S^0}{1 + \alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0} \leq 1\}}$$

$$+ \max\left\{ (\mu + \gamma); (\mu + \lambda) - \frac{\beta S^0}{1 + \alpha S^0} \right\}$$

$$\times \left(\sqrt{\frac{\lambda\gamma(\mu+\gamma)^{-1}}{\mu + \lambda - \frac{\beta S^0}{1 + \alpha S^0}}} - 1 \right) \mathbb{I}_{\{\sqrt{R_0} > 1\}} \tag{3.15}$$

$$+ \frac{\beta}{(1 + \alpha S^0)} \left(\frac{K^2 \sigma_1^2}{2r^2}(r - \mu) \right)^{\frac{1}{2}}$$

$$- \left[2(\sigma_2^{-2} + \sigma_3^{-2}) \right]^{-1},$$

$$\cong v \quad a.s., \tag{3.16}$$

and this is the necessary assertion. Also, if $v < 0$, we get

$$\limsup_{t \to \infty} \frac{1}{t} \ln(I(t)) < 0 \quad a.s.,$$

and

$$\limsup_{t \to \infty} \frac{1}{t} \ln(R(t)) < 0 \quad a.s..$$

Hence

$$\lim_{t \to \infty} I(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} R(t) = 0 \quad a.s..$$

4 The Stationary Distribution

In this section, our primary objective is to rigorously establish empirical evidence substantiating the existence of an ergodic stationary distribution in the solution of the epidemic model represented by equation (1.2). This result is based on the well-established theorem presented by Khasminskii [31]. We investigate the time-homogeneous Markov process $X(t)$ within the d-dimensional space $\mathbb{R}^d$. The process is defined by

$$dX = b(X)dt + \sum_{r=1}^{k} \sigma_r(X)dB_r.$$

The stochastic process, represented by $X(t)$, possesses a diffusion matrix, which can be defined as follows:

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

Lemma 1. *If there exists an open bounded field $U \subset \mathbb{R}^d$ with smooth bounded Γ, then the Markov process $X(t)$ will have a unique ergodic stationary distribution $\mu(.)$, satisfying the following conditions:*
(C_1) The matrix of diffusion A is defined and strictly positive, for each $x \in U$,
(C_2) There is a non-negative $V \in \mathscr{C}^2$, for each $x \in U^c$, such that

$$\mathscr{L}V(x) < 0, \quad x \in U^c.$$

Define the following important parameters

$$\mathscr{R}_0^S = \frac{\beta S^0 \left(1 + \alpha S^0\right)^{-1}}{\left(\lambda + \mu + \frac{\sigma_2^2}{2} + c_3 \frac{S^0 \sigma_1^2}{2} - \frac{\lambda\gamma}{\left(\mu + \gamma + \frac{\sigma_3^2}{2}\right)} \right)},$$

where $S^0 = \frac{K(r - \mu)}{r}$ and c_3 are positive constants that satisfy the criterion of Theorem 4.1.

Theorem 3. *If $\mathscr{R}_0^S > 1$ and there is a constant c_3 such as*

$$c_3 > \frac{\beta K}{r} \left[\alpha + \frac{1}{S^0(1 + \alpha S^0)^2} \right].$$

Then, for each $X(0) \in \mathbb{R}_+^3$, the system (1.2) possesses a unique stationary distribution $\mu(.)$ and is ergodic.

Proof. Let $V_1 \in \mathscr{C}^2 \left(\mathbb{R}_+^3, \mathbb{R} \right)$ such that

$$V_1(X) = \ln\left(\frac{1}{I}\right) + c_1 \ln\left(\frac{1}{R}\right) - c_2 S + c_3 \left[S - S^0 \right.$$

$$\left. - S^0 \ln\left(\frac{S}{S^0}\right) \right] + \frac{c_3 \beta S^0}{\mu}(I + R),$$

where c_1, c_2, and c_3 represent positive constants, which will be determined subsequently. By using the Itô formula,

one obtains

$$\mathscr{L}V_1 = \frac{-1}{I}\left[\frac{\beta SI}{1+\alpha S} - (\lambda+\mu)I + \gamma R\right] + \frac{\sigma_2^2}{2} + \frac{c_1\sigma_3^2}{2}$$
$$-\frac{c_1}{R}[\lambda I - (\mu+\gamma)R] - c_2\left[r\left(1-\frac{S}{K}\right)S - \mu S\right.$$
$$\left.-\frac{\beta SI}{1+\alpha S}\right] + c_3\left[r\left(1-\frac{S}{K}\right)S - \mu S - \frac{\beta SI}{1+\alpha S}\right]$$
$$-\frac{c_3 S^0}{S}\left[r\left(1-\frac{S}{K}\right)S - \mu S - \frac{\beta SI}{1+\alpha S}\right]$$
$$+\frac{c_3 S^0\sigma_1^2}{2} + \frac{c_3\beta^2 S^0 SI}{\mu(1+\alpha S)} - c_3\beta S^0 I - c_3\beta S^0 R,$$
$$\leq \frac{-\beta S}{1+\alpha S} + \lambda+\mu - \frac{\gamma R}{I} + \frac{\sigma_2^2}{2} - \frac{c_1\lambda I}{R} + c_1(\mu+\gamma)$$
$$+\frac{c_1\sigma_3^2}{2} + c_3\left[rS\left(1-\frac{S}{K}\right) - \mu S - rS^0\left(1-\frac{S}{K}\right)\right.$$
$$\left.+\mu S^0\right] - c_2\left[rS\left(1-\frac{S}{K}\right) - \mu S\right] + \frac{c_2\beta SI}{1+\alpha S}$$
$$+\frac{c_3\beta S^0 I}{1+\alpha S} + \frac{c_3 S^0\sigma_1^2}{2} + \frac{c_3\beta^2}{\mu(1+\alpha S)}S^0 SI$$
$$-c_3\beta S^0 I.$$

Hence

$$\mathscr{L}V_1 \leq -2\sqrt{c_1\lambda\gamma} + \left(\lambda+\mu+\frac{\sigma_2^2}{2} + c_3\frac{S^0\sigma_1^2}{2}\right)$$
$$+c_1\left(\gamma+\mu+\frac{\sigma_3^2}{2}\right) - \frac{\beta S}{1+\alpha S} + \frac{rc_2}{K}S\left(S-S^0\right)$$
$$+c_3\left[\frac{-r}{K}\left(S-S^0\right)^2\right] + c_3\beta S^0 I + \frac{c_2\beta SI}{1+\alpha S}$$
$$+\frac{c_3\beta^2 S^0 SI}{\mu(1+\alpha S)} - c_3\beta S^0 I,$$
$$\leq -2\sqrt{c_1\lambda\gamma} + \left[\lambda+\mu+\frac{\sigma_2^2}{2} + c_3\frac{S^0\sigma_1^2}{2}\right]$$
$$+c_1\left(\mu+\gamma+\frac{\sigma_3^2}{2}\right) + \left[\frac{-\beta S}{1+\alpha S} + \frac{c_2 r}{K}\left(S-S^0\right)S\right.$$
$$\left.-\frac{rc_3}{K}\left(S-S^0\right)^2\right] + \left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right)SI.$$

Choosing

$$c_1 = \lambda\gamma\left(\mu+\gamma+\frac{\sigma_3^2}{2}\right)^{-2}.$$

Then

$$\mathscr{L}V_1 \leq -\lambda\gamma\left(\mu+\gamma+\frac{\sigma_3^2}{2}\right)^{-1} + \left[\lambda+\mu\right.$$
$$+\frac{\sigma_2^2}{2} + c_3\frac{S^0\sigma_1^2}{2}\right] - \frac{\beta S^0}{1+\alpha S^0} + \left[\frac{\beta S^0}{1+\alpha S^0}\right.$$
$$-\frac{\beta S}{1+\alpha S} + \frac{rc_2}{K}\left(S-S^0\right)S - \frac{rc_3}{K}\left(S-S^0\right)^2\right]$$
$$+\left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right)SI,$$
$$\simeq -\left(\mathscr{R}_0^S - 1\right)\left[\lambda+\mu+\frac{\sigma_2^2}{2} + c_3\frac{S^0\sigma_1^2}{2} - \lambda\gamma\right.$$
$$\times\left(\mu+\gamma+\frac{\sigma_3^2}{2}\right)^{-1}\right] + \left[\frac{\beta S^0}{1+\alpha S^0} - \frac{\beta S}{1+\alpha S}\right.$$
$$+\frac{rc_2}{K}S\left(S-S^0\right) - \frac{rc_3}{K}\left(S-S^0\right)^2\right]$$
$$+\left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right)SI,$$
$$\cong -\tilde{\lambda} + G(S) + \left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right)SI,$$

where

$$\tilde{\lambda} = \left[\lambda+\mu+\frac{\sigma_2^2}{2} + c_3\frac{S^0\sigma_1^2}{2} - \lambda\gamma\left(\mu+\gamma+\frac{\sigma_3^2}{2}\right)^{-1}\right]$$
$$\times\left(\mathscr{R}_0^S - 1\right) > 0,$$

and

$$G(S) = \frac{\beta S^0}{1+\alpha S^0} - \frac{\beta S}{1+\alpha S} + \frac{rc_2}{K}S\left(S-S^0\right)$$
$$-\frac{rc_3}{K}\left(S-S^0\right)^2.$$

Then

$$G'(S) = \frac{-\beta}{(1+\alpha S)^2} + \frac{rc_2}{K}\left(2S-S^0\right)$$
$$-\frac{2rc_3}{K}\left(S-S^0\right).$$

Let $G'(S)_{|S=S^0} = 0$, we have

$$c_2 = \frac{\beta K}{rS^0\left(1+\alpha S^0\right)^2}.$$

Moreover, we obtain

$$G''(S) = \frac{2\alpha\beta}{(1+\alpha S)^3} + \frac{2rc_2}{K} - \frac{2rc_3}{K},$$
$$\leq 2\alpha\beta + \frac{2r}{K}\left(c_2 - c_3\right) < 0.$$

The validity of the second inequality is established based on the conditions stated in Theorem 4.1. Hence

$$c_3 > c_2 + \frac{\alpha\beta K}{r},$$
$$= \frac{\beta K}{r}\left(\alpha + \frac{1}{S^0(1+\alpha S^0)^2}\right),$$

and

$$G(S) \le G(S^0) = 0.$$

Thus

$$\mathcal{L}V_1 \le -\tilde{\lambda} + \left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right)SI, \qquad (4.1)$$

$$= -\tilde{\lambda} + \tilde{K}SI,$$

where

$$\tilde{K} \cong \left(c_2\beta + \frac{c_3\beta^2 S^0}{\mu}\right) > 0.$$

Define

$$V_2(R) = \ln\left(\frac{1}{R}\right), \quad V_3(X) = \frac{1}{m+1}(S+I+R)^{m+1},$$

where m stands for a positive constant that satisfies

$$0 < m < \frac{\mu}{2 \times 3^m(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)}.$$

Using the formula of Itô to V_2 and V_3 respectively, we get

$$\mathcal{L}V_2 = \mu + \gamma + \frac{\sigma_2^3}{2} - \frac{\lambda I}{R}, \qquad (4.2)$$

and

$$\mathcal{L}V_3 = (S+I+R)^m\left[rS\left(1-\frac{S}{K}\right) - \mu S - \mu I - \mu R\right]$$
$$+ \frac{m}{2}(S+I+R)^{m-1}\left(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 R^2\right),$$
$$\le (S+I+R)^m\left[rS\left(1-\frac{S}{K}\right) - \mu I - \mu R\right]$$
$$+ \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)(S+I+R)^{m+1},$$
$$\le rS(S+I+R)^m - \frac{r}{K}S^{m+2} - \mu I^{m+1} - \mu R^{m+1}$$
$$+ \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)(S+I+R)^{m+1}.$$

Using $\left|\sum_{i=1}^k a_i\right|^n \le k^{n-1}\sum_{i=1}^k |a_i|^n$ for any $n \ge 1$, we get

$$\mathcal{L}V_3 \le \frac{-r}{2K}S^{m+2} - \frac{\mu}{2}I^{m+1} - \frac{\mu}{2}R^{m+1} - \frac{r}{2K}S^{m+2}$$
$$- \frac{\mu}{2}I^{m+1} - \frac{\mu}{2}R^{m+1} + rS(S+I+R)^m$$
$$+ \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)(S^{m+1} + I^{m+1}$$
$$+ R^{m+1}),$$
$$\cong \frac{-r}{2K}S^{m+2} - \frac{\mu}{2}I^{m+1} - \frac{\mu}{2}R^{m+1} - \frac{r}{2K}S^{m+2}$$
$$- \frac{\mu}{4}I^{m+1} + \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)S^{m+1}$$
$$+ r(S+I+R)^m S - \frac{\mu}{4}R^{m+1}$$
$$- \left(\frac{\mu}{4} - \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right)I^{m+1}$$
$$- \left(\frac{\mu}{4} - \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right)R^{m+1},$$
$$\le \frac{-r}{2K}S^{m+2} - \frac{\mu}{2}I^{m+1} - \frac{\mu}{2}R^{m+1} - \frac{r}{2K}S^{m+2}$$
$$+ \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)S^{m+1} + rS(S+I+R)^m$$
$$- \frac{\mu}{4}I^{m+1} - \frac{\mu}{4}R^{m+1},$$

$$\mathcal{L}V_3 \le -\frac{r}{2K}S^{m+2} - \frac{\mu}{2}I^{m+1} - \frac{\mu}{2}R^{m+1} + B, \qquad (4.3)$$

where

$$B = \sup_{(S,I,R)\in\mathbb{R}_+^3}\left\{-\frac{r}{2K}S^{m+2} + \frac{3^m m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right.$$
$$\times S^{m+1} + r(S+I+R)^m S - \frac{\mu}{4}I^{m+1}$$
$$\left.- \frac{\mu}{4}R^{m+1}\right\}.$$

Set the $\mathscr{C}^2-$ function $\tilde{V} : \mathbb{R}_+^3 \longrightarrow \mathbb{R}$

$$\tilde{V}(X) = MV_1(X) + V_2(R) + V_3(X),$$

where M represents a strictly positive number sufficiently large to satisfy the condition:

$$\mu + \gamma + \frac{\sigma_3^2}{2} - M\tilde{\lambda} + B \le -2. \qquad (4.4)$$

Furthermore, it is essential to remark that $\tilde{V}$ is both continuous and unbounded, tending towards infinity as the values of X approach the boundary of $\mathbb{R}_+^3$. As a result, the function must be lower-bounded at a point X_0 within $\mathbb{R}_+^3$. Set the function $V \in \mathscr{C}^2\left(\mathbb{R}_+^3, \mathbb{R}_+ \cup \{0\}\right)$ such that

$$V(S,I,R) = \tilde{V}(S,I,R) - \tilde{V}(S_0,I_0,R_0).$$

Using the formula of Itô, we obtain

$$\mathcal{L}V \le -M\tilde{\lambda} + M\tilde{K}SI - \frac{\lambda I}{R} - \frac{r}{2K}S^{m+2} - \frac{\mu}{2}\left(I^{m+1}\right.$$
$$\left.+ R^{m+1}\right) + B + \mu + \gamma + \frac{\sigma_3^2}{2}.$$

Using the previous results, we can now construct a bounded open set U_ε that satisfies condition $(C2)$ from Lemma 1. Specifically, we define U_ε as bellow

$$U_\varepsilon = \left\{ X \in \mathbb{R}_+^3 : \varepsilon < S < \frac{1}{\varepsilon}, \varepsilon < I < \frac{1}{\varepsilon}, \varepsilon^2 < R < \frac{1}{\varepsilon^2} \right\},$$

where $0 < \varepsilon < 1$ is a sufficiently small number. On the set $\mathbb{R}_+^3 \setminus U_\varepsilon$, ε may be chosen to be sufficiently small for the next conditions

$$\varepsilon < \frac{m+1}{M\tilde{K}m}, \tag{4.5}$$

$$\varepsilon < \frac{\mu(m+1)}{2M\tilde{K}}, \tag{4.6}$$

$$\varepsilon < \frac{m+2}{M\tilde{K}(m+1)}, \tag{4.7}$$

$$\varepsilon < \frac{r(m+2)}{2KM\tilde{K}}, \tag{4.8}$$

$$\frac{-\lambda}{\varepsilon} + D \leq -1, \tag{4.9}$$

$$\frac{-r}{4K\varepsilon^{m+2}} + D \leq -1, \tag{4.10}$$

$$\frac{-\mu}{4\varepsilon^{m+1}} + D \leq -1, \tag{4.11}$$

$$\frac{-\mu}{2\varepsilon^{2m+2}} + D \leq -1, \tag{4.12}$$

where D is a positive constant that we define explicitly as (4.16). For simplicity, we can divide $\mathbb{R}_+^3 \setminus U_\varepsilon$ into six areas:

$$U_1 = \left\{ X \in \mathbb{R}_+^3 : S \leq \varepsilon \right\}, \quad U_2 = \left\{ X \in \mathbb{R}_+^3 : I \leq \varepsilon \right\},$$
$$U_3 = \left\{ X \in \mathbb{R}_+^3 : I > \varepsilon, R \leq \varepsilon^2 \right\},$$
$$U_4 = \left\{ X \in \mathbb{R}_+^3 : S \geq \frac{1}{\varepsilon} \right\}, U_5 = \left\{ X \in \mathbb{R}_+^3 : I \geq \frac{1}{\varepsilon} \right\},$$
$$U_6 = \left\{ X \in \mathbb{R}_+^3 : R \geq \frac{1}{\varepsilon^2} \right\}.$$

We have

$$\mathbb{R}_+^3 \setminus U_\varepsilon = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6.$$

Then, we have to prove that $\mathscr{L}V(X) \leq -1$ on U_ε^c is equivalent to show it for the six areas mentioned above, respectively.

Case 1. For each $(S,I,R) \in U_1$, we have

$$SI \leq \varepsilon I \leq \varepsilon \frac{m + I^{m+1}}{m+1} = \frac{\varepsilon m}{m+1} + \frac{\varepsilon}{m+1} I^{m+1},$$

thus

$$\mathscr{L}V \leq -M\tilde{\lambda} + \frac{M\tilde{K}m\varepsilon}{m+1} - \left(\frac{\mu}{2} - \frac{M\tilde{K}\varepsilon}{m+1} \right) I^{m+1} + B$$
$$+ \mu + \gamma + \frac{\sigma_3^2}{2},$$
$$\leq -1,$$

which is deduced from (4.4), (4.5) and (4.6). Hence

$$\mathscr{L}V(X) \leq -1 \quad \text{for each} \quad X \in U_1. \tag{4.13}$$

Case 2. For every $(S,I,R) \in U_2$, since

$$SI \leq \varepsilon S \leq \varepsilon \frac{m+1+S^{m+2}}{m+2} = \frac{\varepsilon(m+1)}{m+2} + \frac{\varepsilon}{m+2} S^{m+2},$$

hence

$$\mathscr{L}V \leq -M\tilde{\lambda} + \frac{M\tilde{K}(m+1)\varepsilon}{m+2} - \left(\frac{r}{2K} - \frac{M\tilde{K}\varepsilon}{m+2} \right) S^{m+2}$$
$$+ B + \mu + \gamma + \frac{\sigma_3^2}{2},$$
$$\leq -2 + 1 + 0 = -1,$$

which is deduced from (4.5), (4.7) and (4.8). Therefore

$$\mathscr{L}V(X) \leq -1 \quad \text{for all} \quad X \in U_2. \tag{4.14}$$

In addition, we have

$$\mathscr{L}V \leq -\frac{\lambda I}{R} - \frac{r}{4K} S^{m+2} - \frac{\mu}{4} I^{m+1} - \frac{\mu}{2} R^{m+1}$$
$$+ M\tilde{K}SI - \frac{r}{4K} S^{m+2} - \frac{\mu}{4} I^{m+1} + B + \mu$$
$$+ \gamma + \frac{\sigma_3^2}{2}, \tag{4.15}$$
$$\leq -\frac{\lambda I}{R} - \frac{r}{4K} S^{m+2} - \frac{\mu}{4} I^{m+1} - \frac{\mu}{2} R^{m+1} + D,$$

where

$$D \cong \sup_{(S,I,R) \in \mathbb{R}_+^3} \left\{ -\frac{r}{4K} S^{m+2} - \frac{\mu}{4} I^{m+1} + M\tilde{K}SI \right.$$
$$\left. + B + \mu + \gamma + \frac{\sigma_3^2}{2} \right\}. \tag{4.16}$$

Case 3. For every $(S,I,R) \in U_3$, using (4.15), we obtain

$$\mathscr{L}V \leq -\frac{\lambda I}{R} + D,$$
$$\leq -\frac{\lambda \varepsilon}{\varepsilon^2} + D,$$
$$\simeq -\frac{\lambda}{\varepsilon} + D,$$
$$\leq -1,$$

which results from (4.9). Hence

$$\mathscr{L}V(X) \leq -1 \quad \text{for all} \quad X \in U_3. \tag{4.17}$$

Case 4. For each $X \in U_4$, we can conclude from (4.15) that

$$
\begin{aligned}
\mathscr{L}V &\leq \frac{-r}{4K}S^{m+2} + D, \\
&\leq \frac{-r}{4K\varepsilon^{m+2}} + D, \\
&\leq -1,
\end{aligned}
$$

which is derived from (4.10). So

$$\mathscr{L}V(X) \leq -1 \quad \text{for each} \quad X \in U_4. \tag{4.18}$$

Case 5. For each $X \in U_5$, from (4.15), we obtain

$$
\begin{aligned}
\mathscr{L}V &\leq \frac{-\mu}{4}I^{m+1} + D, \\
&\leq \frac{-\mu}{4\varepsilon^{m+1}} + D \leq -1,
\end{aligned}
$$

which results from (4.11). Consequently

$$\mathscr{L}V(X) \leq -1 \quad \text{for every} \quad X \in U_5. \tag{4.19}$$

Case 6. For every $X \in U_6$, we can deduce from (4.15) that

$$
\begin{aligned}
\mathscr{L}V &\leq \frac{-\mu}{2}R^{m+1} + D, \\
&\leq \frac{-\mu}{2\varepsilon^{2m+2}} + D, \\
&\leq -1,
\end{aligned}
$$

which is derived from (4.12). Hence

$$\mathscr{L}V(X) \leq -1 \quad \text{for every} \quad X \in U_6. \tag{4.20}$$

Obviously, by using (4.13), (4.14), (4.17), (4.18), (4.19) and (4.20), we may deduce that given a sufficiently small ε,

$$\mathscr{L}V(X) \leq -1 \quad \text{for every} \quad X \in \mathbb{R}_+^3 \setminus U_\varepsilon.$$

Thus, the condition (C_2) of Lemma 1 is satisfied. The first condition (C_1) of Lemma 1 must be verified. The matrix of diffusion of system (1.2) is written as:

$$
A = \begin{bmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 \\ 0 & 0 & \sigma_3^2 R^2 \end{bmatrix}.
$$

Choosing

$$\tilde{\omega} = \min_{X \in \bar{U}_k \subset \mathbb{R}_+^3} \left\{ \sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 R^2 \right\}.$$

We have for each $X \in \bar{U}_k$, $\xi \in \mathbb{R}^3$

$$\sum_{i,j=1}^{3} a_{ij}(X)\xi_i\xi_j = (\sigma_1 S)^2\xi_1^2 + (\sigma_2 I)^2\xi_2^2 + (\sigma_3 R)^2\xi_3^2,$$

$$\geq \tilde{\omega}\|\xi\|^2,$$

where

$$\bar{U}_k = \left[\frac{1}{k}, k\right] \times \left[\frac{1}{k}, k\right] \times \left[\frac{1}{k}, k\right].$$

Hence, the initial condition, identified as (C_1) in Lemma 1, has been successfully satisfied. Subsequently, through the application of Lemma 1, the system (1.2) is demonstrated to have a distinct ergodic stationary distribution denoted as $\mu(.)$. This concludes the proof.

5 Computational Simulations

To ascertain the efficacy of our findings, we will conduct numerical simulations using the Milstein scheme [32] as the computational method. Specifically, we will discretize equation (1.2) according to the following scheme

$$
\begin{cases}
S_{k+1} = S_k + \left[rS_k\left(1 - \frac{S_k}{K}\right) - \mu S_k - \frac{\beta S_k I_k}{1 + \alpha S_k} \right]\Delta t \\
\qquad + \sigma_1 S_k \sqrt{\Delta t}\,\tau_{1,k} + \frac{\sigma_1^2}{2}S_k(\tau_{1,k}^2 - 1)\Delta t, \\[2mm]
I_{k+1} = I_k + \left[\frac{\beta S_k I_k}{1 + \alpha S_k} - (\mu + \lambda)I_k + \gamma R_k \right]\Delta t \\
\qquad + \sigma_2 I_k \sqrt{\Delta t}\,\tau_{2,k} + \frac{\sigma_2^2}{2}I_k(\tau_{2,k}^2 - 1)\Delta t, \\[2mm]
R_{k+1} = R_k + [\lambda I_k - (\mu + \gamma)R_k]\Delta t \\
\qquad + \sigma_3 R_k \sqrt{\Delta t}\,\tau_{3,k} + \frac{\sigma_3^2}{2}R_k(\tau_{3,k}^2 - 1)\Delta t,
\end{cases}
$$

where $\tau_{i,k}$ $(i = 1,2,3)$ are $\mathcal{N}(0,1)$-distributed independent random variables.

Example 1. We choose $r = 0.7$, $\mu = 0.1$, $\beta = 0.4$, $\alpha = 0.2$, $\lambda = 0.035$, $\gamma = 0.04$, $\sigma_1 = 0.4$, $\sigma_2 = 0.45$ and $\sigma_3 = 0.4$. These values assume Theorem 2 satisfied. Namely,
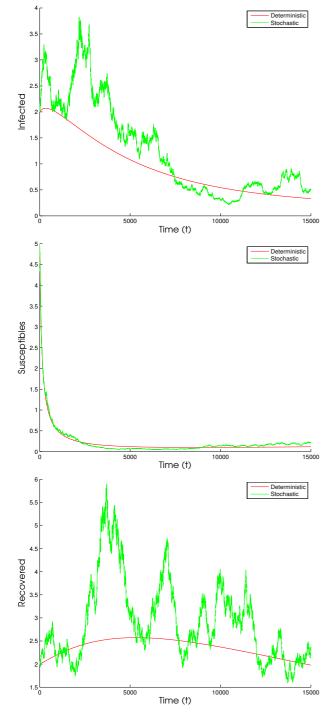
$$r - \mu = 0.6 > \frac{\sigma_1^2}{2}, \quad \nu = -0.58 < 0.$$

Therefore, the Theorem 2 can be obtained. From Fig. 1, it can be easily seen that the disease dies out in the population.
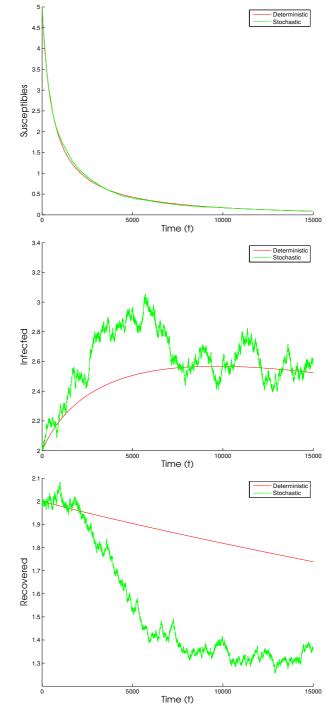
Example 2. We choose $r = 0.7$, $\mu = 0.1$, $\beta = 0.6$, $\alpha = 0.2$, $\lambda = 0.035$, $\gamma = 0.04$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.15$. Through straightforward computation, we have determined that $R_0^s = 2.38 > 1$. Hence, the condition stipulated in Theorem 4.1 is satisfied. Accordingly, we conclude from Theorem 4.1 that the system (1.2) has a unique ergodic stationary distribution $\mu(.)$, which implies the persistence of all individuals. This is illustrated in Fig. 3. It is important to note that the density estimates computed at times $t = 3000$, $t = 5000$, and $t = 7000$ are highly consistent, implying that they can be considered dependable approximations of the stationary distribution of system (1.2).

6 Perspective

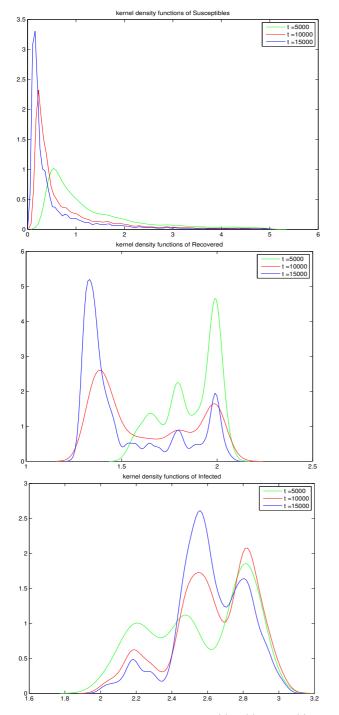Our investigation aimed to analyze the dynamics of a stochastic SIRI epidemic model, considering logistic

Fig. 1. Trajectories of $S(t)$, $I(t)$, and $R(t)$ for system 1.2 in example 1.

Fig. 2. Trajectories of $S(t)$, $I(t)$, and $R(t)$ for system 1.2 in example 2.

growth and saturated incidence. By conducting a comprehensive and rigorous analysis, we have confirmed the existence of positive global solutions for this stochastic epidemic model.

Additionally, we have established adequate conditions under which the disease is expected to become extinct, thereby contributing to a more comprehensive comprehension of system behavior.

Fig. 3. Kernel density functions of $S(t)$, $I(t)$ and $R(t)$ of system 1.2 respectively at time 3000, 5000 and 7000.

Our comprehensive investigation has yielded crucial insights into the dynamics of the stochastic SIRI epidemic model under diverse conditions. Remarkably, when minimizing the level of white noise in the system, a distinct ergodic stationary distribution emerges, signifying the stochastic persistence of the disease. Conversely, an excessive amount of white noise can lead to the extinction of the disease.

The implications of our research are significant for epidemic control and management, as our findings contribute valuable knowledge to ongoing efforts to mitigate the spread of infectious diseases. To support and validate our results, we conducted rigorous numerical simulations.

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In our study, we devised a Lyapunov function to enhance the analysis and establish appropriate criteria for an ergodic stationary distribution concerning the positive solutions within the stochastic epidemic model referred to as (1.2).

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Soulaimane Aznague obtained a Master of Mathematical Engineering degree from the Faculty of Sciences and Technology (FSTT) in Tangier, Morocco. Since 2018, he has pursued a Ph.D. at the Department of Mathematics at Abdelmalek Essaâdi University in the UAE.

Mourad El Idrissi an esteemed scholar, obtained a Master's in Mathematical Engineering from the prestigious Faculty of Sciences and Technology (FSTT) in Tangier, Morocco. Since 2019, he has been engaged in his Ph.D. studies at the distinguished Department of Mathematics at Abdelmalek Essaâdi University in the UAE, demonstrating unwavering dedication to advancing knowledge in the field.

Abdeladim Nait Brahim holds a Master's in Mathematical Engineering from the esteemed Faculty of Sciences and Technology (FSTT) in Tangier, Morocco. Since 2019, he has pursued a Ph.D. at the renowned Department of Mathematics, Abdelmalek Essaâdi University (UAE).

Bilal Harchaoui received a Master of Mathematical Engineering from the Faculty of Sciences and Technology (FSTT), Tangier (Morocco). He has been enrolled for a Ph.D. degree since 2018 at the Department of Mathematics at Abdelmalek Essaâdi University (UAE).

Saloua Boutouil holds a Master's in Mathematical Engineering from the Faculty of Sciences and Technology (FSTT) in Tangier, Morocco. Since 2018, he has been actively pursuing a Ph.D. degree at the Department of Mathematics at Abdelmalek Essaâdi University (UAE).

Adel Settati is an esteemed full professor of mathematics at the Faculty of Sciences and Technology (FSTT) in Tangier, Morocco. He completed his doctoral studies in probability at the renowned University of Rennes 1 in France. Dr. Settati's primary research interests revolve around dynamic systems, applied probability, and stochastic epidemic systems. His research contributions in these fields have been significant, and he has authored numerous research papers in high-impact scientific journals.

Aadil Lahrouz is a distinguished full professor of mathematics at the Faculty of Sciences and Technology (FSTT) located in Tangier, Morocco. Dr. Lahrouz's primary research interests revolve around dynamic systems, applied probability, and stochastic epidemic systems. His research contributions in these fields have been significant, and he has authored numerous research papers in high-impact scientific journals.

Mhamed El Merzguioui holds the distinguished position of a full professor of mathematics within the Faculty of Science and Technology (FSTT) located in Tangier, Morocco. He has earned an Advanced Graduate Degree in the specialized field of mathematical modeling concerning infiltration and the transfer of pollutants through stochastic fissures present amid two distinct porous mediums.

Jamal El Amrani is a highly esteemed full professor of mathematics at the Faculty of Science and Technology (FSTT) in Tangier, Morocco. He holds an Advanced Graduate Degree in Mathematical Modeling and Numerical Mechanics. Since 2012, he has been actively pursuing a Ph.D. in Applied Mathematics at the Department of Mathematics within FSTT.