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Existence and uniqueness of near-coupled coincidence points in partial cone-interval metric spaces

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Abstract: This paper introduces a new type of space called partial cone-interval metric space, and explores some of its topological properties. Using a novel fixed-point technique, we investigate the existence and uniqueness of near-coupled coincidence points in this setting. We provide numerical examples to demonstrate the effectiveness of our approach.

Keywords: Null sets; Fixed-points; Coincidence points; Partial cone-interval; Metric spaces

1 Introduction

From the last two decades, fixed-point theory has been a flourishing area of research work for many mathematicians and researchers since it has many important numerical applications like establishing Picard's Existence Theorem regarding existence and uniqueness of solutions of first order differential equations, integral equations, system of linear equations, initial and boundary value problems involving ordinary, partial and fractional differential equations, functional equations, and variational inequalities.

Over the years, there have been many efforts to generalize the theorems in fixed-point theory for different classes of topological spaces and Banach spaces. We mention the recent results of Sahar [1] that gave more generalizations of many previous results in the field of fixed-point theory for single-valued mappings.

There is a multitude of metric fixed-point theorems for mappings satisfying certain contraction type conditions. There are numerous variations using cone metric spaces and b-cone metric spaces instead, see the results involved in [2,3,4]. On the other side a novel class of generalized α-admissible contraction types of mappings introduced by Nashat Faried et al. [5], they worked in the framework of $\theta-$complete partial satisfactory cone metric spaces and proved the existence and uniqueness of coincidence points for such mappings with some applications.

Guo et al. [6] and Bhaskar et al. [7] were the first researchers who studied the existence of coupled fixed-points for a mixed monotone mapping in metric space satisfying some contractive type conditions. They applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. Since then, mathematicians and researchers have been considerably showing high interest in coupled fixed-point theory regarding their applications to a wide variety of problems, for instance, see [8], and references therein.

In 1966, interval analysis was introduced as a general mathematical tool to deal with interval uncertainty that appears in many mathematical areas and computer models of real-world phenomena. The first monographs dealing with interval analysis were due to Moore [9]. He published his first book named Interval Analysis, which still an important reference to this day. The interval analysis provide an essential tool to tackle uncertainty when the problems in engineering, economics and social sciences are formulated as interval-valued problems. The techniques of functional analysis and non-linear analysis are used to study those interval-valued problems. For more details of interval spaces, we refer the reader to [9, 10].

In 2018, Wu [11] proposed metric interval spaces and normed interval spaces exploiting the null set to study many types of near fixed-point theorems.

In 2020, Ullah et al. [12] studied the near-coincidence point theorems in complete metric interval space and hyperspace via a simulation function.

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In 2021, Sarwar et al. [13] introduced the concept of cone interval b-metric space over Banach algebras and proved some near-fixed-point and near-common fixed-point for self-mappings in such spaces.

In 2022, Joshi and Tomar [14] equipped the b-metric space to the set of closed and bounded real intervals and studied the topological properties of the resulting distance structure, b-interval metric space. Besides, the authors proved that a conventional Banach Contraction Principle may not be demonstrated in a b-interval metric space. Further, they introduced some novel notions such as interval circle, fixed interval circle and its equivalence class. They also established the existence of a near-fixed interval circle and its equivalence interval C-class.

Following up these developments, in this article, using the concept of coupled coincidence point, introduced by Lakshmikantham et al. [15], we first familiarize some of key concepts of the so-called near-coupled fixed-point and near-coupled coincidence point by defining their equivalence interval classes that are based on a novel null set. Further, we build up a new topological structure on distances between closed bounded intervals to elements of the cone in a normed space, namely; partial cone-interval metric spaces. We also very carefully cover the topological aspects under these settings. Among other things, our topological space is fully consistent with two types of convergence, we provide detailed analysis, completely explicit explanations and provide examples of these concepts. All of the concepts introduced in the paper have found interesting examples. Our new results unified some of the early findings in fixed-point theory.

2 Prelude and Relevant Pre-requisite

Throughout this study, we have denoted the one-dimensional Euclidean space by $\mathbb{R}$, the set of non-negative reals by $\mathbb{R}_0^+$, the set of cardinal numbers by $\mathbb{N}$.

We review some notations, working hypotheses and necessary background materials for a better understanding of an interval circumference. The key references for this section are [9,10,16].

Let $\mathscr{I}(\mathbb{R}) := \{[a,b] : a,b \in \mathbb{R} \text{ and } a \leq b\}$ be the set of all real intervals. Identifying $a \in \mathbb{R}$ with the degenerate interval $[a,a] \in \mathscr{I}(\mathbb{R})$, we consider $\mathbb{R}$ as a subset of $\mathscr{I}(\mathbb{R})$.

Interval arithmetic is a natural extension of real arithmetic. The basic interval operations in $\mathscr{I}(\mathbb{R})$ are denoted by $\oplus, \ominus, \otimes$, and are formulated in terms of the interval's endpoints. The corresponding interval addition in $\mathscr{I}(\mathbb{R})$ is formulated as:

$$[a,b] \oplus [c,d] := [a+c, b+d] \in \mathscr{I}(\mathbb{R}).$$

The scalar multiplication in $\mathscr{I}(\mathbb{R})$ is calculated as:

$$k[a,b] := \begin{cases} [ka,kb], & \text{if } k \geq 0, \\ [kb,ka], & \text{if } k < 0. \end{cases}$$

The interval negation in $\mathscr{I}(\mathbb{R})$ is defined by

$$-[a,b] = [-b,-a],$$

where $-[a,b]$ means $(-1)[a,b]$.

In accordance with the above definitions, the subtraction in $\mathscr{I}(\mathbb{R})$ for any two real interval $[a,b]$ and $[c,d]$ is defined as follows:

$$[a,b] \ominus [c,d] := [a,b] \oplus (-[c,d]) = [a,b] \oplus [-d,-c] = [a-d, b-c] \in \mathscr{I}(\mathbb{R}).$$

It is evident that $\mathscr{I}(\mathbb{R})$ is not a real vector space in the conventional sense (under the aforementioned addition and scalar multiplication). The main reason for this is due to the lack of inverse elements for non-degenerated closed intervals (there will be no additive inverse element for each interval).

It is clear that $[0,0] \in \mathscr{I}(\mathbb{R})$ is a zero element. However, for any $[a,b] \in \mathscr{I}(\mathbb{R})$, the substraction

$$[a,b] \ominus [a,b] = [a-b, b-a] = [a-b, -(a-b)] \neq [0,0]$$

for any non-degenerate interval $[a,b]$. In other words, the inverse element does not exist for each interval in general.

Instead of considering the zero element $[0,0]$, we define the null set Ω of $\mathscr{I}(\mathbb{R})$ as follows:

$$\Omega = \{[a,b] \ominus [a,b] : [a,b] \in \mathscr{I}(\mathbb{R})\}.$$

It is noteworthy that

$$\Omega = \{[-k,k] : k \geq 0\}.$$

It may also be determined that $[-1,1]$ generates Ω via non-negative scalar multiplication, as demonstrated below:

$$\Omega = \{k[-1,1] : k \geq 0\}.$$

The interval $[-1,1]$ is called a generator of the null set Ω.

We write $[a,b] \stackrel{\Omega}{=} [c,d]$ if, and only if, there exist $w_1, w_2 \in \Omega$ such that $[a,b] \oplus w_1 = [c,d] \oplus w_2$.

Remark.[10] The binary relation $\stackrel{\Omega}{=}$ is an equivalence relation.

According to the binary relation $\stackrel{\Omega}{=}$, for any $[a,b] \in \mathscr{I}(\mathbb{R})$, we define the class

$$\langle [a,b] \rangle := \left\{ [c,d] \in \mathscr{I}(\mathbb{R}) : [a,b] \stackrel{\Omega}{=} [c,d] \right\}. \tag{1}$$

The family of all classes $\langle [a,b] \rangle$ for $[a,b] \in \mathscr{I}(\mathbb{R})$ is denoted by $\langle \mathscr{I}(\mathbb{R}) \rangle$.

Remark 2 says that the classes defined in (1) form the equivalence interval classes. In this case, the family $\langle \mathscr{I}(\mathbb{R}) \rangle$ is called the quotient set of $\mathscr{I}(\mathbb{R})$. We also have that $[c,d] \in \langle [a,b] \rangle$ implies $\langle [a,b] \rangle = \langle [c,d] \rangle$. In other words, the family of all equivalence classes form a partition of the whole set $\mathscr{I}(\mathbb{R})$. This is an important fact, which we discuss in the following lines:

$$\langle \mathscr{I}(\mathbb{R}) \rangle := \{\langle [a,b] \rangle : a,b \in \mathbb{R} \text{ and } a \leq b\}$$

$$= \mathscr{I}(\mathbb{R})/\Omega$$
$$= \{[a,b]+\Omega : a,b \in \mathbb{R} \text{ and } a \leq b\}.$$

It is mildly interesting that

$$[a,b]+\Omega := \{[a,b] \oplus [-k,k] : k \geq 0\}$$
$$= \{[a-k,b+k] : k \geq 0\}$$
$$= \langle [a,b] \rangle .$$

It seems evident that $\langle [a,b] \rangle$ can be rewritten as:

$$\langle [a,b] \rangle := \{[a-k,b+k] : k \geq 0\}.$$

*Remark.*We have that $[c,d] \in \langle [a,b] \rangle$ implies $\langle [a,b] \rangle = \langle [c,d] \rangle$.

Remark.[17] In an interval space $\mathscr{I}(\mathbb{R})$ with the null set Ω, $[a,b] \overset{\Omega}{=} [c,d]$ if, and only if, $c-a=b-d$. Thus, we obtain that

$$\langle [a,b] \rangle := \{[c,d] \in \mathscr{I}(\mathbb{R}) : a+b=c+d\}.$$

Interval multiplication of $[a,b]$ and $[c,d]$, denoted by $[a,b] \otimes [c,d]$, is defined by the formula
$[a,b] \otimes [c,d] := [\min\{ac,ad,bc,bd\}, \max\{ac,ad,bc,bd\}] \in \mathscr{I}(\mathbb{R})$, where max and min are the $\leq$-maximal and $\leq$-minimal, respectively. In order to preserve standard notation, the multiplication sign $\otimes$ is usually dropped and we write $[a,b][c,d]$ for $[a,b] \otimes [c,d]$.

Interval multiplication is commutative, associative and $[1,1]$ is the identity element. Multiplicative inverses do not exist in general.

In the case of division $0 \notin [c,d]$ is assumed, we can define $\frac{1}{[a,b]}$ by the rule

$$\frac{1}{[a,b]} := \left[\frac{1}{b}, \frac{1}{a}\right] \in \mathscr{I}(\mathbb{R}) \quad (\text{if } a>0).$$

We can define an interval division $\frac{[a,b]}{[c,d]} = [a,b] \otimes \frac{1}{[c,d]}$, where $0 \notin [c,d]$, as follows:
$\frac{[a,b]}{[c,d]} = [a,b] \otimes \left[\frac{1}{d}, \frac{1}{c}\right] := \left[\min\left\{\frac{a}{d}, \frac{a}{c}, \frac{b}{d}, \frac{b}{c}\right\}, \max\left\{\frac{a}{d}, \frac{a}{c}, \frac{b}{d}, \frac{b}{c}\right\}\right] \in \mathscr{I}(\mathbb{R})$.

An interval real vector or an axis-aligned box-valued is a vector which has n components, each of which is an interval in $\mathscr{I}(\mathbb{R})$.
Let $\mathscr{I}^n(\mathbb{R}) = \{(I_1, I_2, \cdots, I_n) : I_i = [a_i, b_i], 1 \leq i \leq n\}$ be the n-dimensional real interval vector space. An interval vector $(I_1, I_2, \cdots, I_n)$ has n interval components and can be interpreted geometrically as an n-dimensional rectangular convex polytope or box. In the present work, we will stick to the two-component interval vector $([a,b],[c,d]) \in \mathscr{I}^2(\mathbb{R})$. It can be represented geometrically as a closed rectangular region in the xy-plane, whose sides are parallel to the coordinates. The arithmetical operations between interval vectors are direct extensions of the same operations for punctual vectors.

We define the null set $\Omega \times \Omega$ in $\mathscr{I}^2(\mathbb{R})$ by
$$\Omega \times \Omega = \{([a,b],[c,d]) - ([a,b],[c,d]) : ([a,b],[c,d]) \in \mathscr{I}^2(\mathbb{R})\}$$
$$= \{([-k_1,k_1],[-k_2,k_2]) : k_1,k_2 \geq 0\}$$
$$= \{(k_1[-1,1], k_2[-1,1]) : k_1,k_2 \geq 0\}.$$

Proposition 1*In an interval vector space $\mathscr{I}^2(\mathbb{R})$ with the null set $\Omega \times \Omega$, $([a,b],[c,d]) \overset{\Omega\times\Omega}{=} ([e,f],[g,h])$ if, and only if,*
$$a+b=e+f \text{ and } c+d=g+h.$$

*Proof.*The result can be readily attained by doing the followings:
$([a,b],[c,d]) \overset{\Omega\times\Omega}{=} ([e,f],[g,h])$ if, and only if, $([-k_1,k_1],[-k_2,k_2]), ([-l_1,l_1],[-l_2,l_2]) \in \Omega \times \Omega$ such that $([a,b],[c,d]) + ([-k_1,k_1],[-k_2,k_2]) = ([e,f],[g,h]) + ([-l_1,l_1],[-l_2,l_2])$ if, and only if, $([a-k_1,b+k_1],[c-k_2,d+k_2]) = ([e-l_1,f+l_1],[g-l_2,h+l_2])$ if, and only if, $[a-k_1,b+k_1] = [e-l_1,f+l_1]$ and $[c-k_2,d+k_2] = [g-l_2,h+l_2]$ if, and only if, $a-k_1 = e-l_1$, $b+k_1 = f+l_1$, $c-k_2 = g-l_2$, and $d+k_2 = h+l_2$ if, and only if, $a-e = k_1-l_1$, $b-f = -k_1+l_1 = -(k_1-l_1)$, $c-g = k_2-l_2$, and $d-h = -k_2+l_2 = -(k_2-l_2)$ if, and only if, $a-e = -(b-f)$ and $c-g = -(d-h)$ if, and only if, $a+b=e+f$ and $c+d=h+g$.

*Remark.*The binary relation $\overset{\Omega\times\Omega}{=}$ is an equivalence relation.

Corollary 2*Let $\mathscr{I}^2(\mathbb{R})$ be an interval vector space over $\mathbb{R}^2$. We give equivalent characterizations of the equivalence relation $\overset{\Omega\times\Omega}{=}$ as follows:*

(i)$([a,b],[c,d]) \overset{\Omega\times\Omega}{=} ([e,f],[g,h])$;

(ii)There exist $w_1', w_2' \in \Omega \times \Omega$ such that
$$([a,b],[c,d]) + w_1' = ([e,f],[g,h]) + w_2';$$

(iii)There exists $w' \in \Omega \times \Omega$ such that
$([a,b],[c,d]) = ([e,f],[c,d]) + w'$ *or* $([a,b],[c,d]) + w' = ([e,f],[c,d])$;

(iv)$[a,b] \overset{\Omega}{=} [e,f]$ *and* $[c,d] \overset{\Omega}{=} [g,h]$.

*Remark.*According to the equivalence relation $\overset{\Omega\times\Omega}{=}$, the equivalence class $\langle ([a,b],[c,d]) \rangle_{\Omega\times\Omega}$ for any interval vector $([a,b],[c,d])$ in $\mathscr{I}^2(\mathbb{R})$ is stated in the form $\langle ([a,b],[c,d]) \rangle_{\Omega\times\Omega} = \left\{([e,f],[g,h]) \in \mathscr{I}^2(\mathbb{R}) : ([a,b],[c,d]) \overset{\Omega\times\Omega}{=} ([e,f],[g,h])\right\}$.

*Remark.*We can redefine the class $\langle ([a,b],[c,d]) \rangle_{\Omega\times\Omega}$ by $\langle ([a,b],[c,d]) \rangle_{\Omega\times\Omega} := \{([a-k_1,b+k_1],[c-k_2,d+k_2]) : k_1,k_2 \geq 0\}$.

The family of all equivalence classes $\langle ([a,b],[c,d]) \rangle_{\Omega\times\Omega}$ for $([a,b],[c,d]) \in \mathscr{I}^2(\mathbb{R})$ will be denoted by $\langle \mathscr{I}^2(\mathbb{R})\rangle_{\Omega\times\Omega}$.

In the following, we give a brief account of some needed terminologies from fixed-point theory.

A set $\mathfrak{C}$ is a cone in a real Banach space $E := (E, \|.\|)$ if, $\mathfrak{C}$ is closed, non-empty, the sum of two elements of $\mathfrak{C}$ is an element of $\mathfrak{C}$, non-negative scalar multiples of elements of $\mathfrak{C}$ are elements of $\mathfrak{C}$, and $\mathfrak{C}\cap(-\mathfrak{C}) = \{\theta\}$.

Any cone $\mathfrak{C}\subset E$ defines the following partial ordered relations:

$$x \preceq y \text{ if, and only if, } y - x \in \mathfrak{C};$$

$$x \prec y \text{ if, and only if, } y - x \in \mathfrak{C} - \{\theta\},$$

where θ is the zero element in E;
$x \ll y$ if, and only if, $y - x \in \mathrm{Int}(\mathfrak{C})$, where $\mathrm{Int}(\mathfrak{C})$ denotes the topological interior of $\mathfrak{C}$—if there are any.

Definition 3[18] *A cone $\mathfrak{C}$ of a real Banach space $(E, \|.\|)$ is solid if, and only if, $Int(\mathfrak{C}) \neq \emptyset$, and it is normal if, and only if, there exists a real number $K > 0$ such that $\|x\| \le K\|y\|$ for every $x, y \in E$ with $\theta \preceq x \preceq y$. The smallest positive constant K for which the above inequality holds is called the normal constant of $\mathfrak{C}$.*

Definition 4[19] *Let $\mathfrak{C}$ be a solid cone of the normed space E. A sequence $\{u_n\}_{n\in\mathbb{N}}$ is said to be a $c-$sequence, if for each $c \gg \theta$, there exists a natural number n_0 such that $u_n \ll c$ for all $n \ge n_0$.*

Lemma 5[20] *Let $\mathfrak{C}$ be a solid cone of the normed space $(E, \|.\|)$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in E. Then, $u_n \overset{\|\cdot\|}{\to} \theta$ implies that $\{u_n\}_{n\in\mathbb{N}}$ is a $c-$sequence.*

Proposition 6[20] *Let $\mathfrak{C}$ be a solid cone in a real Banach space E. If $\theta \preceq u \ll c$ holds for any $c \in Int(\mathfrak{C})$, then $u = \theta$.*

Lemma 7[21] *If E is a real Banach space with a cone $\mathfrak{C}$ and if $a \preceq \lambda a$ with $a \in \mathfrak{C}$ and $0 \le \lambda < 1$, then $a = \theta$.*

Next, we are going to define the partial cone metric for vector intervals in $\mathscr{I}^2(\mathbb{R})$.

Definition 8*Let E be a real Banach space ordered by the cone $\mathfrak{C}$. The interval space $\mathscr{I}(\mathbb{R})$ with the null set Ω and the correspondence $\wp : \mathscr{I}^2(\mathbb{R}) \to \mathfrak{C}$ is called a partial cone-interval metric on $\mathscr{I}^2(\mathbb{R})$, if the following conditions hold:*

(PCIM$_1$) :$\theta \preceq \wp([a,b],[a,b]) \preceq \wp([a,b],[c,d])$ *for all* $[a,b],[c,d] \in \mathscr{I}(\mathbb{R})$;

(PCIM$_2$) :$\wp([a,b],[a,b]) = \wp([c,d],[c,d]) = \wp([a,b],[c,d])$ *if, and only if,* $[a,b] \overset{\Omega}{=} [c,d]$;

(PCIM$_3$) :$\wp([a,b],[c,d]) = \wp([c,d],[a,b])$ *for all* $[a,b],[c,d] \in \mathscr{I}(\mathbb{R})$;

(PCIM$_4$) :
$\wp([a,b],[c,d]) \preceq \wp([a,b],[e,f]) + \wp([e,f],[c,d]) - \wp([e,f],[e,f])$ *for any three given intervals* $[a,b],[c,d],[e,f] \in \mathscr{I}(\mathbb{R})$ *such that* $[e,f] \overset{\Omega}{\neq} [a,b]$ *and* $[e,f] \overset{\Omega}{\neq} [c,d]$.

Then, the quadruple $(\mathscr{I}(\mathbb{R}), E, \mathfrak{C}, \wp)$ is known as a partial cone-interval metric space.

Different partial cone-interval metrics could be defined on $\mathscr{I}(\mathbb{R})$ giving rise to different partial cone-interval metric spaces.

Now, we make some useful observations.

*Remark.*For any $w_1, w_2 \in \Omega$ and for any $[a,b],[c,d] \in \mathscr{I}(\mathbb{R})$, we have

$$\wp([a,b]\oplus w_1, [c,d]\oplus w_2) = \wp([a,b],[c,d]).$$

Remark. (i)If $\wp([a,b],[c,d]) = \theta$, then $[a,b] \overset{\Omega}{=} [c,d]$.
(ii)$\theta \prec \wp([a,b],[c,d])$ for all $[a,b],[c,d] \in \mathscr{I}(\mathbb{R})$ with $[a,b] \overset{\Omega}{\neq} [c,d]$.

For convenience of writing, the points of the set of the intervals of real numbers will be denoted by capital letters I, J, and K.

Example 9*Let $E := M_{2\times2}(\mathbb{R})$ be the set of all real two-rowed square matrices with the norm $\|A\| := \max\{|a_{ij}|\}$, where a_{ij}'s are the inputs of $A \in M_{2\times2}(\mathbb{R})$ for $1 \le i,j \le 2$. The null matrix $\theta_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ represents the neutral element in $M_{2\times2}(\mathbb{R})$. Moreover, $\mathfrak{C} := M_{2\times2}(\mathbb{R}_0^+)$ is a non-empty solid normal cone with a normal constant $K = 1$. Define an order $\preceq$ on $M_{2\times2}(\mathbb{R})$ as follows:*

$[a_{ij}]_{2\times2} \preceq [b_{ij}]_{2\times2}$ *if, and only if,* $a_{ij} \le b_{ij}$ *for* $1 \le i,j \le 2$.

We demonstrate $\wp : \mathscr{I}^2(\mathbb{R}) \to \mathfrak{C}$ as

$$\wp(I,J) = \begin{bmatrix} |a-c| & 0 \\ 0 & |b-d| \end{bmatrix} \in \mathfrak{C},$$

where $I = [a,b], J = [c,d] \in \mathscr{I}(\mathbb{R})$.
We attest that $(\mathscr{I}(\mathbb{R}), M_{2\times2}(\mathbb{R}), M_{2\times2}(\mathbb{R}_0^+), \wp)$ is a partial cone-interval metric space.

(PCIM$_1$)
$\wp(I,J) \succeq \wp(I,I) \Leftrightarrow \begin{bmatrix} |a-c| & 0 \\ 0 & |b-d| \end{bmatrix} \succeq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} |a-c| & 0 \\ 0 & |b-d| \end{bmatrix} \in \mathfrak{C} \Leftrightarrow |a-c| \ge 0$ *and* $|b-d| \ge 0$.

(PCIM$_2$) *For any* $I,J \in \mathscr{I}(\mathbb{R})$, *we have*
$\wp(I,I) = \wp(J,J) = \wp(I,J) \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} |a-c| & 0 \\ 0 & |b-d| \end{bmatrix} \Leftrightarrow$
$|a-c| = 0$ *and* $|b-d| \Leftrightarrow a = c$ *and* $b = d \Leftrightarrow I \overset{\Omega}{=} J$.

The validity of (PCIM$_3$) *can easily be proved.*

(PCIM$_4$) *Clearly,* $|a-e|+|e-c|\geq|a-c|$ *and* $|b-f|+|f-d|\geq|b-d|$ *for any real numbers* $a,b,c,d,e,$ *and* f. *Therefore, we attain that*

$$\wp(I,K)+\wp(K,J)-\wp(K,K)-\wp(I,J)$$

$$=\begin{bmatrix}|a-e| & 0\\ 0 & |b-f|\end{bmatrix}+\begin{bmatrix}|e-c| & 0\\ 0 & |f-d|\end{bmatrix}-\begin{bmatrix}|a-c| & 0\\ 0 & |b-d|\end{bmatrix}$$
$$=\begin{bmatrix}|a-e|+|e-c|-|a-c| & 0\\ 0 & |b-f|+|f-d|-|b-d|\end{bmatrix}\in\mathfrak{C}$$

for any $I=[a,b],J=[c,d],K=[e,f]\in\mathscr{I}(\mathbb{R})$.

From all of this, it is easy to see that $\wp$ *meets all the axiom schemes of Definition 8. Henceforth,* $\wp$ *is partial cone-interval metric over the given Banach space* $M_{2\times2}(\mathbb{R})$ *and* $\left(\mathscr{I}(\mathbb{R}),M_{2\times2}(\mathbb{R}),M_{2\times2}\left(\mathbb{R}_0^+\right),\wp\right)$ *is partial cone-interval metric space.*

In the sequel, we always suppose that $\mathfrak{C}$ is cone with non-empty interior in E. Now, we express some essential topological properties of partial cone-interval metric space, as declared follows.

Definition 10 *Let* $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ *be a partial cone-interval metric space. The set* $B_{\wp}([a_0,b_0];c):=\{[a,b]\in\mathscr{I}(\mathbb{R}):\wp([a_0,b_0],[a,b])\ll c+\wp([a_0,b_0],[a_0,b_0])\}$ *is the interval disc centred at* $[a_0,b_0]$ *and radius* $c\in Int(\mathfrak{C})$.

Definition 11 *Let* $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ *be a partial cone-interval metric space,* $[a,b]\in\mathscr{I}(\mathbb{R})$ *and* $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *be a sequence of intervals in* $\mathscr{I}(\mathbb{R})$. *Then,*

(i) $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is convergent to the interval* $[a,b]$, *briefly denoted by* $[a_n,b_n]\overset{\tau_{\wp}}{\to}[a,b]$, *whenever for every* $c\in E$ *with* $\theta\ll c$, *there is* $n_0\in\mathbb{N}$ *such that*

$$\wp([a_n,b_n],[a,b])\ll\wp([a,b],[a,b])+c \text{ for all } n\geq n_0.$$

To put it in another way, the sequence $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is convergent to the interval* $[a,b]$ *if, and only if,* $\{\wp([a_n,b_n],[a,b])-\wp([a,b],[a,b])\}_{n\in\mathbb{N}}$ *is a* $c-$*sequence in* $\mathfrak{C}$.

(ii) $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is strongly convergent to* $[a,b]$, *briefly denoted by* $[a_n,b_n]\overset{s-\tau_{\wp}}{\to}[a,b]$, *if, and only if,* $\lim_{n\to\infty}\wp([a_n,b_n],[a,b])=\lim_{n\to\infty}\wp([a_n,b_n],[a_n,b_n])=\wp([a,b],[a,b])$. *Equivalently;* $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is strongly convergent to* $[a,b]$ *if, and only if,* $\{\wp([a_n,b_n],[a,b])-\wp([a,b],[a,b])\}_{n\in\mathbb{N}}$ *is convergent to* θ *with respect to the norm topology of* E. *That is;* $\lim_{n\to\infty}\|\wp([a_n,b_n],[a,b])-\wp([a,b],[a,b])\|=0$.

(iii) $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is a* $\theta-$*Cauchy if, given* $c\in E$ *with* $\theta\ll c$, *there is* $n_0\in\mathbb{N}$ *such that* $\wp([a_n,b_n],[a_m,b_m])\ll c$ *whenever* $m>n\geq n_0$.

(iv) The partial cone-interval metric space $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ *is said to be* $\theta-$*complete, in case each* $\theta-$*Cauchy interval sequence* $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *of* $\mathscr{I}(\mathbb{R})$ *converges to the interval* $[a,b]$ *such that* $\wp([a,b],[a,b])=\theta$.

(v) $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *is a Cauchy sequence if, there is an element* $u\in\mathfrak{C}$ *such that* $\lim_{n,m\to\infty}\wp([a_n,b_n],[a_m,b_m])=u$.

(vi) The partial cone-interval metric space $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ *is complete if, each Cauchy interval sequence* $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *in* $\mathscr{I}(\mathbb{R})$ *is strongly convergent towards* $[a,b]\in\mathscr{I}(\mathbb{R})$ *such that* $\wp([a,b],[a,b])=u$.

Remark. In the case when the underlying cone $\mathfrak{C}$ is solid, it is easy to prove that every strongly convergent sequence of intervals in $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ is convergent. The converse does not hold in general. If cone $\mathfrak{C}$ is solid and normal, then the two types of convergence are equivalent.

The example below covers all single steps to prove the convergence of an interval sequence in the framework of partial cone-interval metric space $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$.

Example 12 *On* $E:=\mathbb{R}^2$, *we define a norm* $\|(x,y)\|_1:=|x|+|y|$ *for all* $(x,y)\in\mathbb{R}^2$. *Take a cone* $\mathfrak{C}:=\left\{(x,y)\in\mathbb{R}^2:x,y\geq0\right\}$ *in* $\mathbb{R}^2$. *Clearly,* $\mathfrak{C}$ *is a non-empty normal solid cone on* $\mathbb{R}^2$, *wherein the partial ordering on E induced by* $\mathfrak{C}$ *is defined by:*

$$(x,y)\preceq(u,v) \text{ if, and only if, } x\leq u \text{ and } y\leq v,$$

where $\leq$ *is the usual order on the elements of* $\mathbb{R}$. *Define* $\wp:\mathscr{I}^2(\mathbb{R})\to\mathfrak{C}$ *such as for every* $I=[a,b],J=[c,d]\in\mathscr{I}(\mathbb{R})$, *we have*

$$\wp(I,J)=(|a+b-c-d|,k|a+b-c-d|)\in\mathfrak{C},$$

where $k\geq1$. *Therefore,* $\left(\mathscr{I}(\mathbb{R}),\mathbb{R}^2,\mathfrak{C},\wp\right)$ *is regarded as a partial cone-interval metric space. Consider the interval sequence* $\{[a_n,b_n]\}_{n\in\mathbb{N}}:=\left\{\left[\frac{1}{n},1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$. *We claim that* $\left[\frac{1}{n},1+\frac{1}{n}\right]\overset{\tau_{\wp}}{\to}\left[\frac{-1}{2},\frac{3}{2}\right]$. *To see this, let* $c=(c_1,c_2)\in E$ *with* $c\gg\theta$ *be given arbitrary. Then, it should be* $c_1,c_2>0$. *It is an easy task to find some natural number* n_0 *such that* $\wp\left(\left[\frac{1}{n},1+\frac{1}{n}\right],\left[\frac{-1}{2},\frac{3}{2}\right]\right)-\wp\left(\left[\frac{-1}{2},\frac{3}{2}\right],\left[\frac{-1}{2},\frac{3}{2}\right]\right)\ll c$ *for all* $n\geq n_0$. *With* $c=(c_1,c_2)$ *being given provided that* $c_1,c_2>0$, *observe the following considerations:*

$$\wp\left(\left[\tfrac{1}{n},1+\tfrac{1}{n}\right],\left[\tfrac{-1}{2},\tfrac{3}{2}\right]\right)-\wp\left(\left[\tfrac{-1}{2},\tfrac{3}{2}\right],\left[\tfrac{-1}{2},\tfrac{3}{2}\right]\right)\ll c$$

if, and only if, $(c_1,c_2)-\wp\left(\left[\frac{1}{n},1+\frac{1}{n}\right],\left[\frac{-1}{2},\frac{3}{2}\right]\right)\in Int(\mathfrak{C})$

if, and only if, $(c_1,c_2)-\left(\frac{2}{n},\frac{2k}{n}\right)\in Int(\mathfrak{C})$

if, and only if, $\left(c_1-\frac{2}{n},c_2-\frac{2k}{n}\right)\in Int(\mathfrak{C})$

implies that $c_1>\frac{2}{n}$ *and* $c_2>\frac{2k}{n}$.

In the sense of real sequences, we have $\frac{2}{n} \underset{n\to\infty}{\longrightarrow} 0$. *Thus, for* $c_1 > 0$ *considered above, there exists* $n_1 = n_1(c_1) \in \mathbb{N}$ *such that for all* $n \geq n_1$, *we have* $\frac{2}{n} < c_1$. *For* $n \geq n_1$, *choose* $n_1 := \left[\frac{2}{c_1}\right] + 1$ *provided that* $\frac{2}{n} \leq \frac{2}{n_1} < c_1$. *In the same manner, we have* $\frac{2k}{n} \underset{n\to\infty}{\longrightarrow} 0$. *Thus, for the above* $c_2 > 0$, *there exists* $n_2 = n_2(c_2) \in \mathbb{N}$ *such that* $\frac{2k}{n} < c_2$ *for all* $n \geq n_2$. *Since* $n \geq n_2$, *we can choose* $n_2 := \left[\frac{2k}{c_2}\right] + 1$ *such that* $\frac{2k}{n} \leq \frac{2k}{n_2} < c_2$. *We denote by* $n_0 := \max\{n_1, n_2\}$, *then for any arbitrary* $c = (c_1, c_2) \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *that depends on* c_1 *or* c_2, *such that for all* $n \geq n_0$, *we have that* $c_1 > \frac{2}{n}$ *and* $c_2 > \frac{2k}{n}$. *This shows that* $\{\wp([\frac{1}{n}, 1+\frac{1}{n}], [\frac{-1}{2}, \frac{3}{2}]) - \wp([\frac{-1}{2}, \frac{3}{2}], [\frac{-1}{2}, \frac{3}{2}])\}_{n\in\mathbb{N}}$ *is a* $c-$*sequence. In this sense, we conclude that* $\left[\frac{1}{n}, 1+\frac{1}{n}\right] \overset{\tau_\wp}{\to} \left[\frac{-1}{2}, \frac{3}{2}\right]$. *In an obvious way, it is easy to prove that* $\left[\frac{1}{n}, 1+\frac{1}{n}\right] \overset{\tau_\wp}{\to} [a,b]$ *for any* $a, b \in \mathbb{R}$ *with* $a \leq b$ *and* $a + b = 1$. *Explicitly,* $\left\langle\left[\frac{-1}{2}, \frac{3}{2}\right]\right\rangle$ *is the class limit of* $\left\{\left[\frac{1}{n}, 1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$.

The point made in the above example demonstrates that the given interval sequence is convergent to infinitely many points (more precisely, the sequence is convergent to all elements in the equivalence class of the limit point). However, this is not always true. To back up this fact, it suffices to give the following counterexample.

Example 13*Consider* $E := \mathbb{R}^2$ *normed by* $\|(x,y)\|_2 := \left(x^2+y^2\right)^{\frac{1}{2}}$ *for all ordered pairs* $(x,y) \in \mathbb{R}^2$. *Let* $\mathfrak{C} := \left\{(x,y) \in \mathbb{R}^2 : x, y \geq 0\right\}$. *Thus,* $\mathfrak{C} \subset E$ *is the underlying cone in* E. *It appears to have a non-empty interior and it is a normal cone. Let a partial ordering on* $\mathfrak{C}$ *be defined as in Example 12. We define* $\wp : \mathscr{I}^2(\mathbb{R}) \to \mathfrak{C}$ *by*
$\wp(I,J) = (|a-c| + |b-d|, |a-c| + |b-d| + \beta) \in \mathfrak{C}$, *where* $I = [a,b], J = [c,d] \in \mathscr{I}(\mathbb{R})$ *and* $\beta > 0$ *is any constant. Therefore,* $\left(\mathscr{I}(\mathbb{R}), \mathbb{R}^2, \mathfrak{C}, \wp\right)$ *is a partial cone-interval metric space.*
Choosing $\{[a_n, b_n]\}_{n\in\mathbb{N}} := \left\{\left[\frac{1}{n}, 1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$. *Obviously,* $\wp\left(\left[\frac{1}{n}, 1+\frac{1}{n}\right], [0,1]\right) - \wp([0,1],[0,1]) = \left(\frac{2}{n}, \frac{2}{n}\right)$. *It is easily verifiable that*
$\left\|\wp\left(\left[\frac{1}{n}, 1+\frac{1}{n}\right], [0,1]\right) - \wp([0,1],[0,1])\right\|_2 = \left\|\left(\frac{2}{n}, \frac{2}{n}\right)\right\|_2 = \frac{2\sqrt{2}}{n} \underset{n\to\infty}{\longrightarrow} 0.$
Accordingly, $\{\wp([\frac{1}{n}, 1+\frac{1}{n}], [0,1]) - \wp([0,1],[0,1])\}_{n\in\mathbb{N}}$ *is a* $c-$*sequence. In this manner, we obtain that* $\left[\frac{1}{n}, 1+\frac{1}{n}\right] \overset{\tau_\wp}{\to} [0,1]$. *By using a contradiction argument, we will now prove that* $\left[\frac{1}{n}, 1+\frac{1}{n}\right] \overset{\tau_\wp}{\nrightarrow} [-1,2]$. *As we know, if* $\left[\frac{1}{n}, 1+\frac{1}{n}\right] \overset{\tau_\wp}{\to} [-1,2]$, *then for any arbitrary* $c \in E$ *with* $c \gg \theta$, *there exists* $n_0 \in \mathbb{N}$ *provided that for every* $n \geq n_0$, *for which we have* $\wp([\frac{1}{n}, 1+\frac{1}{n}], [-1,2]) - \wp([-1,2],[-1,2]) \ll c$. *To consider it through explanation, let* $c = (c_1, c_2)$ *with* $c_1, c_2 > 0$ *be given. We observe that*
$\wp\left(\left[\frac{1}{n}, 1+\frac{1}{n}\right], [-1,2]\right) - \wp([-1,2],[-1,2]) \ll c$

if, and only if, $(2, 2+\beta) - (0, \beta) \ll (c_1, c_2)$

if, and only if, $(c_1 - 2, c_2 - 2) \in Int(\mathfrak{C})$

implies that $c_1 > 2$ *and* $c_2 > 2$.
Thus, $\wp\left(\left[\frac{1}{n}, 1+\frac{1}{n}\right], [-1,2]\right) - \wp([-1,2],[-1,2]) \ll c$ *can occur with appropriate* $c_1 > 2$ *and* $c_2 > 2$, *while* $c = (c_1, c_2) \gg \theta$ *was arbitrary. Thus, this conclusion causes a contradiction. Therefore, the interval sequence* $\left\{\left[\frac{1}{n}, 1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ *is not convergent to the fix-interval* $[-1,2]$. *From this point, one can readily see that*
$[0,1] \overset{\Omega}{=} [-1,2]$ *and* $[\frac{1}{n}, 1+\frac{1}{n}] \overset{\tau_\wp}{\to} [0,1]$, *but* $[\frac{1}{n}, 1+\frac{1}{n}] \overset{\tau_\wp}{\nrightarrow} [-1,2]$. *Certainly, the interval sequence* $\left\{\left[\frac{1}{n}, 1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ *is not convergent to any point* $[a,b] \in \langle[0,1]\rangle$ *and* $[0,1]$ *is the only interval limit of* $\left\{\left[\frac{1}{n}, 1+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$.

Continued from the previous two examples, if $[a,b]$ is a limit interval of $\{[a_n, b_n]\}_{n\in\mathbb{N}}$, then it is not necessarily that $[a_n, b_n] \overset{\tau_\wp}{\to} \langle[a,b]\rangle$. As a matter of fact, the statement is satisfied under a certain condition will be declared in details below.

Proposition 14*Consider the partial cone-interval metric space* $(\mathscr{I}(\mathbb{R}), E, \mathfrak{C}, \wp)$ *with* $\wp(I,I) = \theta$ *for any point* $I \in \mathscr{I}(\mathbb{R})$. *For any interval sequence* $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ *in* $\mathscr{I}(\mathbb{R})$, *if* $[a_n, b_n] \overset{\tau_\wp}{\to} [a,b]$, *then* $[a_n, b_n] \overset{\tau_\wp}{\to} \langle[a,b]\rangle$.

*Proof.*For any $[c,d] \in \langle[a,b]\rangle$, we can write $[c,d] := [a-k, b+k]$ for any $k \geq 0$. Need to show that $[a_n, b_n] \overset{\tau_\wp}{\to} [c,d]$. Due to (PCIM$_4$), we have
$\wp([a_n, b_n], [c,d]) \preceq \wp([a_n, b_n], [a,b]) + \wp([a,b], [c,d])$.
Since $[a_n, b_n] \overset{\tau_\wp}{\to} [a,b]$, then for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that $\wp([a_n, b_n], [a,b]) \ll c$ for all $n \geq n_0$. With arbitrary $c \gg \theta$ and for sufficiently large n, we have

$$
\begin{aligned}
\wp([a_n, b_n], [c,d]) &\ll c + \wp([a,b], [c,d]) \\
&= c + \wp([a,b], [a-k, b+k]) \\
&= c + \wp([a,b] \oplus [0,0], [a,b] \oplus [-k,k]) \\
&= c + \wp([a,b], [a,b]) \\
&= c.
\end{aligned}
$$

For strongly-type convergence, it is worth pointing that the limit interval of an interval sequence need not be unique. The following example makes this clear.

Example 15*Consider that* $E := \mathbb{R}^2$ *dealing with the norm* $\|(x,y)\|_\infty := \max\{|x|, |y|\}$ *in the order of the cone* $\mathfrak{C} := \left\{(x,y) \in \mathbb{R}^2 : x, y \geq 0\right\}$. *Let a partial ordering on* $\mathfrak{C}$ *be defined as in Example 13. Choose a mapping* $\wp : \mathscr{I}^2(\mathbb{R}) \to \mathfrak{C}$ *defined as*

$$\wp(I,J) = (|a+b-c-d|, 0) \in \mathfrak{C}$$

for any two real intervals I *and* J *in* $\mathscr{I}(\mathbb{R})$, *write* $I = [a,b]$ *and* $J = [c,d]$. *It could be easily seen that* $\left(\mathscr{I}(\mathbb{R}), \mathbb{R}^2, \mathfrak{C}, \wp\right)$ *forms a partial cone-interval metric* *space. Construct the sequence of intervals* $\{[a_n,b_n]\}_{n\in\mathbb{N}} := \left\{\left[1-\frac{1}{n+1}, 2+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$. *The computations yield*

$$\|\wp([a_n,b_n],[0,3]) - \wp([0,3],[0,3])\|_\infty = \left\|\wp\left(\left[1-\frac{1}{n+1},2+\frac{1}{n}\right],[0,3]\right) - \wp([0,3],[0,3])\right\|_\infty$$
$$= \left\|\left(\left|1-\frac{1}{n+1}+2+\frac{1}{n}-0-3\right|,0\right)\right\|_\infty$$
$$= \left\|\left(\frac{1}{n(n+1)},0\right)\right\|_\infty$$
$$= \frac{1}{n(n+1)} \underset{n\to\infty}{\to} 0.$$

Hence, we obtain that $\left[1-\frac{1}{n+1},2+\frac{1}{n}\right] \overset{s-\tau_\wp}{\to} [0,3]$. *In a fairly direct manner, we find that* $\left[1-\frac{1}{n+1},2+\frac{1}{n}\right] \overset{s-\tau_\wp}{\to} \left[\frac{1}{2},\frac{5}{2}\right]$. *Somewhat more generally,* $\left[1-\frac{1}{n+1},2+\frac{1}{n}\right] \overset{s-\tau_\wp}{\to} [a,b]$ *for any* $a,b\in\mathbb{R}$ *with* $a\le b$ *and* $a+b=3$. *Therefore, the real-valued interval sequence* $\left\{\left[1-\frac{1}{n+1},2+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ *have infinitely many limit intervals. Therefore, the generated topology* $\tau_\wp$ *is not Hausdorff.*

We proceed as follows:

Proposition 16*Let* $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ *be a partial cone-interval metric space and* $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ *be a sequence of intervals in* $\mathscr{I}(\mathbb{R})$. *If there exist* $[a,b],[c,d]\in\mathscr{I}(\mathbb{R})$ *such that* $[a_n,b_n] \overset{s-\tau_\wp}{\to} [a,b]$ *and* $[a_n,b_n] \overset{s-\tau_\wp}{\to} [c,d]$, *then we must have* $[a,b] \overset{\Omega}{=} [c,d]$.

*Proof.*Applying the criteria (PCIM_4), we have $\wp([a,b],[c,d]) \preceq \wp([a,b],[a_n,b_n]) + \wp([a_n,b_n],[c,d]) - \wp([a_n,b_n],[a_n,b_n])$. Taking the limit as $n\to\infty$, we get $\wp([a,b],[c,d]) \preceq \wp([a,b],[a,b]) + \wp([c,d],[c,d]) - \wp([a,b],[a,b]) = \wp([c,d],[c,d])$. From (PCIM_1), we know that $\wp([c,d],[c,d]) \preceq \wp([a,b],[c,d])$. Thus, $\wp([c,d],[c,d]) = \wp([a,b],[c,d])$. Running through the above arguments with $\lim_{n\to\infty}\wp([a_n,b_n],[a_n,b_n]) = \wp([c,d],[c,d])$, so that we get $\wp([a,b],[a,b]) = \wp([a,b],[c,d])$. Accordingly, we find that $\wp([a,b],[a,b]) = \wp([c,d],[c,d]) = \wp([a,b],[c,d])$, which implies $[a,b] \overset{\Omega}{=} [c,d]$.

The sequence $\{[a_n,b_n]\}_{n\in\mathbb{N}} := \left\{\left[1-\frac{1}{n+1},2+\frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ of Example 15 is in agreement with Proposition 16.

Lemma 17*The class limit in the partial cone-interval metric space is unique.*

The concept of near-coupled fixed-point is defined below.

Definition 18*Let* $T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R})$ *be a well-defined mapping. A constant interval vector* $([a,b],[c,d])\in\mathscr{I}^2(\mathbb{R})$ *is called a near-coupled fixed-point of T if, and only if,*

$$T([a,b],[c,d]) \overset{\Omega}{=} [a,b] \text{ and } T([c,d],[a,b]) \overset{\Omega}{=} [c,d].$$

To provide a simple illustration of this concept, we invoke the following example.

Example 19*Let* $T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R})$ *be defined as*

$$T([a,b],[c,d]) = \begin{cases} \frac{[a,b]}{[1,1]\ominus[c,d]}, & \text{if } c\le d<1 \text{ or } 1<c\le d, \\ [a,b], & \text{if } c\le 1\le d. \end{cases}$$

Note that the mapping T above is well-defined. By using interval arithmetic operations, we find that there are many infinitely near-coupled fixed-points of the mapping T. For instance, the pairs $\left(\left[\frac{-1}{2},\frac{1}{2}\right],[0,0]\right)$ *and* $([2,3],[0,0])$ *are near-coupled fixed-points of the mapping T in* $\mathscr{I}^2(\mathbb{R})$. *Nevertheless, we see that* $([2,3],[0,0]) \notin \left\langle\left(\left[\frac{-1}{2},\frac{1}{2}\right],[0,0]\right)\right\rangle_{\Omega\times\Omega}$.

In this paper, we are systematically interested in solving the following system of interval equations:

$$\begin{cases} S([a,b]) \overset{\Omega}{=} T([a,b],[c,d]) \\ S([c,d]) \overset{\Omega}{=} T([c,d],[a,b]), \end{cases} \tag{2}$$

where $T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R})$ and $S:\mathscr{I}(\mathbb{R})\to\mathscr{I}(\mathbb{R})$. The solution $([a,b],[c,d])\in\mathscr{I}^2(\mathbb{R})$ of the above system, if it exists, is called a near-coupled coincidence point of the mappings T and S, and $(S[a,b],S[c,d])$ is called near-coupled point of coincidence.

The following example illustrates the above-mentioned concept.

Example 20*Let the interval-vector mapping T be represented in the form*

$$\begin{cases} T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R}) \\ ([a,b],[c,d]) \mapsto \left[a+c-\frac{1}{2},|b+d|+\frac{1}{2}\right]. \end{cases}$$

Let S be an interval-valued mapping of the interval variable $[a,b]$ *such that*

$$\begin{cases} S:\mathscr{I}(\mathbb{R})\to\mathscr{I}(\mathbb{R}) \\ [a,b]\mapsto [a,|b|]. \end{cases}$$

By simple substitution, resulting in $T\left(\left[\frac{-1}{2},\frac{1}{2}\right],[-2,2]\right) = [-3,3]$, $S\left(\left[\frac{-1}{2},\frac{1}{2}\right]\right) = \left[\frac{-1}{2},\frac{1}{2}\right]$, $T\left([-2,2],\left[\frac{-1}{2},\frac{1}{2}\right]\right) = [-3,3]$ *and* $S([-2,2]) = [-2,2]$. *We simply try to show that* $[-3,3] \overset{\Omega}{=} \left[-\frac{1}{2},\frac{1}{2}\right]$ *and* $[-3,3] \overset{\Omega}{=} [-2,2]$. *For justifying, choose* $w_1,w_2\in\Omega$ *such that* $w_1 = \left[\frac{-5}{2},\frac{5}{2}\right]$ *and* $w_2 = [-1,1]$. *It is easy to evaluate the following resulting interval expressions:* $[-3,3] = \left[\frac{-1}{2},\frac{1}{2}\right]\oplus w_1$ *and* $[-3,3] = [-2,2]\oplus w_2$. *Therefore, the point* $\left(\left[\frac{-1}{2},\frac{1}{2}\right],[-2,2]\right)$ *defines a near-coupled coincidence point of the mappings T and S. Patently, for this example, the interval equations* (2) *have infinitely many solutions. More precisely, points on the form* $([-k_1,k_1],[-k_2,k_2])\in\Omega\times\Omega$, *where* $k_1,k_2\ge 0$, *are near-coupled coincidence points of the mappings T and S.*

3 Main Results: On The Existence and Uniqueness of Equivalence Class of Near-Coupled Coincidence Points

In the present section, we are mainly concerned with the existence of near-coupled coincidence points and the uniqueness of their equivalence class.

The following proposition prepares the way for the main result of this section, namely; Theorem 22.

Proposition 21 *Let $(\mathscr{I}(\mathbb{R}), E, \mathfrak{C}, \wp)$ be a partial cone-interval metric space. Define the mappings $T : \mathscr{I}^2(\mathbb{R}) \to \mathscr{I}(\mathbb{R})$ and $S : \mathscr{I}(\mathbb{R}) \to \mathscr{I}(\mathbb{R})$. Assume that the following assertions are satisfied:*

(i) $T(\mathscr{I}^2(\mathbb{R})) \subseteq S(\mathscr{I}(\mathbb{R}))$;

(ii) There exists a constant $0 < \lambda < 1$ such that $\wp(T([a,b],[c,d]), T([e,f],[g,h])) \preceq \frac{\lambda}{2}(\wp(S([a,b]), S([e,f])) + \wp(S([c,d]), S([g,h])))$ satisfies for all intervals $[a,b],[c,d],[e,f],[g,h] \in \mathscr{I}(\mathbb{R})$;

(iii) There exist sequences $\{S([a_n,b_n])\}_{n\in\mathbb{N}}$ and $\{S([c_n,d_n])\}_{n\in\mathbb{N}}$ of successive approximations of T starting from $S([a_0,b_0])$ and $S([c_0,d_0])$, respectively.

Then, for all $n \in \mathbb{N}$ and for some $0 < \lambda < 1$, the following estimations hold simultaneously:

$$\begin{cases} \wp(S([a_n,b_n]), S([a_{n+1},b_{n+1}])) \preceq \frac{\lambda^n}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1]))), \\ \wp(S([c_n,d_n]), S([c_{n+1},d_{n+1}])) \preceq \frac{\lambda^n}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1]))). \end{cases} \quad (3)$$

Proof. In view of the given assumption $T(\mathscr{I}^2(\mathbb{R})) \subseteq S(\mathscr{I}(\mathbb{R}))$, we can choose the points $[a_1,b_1],[c_1,d_1] \in \mathscr{I}(\mathbb{R})$ such that $S([a_1,b_1]) \stackrel{\Omega}{=} T([a_0,b_0],[c_0,d_0])$ and $S([c_1,d_1]) \stackrel{\Omega}{=} T([c_0,d_0],[a_0,b_0])$. If we carry over this way, we constitute the sequences $[a_{n+1},b_{n+1}]$ and $[c_{n+1},d_{n+1}]$ in $\mathscr{I}(\mathbb{R})$ by

$$\begin{cases} S([a_{n+1},b_{n+1}]) \stackrel{\Omega}{=} T([a_n,b_n],[c_n,d_n]), \\ S([c_{n+1},d_{n+1}]) \stackrel{\Omega}{=} T([c_n,d_n],[a_n,b_n]) \end{cases} \quad (4)$$

for all $n \in \mathbb{N} \cup \{0\}$. Toward proving (3), we will use the mathematical induction. We argue as follows:
For the value $n = 1$, we have
$\wp(S([a_1,b_1]), S([a_2,b_2])) = \wp(T([a_0,b_0],[c_0,d_0]), T([a_1,b_1],[c_1,d_1]))$
$\preceq \frac{\lambda}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])))$.
Exactly in the similar manner as above, we have
$\wp(S([c_1,d_1]), S([c_2,d_2])) \preceq \frac{\lambda}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))$.
Thus, the statement is true for $n = 1$. If it is true for $n = k$, then we have
$\wp(S([a_k,b_k]), S([a_{k+1},b_{k+1}])) \preceq \frac{\lambda^k}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])))$,
$\wp(S([c_k,d_k]), S([c_{k+1},d_{k+1}])) \preceq \frac{\lambda^k}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))$.
Now, at step $n = k+1$, we come by
$\wp(S([a_{k+1},b_{k+1}]), S([a_{k+2},b_{k+2}])) = \wp(T([a_k,b_k],[c_k,d_k]), T([a_{k+1},b_{k+1}],[c_k,d_k]))$
$\preceq \frac{\lambda}{2}(\wp(S([a_k,b_k]), S([a_{k+1},b_{k+1}])) + \wp(S([c_k,d_k]), S([c_{k+1},d_{k+1}])))$
$\preceq \frac{\lambda}{2}\left(\frac{\lambda^k}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1]))) + \frac{\lambda^k}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))\right)$
$= \frac{\lambda}{2}(\lambda^k(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1]))))$
$= \frac{\lambda^{k+1}}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])))$. In a similar manner, it is seen that
$\wp(S([c_{k+1},d_{k+1}]), S([c_{k+2},d_{k+2}])) \preceq \frac{\lambda^{k+1}}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))$,
which is true for $n = k+1$. This completes the induction argument on n.

Theorem 22. *Consider a $\theta-$complete partial cone-interval metric space $(\mathscr{I}(\mathbb{R}), E, \mathfrak{C}, \wp)$ relative to a solid cone $\mathfrak{C}$. If there exist two mappings $T : \mathscr{I}^2(\mathbb{R}) \to \mathscr{I}(\mathbb{R})$ and $S : \mathscr{I}(\mathbb{R}) \to \mathscr{I}(\mathbb{R})$ so that*

(i) $\wp(T([a,b],[c,d]), T([e,f],[g,h])) \preceq \frac{\lambda}{2}(\wp(S([a,b]), S([e,f])) + \wp(S([c,d]), S([g,h])))$ satisfies for all intervals $[a,b],[c,d],[e,f],[g,h] \in \mathscr{I}(\mathbb{R})$ with $0 < \lambda < 1$;

(ii) $T(\mathscr{I}^2(\mathbb{R})) \subseteq S(\mathscr{I}(\mathbb{R}))$;

(iii) $S(\mathscr{I}(\mathbb{R}))$ is closed subset of $\mathscr{I}(\mathbb{R})$,

then T and S have precisely a unique equivalence class of near-coupled coincidence points in $\mathscr{I}^2(\mathbb{R})$.

Proof. Let us start from two generic points $S([a_0,b_0]), S([c_0,d_0])$ in $S(\mathscr{I}(\mathbb{R}))$ and consider the coupled Picard pair iterative scheme

$$\begin{cases} S([a_{n+1},b_{n+1}]) \stackrel{\Omega}{=} T([a_n,b_n],[c_n,d_n]), \\ S([c_{n+1},d_{n+1}]) \stackrel{\Omega}{=} T([c_n,d_n],[a_n,b_n]) \end{cases} \quad (5)$$

for all $n \in \mathbb{N} \cup \{0\}$. For the purpose at hand, we distinguish two cases:
Case (i): If we choose $\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])) = \theta$, then we have $S([a_0,b_0]) \stackrel{\Omega}{=} T([a_0,b_0],[c_0,d_0])$ and $S([c_0,d_0]) \stackrel{\Omega}{=} T([c_0,d_0],[a_0,b_0])$. This means that $([a_0,b_0],[c_0,d_0])$ is a near-coupled coincidence point of T and S, and so there is nothing to prove.
Hence, we may suppose that $\wp(S([a_0,b_0]), S([a_1,b_1])) \succ \theta$ and $\wp(S([c_0,d_0]), S([c_1,d_1])) \succ \theta$.
Case (ii): If $S([a_0,b_0]) \stackrel{\Omega}{=} S([a_n,b_n])$ and $S([c_0,d_0]) \stackrel{\Omega}{=} S([c_n,d_n])$ for any $n \geq 2$, then $S([a_0,b_0]) \oplus w_1 = S([a_n,b_n]) \oplus w_2$ and $S([c_0,d_0]) \oplus w_3 = S([c_n,d_n]) \oplus w_4$ for some w_1, w_2, w_3 and w_4 in Ω.
For sufficiently large n, we observe the following:
$\wp(S([a_0,b_0]), S([a_1,b_1])) = \wp(S([a_n,b_n]), S([a_1,b_1]))$
$= \wp(T([a_{n-1},b_{n-1}],[c_{n-1},d_{n-1}]), T([a_0,b_0],[c_0,d_0]))$
$\preceq \frac{\lambda}{2}(\wp(S([a_{n-1},b_{n-1}]), S([a_0,b_0])) + \wp(S([c_{n-1},d_{n-1}]), S([c_0,d_0])))$
$= \frac{\lambda}{2}(\wp(S([a_{n-1},b_{n-1}]), S([a_0,b_0]) \oplus w_1) + \wp(S([c_{n-1},d_{n-1}]), S([c_0,d_0]) \oplus w_3))$
$= \frac{\lambda}{2}(\wp(S([a_{n-1},b_{n-1}]), S([a_n,b_n]) \oplus w_2) + \wp(S([c_{n-1},d_{n-1}]), S([c_n,d_n]) \oplus w_4))$
$= \frac{\lambda}{2}(\wp(S([a_{n-1},b_{n-1}]), S([a_n,b_n])) + \wp(S([c_{n-1},d_{n-1}]), S([c_n,d_n])))$
$\preceq \frac{\lambda}{2}(\lambda^{n-1}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1]))))$
$\preceq \frac{\lambda^n}{2}(\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])))$.
Proceeding along the same lines as above, we attain
$\wp(S([c_0,d_0]), S([c_1,d_1])) \preceq \frac{\lambda^n}{2}(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))$.
Adding up, we get
$\wp(S([a_0,b_0]), S([a_1,b_1])) + \wp(S([c_0,d_0]), S([c_1,d_1])) \preceq \lambda^n(\wp(S([c_0,d_0]), S([c_1,d_1])) + \wp(S([a_0,b_0]), S([a_1,b_1])))$.
From this inequality and by taking advantage of Lemma 7, we have $\wp(S([a_0,b_0]), S([a_1,b_1])) = \theta$ and $\wp(S([c_0,d_0]), S([c_1,d_1])) = \theta$.

Thereby, it must be the case that
$S([a_0,b_0]) \overset{\Omega}{=} S([a_1,b_1]) \overset{\Omega}{=} T([a_0,b_0],[c_0,d_0])$ and $S([c_0,d_0]) \overset{\Omega}{=} S([c_1,d_1]) \overset{\Omega}{=} T([c_0,d_0],[a_0,b_0])$.
So that $([a_0,b_0],[c_0,d_0])$ is near-coupled coincidence point of T and S. Keeping generality in mind, we presume that $\{(S([a_n,b_n]),S([c_n,d_n]))\}_{n\in\mathbb{N}} \in S(\mathscr{I}(\mathbb{R}))\times S(\mathscr{I}(\mathbb{R}))$ contains no near-coupled point of coincidence; that is, $(S([a_n,b_n]),S([c_n,d_n])) \overset{\Omega\times\Omega}{\neq} (S([a_{n+1},b_{n+1}]),S([c_{n+1},d_{n+1}]))$ for $n \in \mathbb{N}\cup\{0\}$. Thus, we suppose that $\wp(S([a_n,b_n]),S([a_{n+1},b_{n+1}])) \succ \theta$ and $\wp(S([c_n,d_n]),S([c_{n+1},d_{n+1}])) \succ \theta$. Otherwise, $([a_n,b_n],[c_n,d_n])$ is a near-coupled coincidence point of T and S.

One way to think about the previous two cases is that if $\{(S([a_n,b_n]),S([c_n,d_n]))\}_{n\in\mathbb{N}}$ contains no near-coupled point of coincidence, then $\{(S([a_n,b_n]),S([c_n,d_n]))\}_{n\in\mathbb{N}}$ approaches to the desired near-coupled point of coincidence.

The key step in proving the existence of a near-coupled point of coincidence is just to show that the sequences $\{S([a_n,b_n])\}_{n\in\mathbb{N}}$ and $\{S([c_n,d_n])\}_{n\in\mathbb{N}}$ represent $\theta-$Cauchy (and then imposing the $\theta-$completeness). To each indicates $n,p\in\mathbb{N}$, we investigate that

$$\wp(S([a_n,b_n]),S([a_{n+p},b_{n+p}])) \overset{(\mathrm{PCIM}_4)}{\preceq} \sum_{i=n}^{n+p-1}\wp(S([a_i,b_i]),S([a_{i+1},b_{i+1}])) - \sum_{i=n+1}^{n+p-1}\wp(S([a_i,b_i]),S([a_i,b_i]))$$
$$\preceq \sum_{i=n}^{n+p-1}\wp(S([a_i,b_i]),S([a_{i+1},b_{i+1}]))$$
$$\preceq \frac{\lambda^n}{2}+\frac{\lambda^{n+1}}{2}+\ldots+\frac{\lambda^{n+p-1}}{2}\left(\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))\right)$$
$$= \frac{\lambda^n-\lambda^{n+p-1}}{2(1-\lambda)}\left(\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))\right)$$
$$\prec \frac{\lambda^n}{2(1-\lambda)}\left(\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))\right).$$

Henceforth, we come to the conclusion that
$\wp(S([a_n,b_n]),S([a_{n+p},b_{n+p}])) \preceq \frac{\lambda^n}{2(1-\lambda)}\left(\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))\right)$,
for certain $0<\lambda<1$. For all n, passage to the limit yields $\frac{\lambda^n}{2(1-\lambda)}\left(\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))\right) \underset{n\to\infty}{\to} \theta$. This is immediate since $0<\lambda<1$ and $\wp(S([a_0,b_0]),S([a_1,b_1]))+\wp(S([c_0,d_0]),S([c_1,d_1]))$ is fixed. The above proof concludes that $\lim_{n\to\infty}\wp(S([a_n,b_n]),S([a_{n+p},b_{n+p}]))=\theta$. In this sense, $\{\wp(S([a_n,b_n]),S([a_{n+p},b_{n+p}]))\}_{n,p\in\mathbb{N}}$ is a $c-$sequence for any $c\gg\theta$. This displays that $\{S([a_n,b_n])\}_{n\in\mathbb{N}}$ is $\theta-$Cauchy. Performing the same process one can show that $\{S([c_n,d_n])\}_{n\in\mathbb{N}}$ is $\theta-$Cauchy. From this and the $\theta-$completeness of $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$, it holds that there corresponds some elements $[\bar{a},\bar{b}],[\bar{c},\bar{d}]\in\mathscr{I}(\mathbb{R})$ in such a way that $S([a_n,b_n]) \overset{\tau_\wp}{\to} [\bar{a},\bar{b}]$ and $S([c_n,d_n]) \overset{\tau_\wp}{\to} [\bar{c},\bar{d}]$ with $\wp([\bar{a},\bar{b}],[\bar{a},\bar{b}])=\theta$ and $\wp([\bar{c},\bar{d}],[\bar{c},\bar{d}])=\theta$. Since $\{S([a_n,b_n])\}_{n\in\mathbb{N}}\subseteq S(\mathscr{I}(\mathbb{R}))$ and $S(\mathscr{I}(\mathbb{R}))$ is closed, it follows that $[\bar{a},\bar{b}]$ belongs surely to $S(\mathscr{I}(\mathbb{R}))$. Thus, it should be some element $[a',b']\in\mathscr{I}(\mathbb{R})$ such that $S([a',b'])=[\bar{a},\bar{b}]$. In a fairly direct manner, we have $S([c',d'])=[\bar{c},\bar{d}]$ for some $[c',d']\in\mathscr{I}(\mathbb{R})$. It insures that $S[a_n,b_n] \overset{\tau_\wp}{\to} S([a',b'])$ and $S[c_n,d_n] \overset{\tau_\wp}{\to} S([c',d'])$ with $\wp(S([a',b']),S([a',b']))=\theta$ and $\wp(S([c',d']),S([c',d']))=\theta$. Corresponding to any $c\in E$ with $c\gg\theta$, we can successively find $n_1,n_2\in\mathbb{N}$ such that

$$\begin{cases} \frac{\lambda}{2}\wp(S([a',b']),S([a_n,b_n])) \ll \frac{c}{3} \text{ for all } n\geq n_1; \\ \wp(S([a',b']),S([a_{n+1},b_{n+1}])) \ll \frac{c}{3} \text{ for all } n\geq n_1; \\ \frac{\lambda}{2}\wp(S([c',d']),S([c_n,d_n])) \ll \frac{c}{3} \text{ for all } n\geq n_2. \end{cases}$$

Denoting $n_0:=\max\{n_1,n_2\}$, hence with any arbitrary $c\gg\theta$ whenever $n\geq n_0$, we get

$$\begin{cases} \frac{\lambda}{2}\wp(S([a',b']),S([a_n,b_n])) \ll \frac{c}{3}; \\ \wp(S([a',b']),S([a_{n+1},b_{n+1}])) \ll \frac{c}{3}; \\ \frac{\lambda}{2}\wp(S([c',d']),S([c_n,d_n])) \ll \frac{c}{3}. \end{cases}$$

Following that, we claim that any $(S([\tilde{a},\tilde{b}]),S([\tilde{c},\tilde{d}]))\in\langle(S([a',b']),S([c',d']))\rangle_{\Omega\times\Omega}$ defines a near-coupled point of coincidence. Thus, there exist $w_1,w_2,w_3,w_4\in\Omega$ such that
$S([\tilde{a},\tilde{b}])\oplus w_1=S([a',b'])\oplus w_2$ and $S([\tilde{c},\tilde{d}])\oplus w_3=S([c',d'])\oplus w_4$.
It is quite easy to prove the following:

$$\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}])) = \wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}])\oplus w_1)$$
$$\overset{(\mathrm{PCIM}_4)}{\preceq} \wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([a_{n_0+1},b_{n_0+1}]))+\wp(S([a_{n_0+1},b_{n_0+1}]),S([\tilde{a},\tilde{b}])\oplus w_1)$$
$$= \wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),T([a_{n_0},b_{n_0}],[c_{n_0},d_{n_0}]))+\wp(S([a_{n_0+1},b_{n_0+1}]),S([\tilde{a},\tilde{b}])\oplus w_1)$$
$$\preceq \frac{\lambda}{2}\left(\wp(S([\tilde{a},\tilde{b}]),S([a_{n_0},b_{n_0}]))+\wp(S([\tilde{c},\tilde{d}]),S([c_{n_0},d_{n_0}]))\right)$$
$$+\wp(S([a_{n_0+1},b_{n_0+1}]),S([\tilde{a},\tilde{b}])\oplus w_1)$$
$$= \frac{\lambda}{2}\left(\wp(S([\tilde{a},\tilde{b}])\oplus w_1,S([a_{n_0},b_{n_0}]))+\wp(S([\tilde{c},\tilde{d}])\oplus w_3,S([c_{n_0},d_{n_0}]))\right)$$
$$+\wp(S([a_{n_0+1},b_{n_0+1}]),S([\tilde{a},\tilde{b}])\oplus w_1)$$
$$= \frac{\lambda}{2}\left(\wp(S([a',b'])\oplus w_2,S([a_{n_0},b_{n_0}]))+\wp(S([c',d'])\oplus w_4,S([c_{n_0},d_{n_0}]))\right)$$
$$+\wp(S([a_{n_0+1},b_{n_0+1}]),S([a',b'])\oplus w_2)$$
$$= \frac{\lambda}{2}\left(\wp(S([a',b']),S([a_{n_0},b_{n_0}]))+\wp(S([c',d']),S([c_{n_0},d_{n_0}]))\right)$$
$$+\wp(S([a_{n_0+1},b_{n_0+1}]),S([a',b']))$$
$$\ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3}$$
$$= c.$$

Hence, with any $c\gg\theta$, we get $\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\ll c$. From the arbitrary choice of c, we procure $\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\ll\frac{c}{m}$ for any $\frac{c}{m}\gg\theta$ and for any $m\in\mathbb{N}$. According to this, we bear that $\{\frac{c}{m}-\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\}_{m\in\mathbb{N}}\subseteq\mathfrak{C}$. Since $\mathfrak{C}$ is a closed subset in E, then we have $-\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\in\mathfrak{C}$. We also know that $\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\in\mathfrak{C}$.
Thus, $\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))\in\mathfrak{C}\cap(-\mathfrak{C})=\{\theta\}$. From which it follows that $\wp(T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}]),S([\tilde{a},\tilde{b}]))=\theta$, and thus $T([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}])\overset{\Omega}{=}S([\tilde{a},\tilde{b}])$ for every $(S([\tilde{a},\tilde{b}]),S([\tilde{c},\tilde{d}]))$ in $\langle(S([a',b']),S([a',b']))\rangle_{\Omega\times\Omega}$.
In a similar way, we can show that $T([\tilde{c},\tilde{d}],[\tilde{a},\tilde{b}])\overset{\Omega}{=}S([\tilde{c},\tilde{d}])$ for any $(S([\tilde{a},\tilde{b}]),S([\tilde{c},\tilde{d}]))$ in $\langle(S([a',b']),S([c',d']))\rangle_{\Omega\times\Omega}$.

Lastly, it remains to show that $\langle(S([a',b']),S([c',d']))\rangle_{\Omega\times\Omega}$ is the unique equivalence class of near-coupled points of coincidence. Make a counter-hypothesis: assume that $(S([\hat{a},\hat{b}]),S([\hat{c},\hat{d}]))$ is one more near-coupled point of coincidence such that $(S([\hat{a},\hat{b}]),S([\hat{c},\hat{d}]))\notin\langle(S([a',b']),S([c',d']))\rangle_{\Omega\times\Omega}$.
Analogously,
$T([\hat{a},\hat{b}],[\hat{c},\hat{d}])\overset{\Omega}{=}S([\hat{a},\hat{b}])$ and $T([\hat{c},\hat{d}],[\hat{a},\hat{b}])\overset{\Omega}{=}S([\hat{c},\hat{d}])$.
Thus, there exist $w_i\in\Omega$ for $1\leq i\leq 8$ provided that

$$\begin{cases} T\left(\left[\hat{a},\hat{b}\right],\left[\hat{c},\hat{d}\right]\right)\oplus w_1 = S\left(\left[\hat{a},\hat{b}\right]\right)\oplus w_2, \\ T\left(\left[\hat{c},\hat{d}\right],\left[\hat{a},\hat{b}\right]\right)\oplus w_3 = S\left(\left[\hat{c},\hat{d}\right]\right)\oplus w_4, \\ T\left([a',b'],[c',d']\right)\oplus w_5 = S\left([a',b']\right)\oplus w_6, \\ T\left([c',d'],[a',b']\right)\oplus w_7 = S\left([c',d']\right)\oplus w_8. \end{cases}$$

For this, we proceed as follows:

$\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right) = \wp\left(S\left(\left[\hat{a},\hat{b}\right]\right)\oplus w_2, S\left([a',b']\right)\oplus w_6\right)$
$= \wp\left(T\left(\left[\hat{a},\hat{b}\right],\left[\hat{c},\hat{d}\right]\right)\oplus w_1, T\left([a',b'],[c',d']\right)\oplus w_5\right)$
$= \wp\left(T\left(\left[\hat{a},\hat{b}\right],\left[\hat{c},\hat{d}\right]\right), T\left([a',b'],[c',d']\right)\right)$
$\preceq \frac{\lambda}{2}\left(\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)+\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)\right).$

Hence,
$\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right) \preceq \frac{\lambda}{2}\left(\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)+\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)\right).$
On the other hand,
$\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right) = \wp\left(S\left(\left[\hat{c},\hat{d}\right]\right)\oplus w_4, S\left([c',d']\right)\oplus w_8\right)$
$= \wp\left(T\left(\left[\hat{c},\hat{d}\right],\left[\hat{a},\hat{b}\right]\right)\oplus w_3, T\left([c',d'],[a',b']\right)\oplus w_7\right)$
$= \wp\left(T\left(\left[\hat{c},\hat{d}\right],\left[\hat{a},\hat{b}\right]\right), T\left([c',d'],[a',b']\right)\right)$
$\preceq \frac{\lambda}{2}\left(\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)+\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)\right).$

Therefore,
$\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right) \preceq \frac{\lambda}{2}\left(\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)+\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)\right).$
Adding up, we get
$\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)+\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right) \preceq \lambda\left(\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)+\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)\right).$
But since $0<\lambda<1$, by Lemma 7, it immediately follows $\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)+\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)=\theta$, which means that $\wp\left(S\left(\left[\hat{a},\hat{b}\right]\right),S\left([a',b']\right)\right)=\theta$ and $\wp\left(S\left(\left[\hat{c},\hat{d}\right]\right),S\left([c',d']\right)\right)=\theta$. We accordingly have
$S\left(\left[\hat{a},\hat{b}\right]\right)\overset{\Omega}{=}S\left([a',b']\right)$ and $S\left(\left[\hat{c},\hat{d}\right]\right)\overset{\Omega}{=}S\left([c',d']\right)$, which signifies a contradiction. This contradiction proves the expected uniqueness, which ends the proof of the theorem.

Corollary 23 *Consider a $\theta-$complete partial cone-interval metric space $(\mathscr{I}(\mathbb{R}),E,\mathfrak{C},\wp)$ relative to a solid cone $\mathfrak{C}$. Define the mapping $T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R})$ such that the contractive inequality*
$\wp\left(T\left([a,b],[c,d]\right),T\left([e,f],[g,h]\right)\right)\preceq\frac{\lambda}{2}\left(\wp\left([a,b],[e,f]\right)+\wp\left([c,d],[g,h]\right)\right)$
satisfies for all intervals $[a,b],[c,d],[e,f],[g,h]\in\mathscr{I}(\mathbb{R})$ with $0<\lambda<1$.
Then, T has precisely a unique equivalence class of near-coupled fixed-points in $\mathscr{I}^2(\mathbb{R})$.

Proof. Assume the notation of Theorem 22 with $S=I_{\mathscr{I}(\mathbb{R})}$, the identity mapping on $\mathscr{I}(\mathbb{R})$.

In the favor of the above-mentioned theorem, the following justify example substantiates the result.

Example 24 *With reference to the partial cone-interval metric space of Example 15. In practice, this distance structure is $\theta-$complete. Postulate that the mappings T and S are defined by*

$$\begin{cases} T:\mathscr{I}^2(\mathbb{R})\to\mathscr{I}(\mathbb{R}) \\ ([a,b],[c,d])\mapsto\left[a+c-\frac{1}{2},b+d+\frac{1}{2}\right] \end{cases}$$

$$\begin{cases} S:\mathscr{I}(\mathbb{R})\to\mathscr{I}(\mathbb{R}) \\ [a,b]\mapsto[8a,8b]. \end{cases}$$

Fix $\lambda=\frac{1}{2}\in(0,1)$.
For any real intervals $[a,b],[c,d],[e,f]$, and $[g,h]$ in $\mathscr{I}(\mathbb{R})$, we get
$\wp\left(T\left([a,b],[c,d]\right),T\left([e,f],[g,h]\right)\right)=\wp\left(\left[a+c-\frac{1}{2},b+d+\frac{1}{2}\right],\left[e+g-\frac{1}{2},f+h+\frac{1}{2}\right]\right)$
$=\left(|a+c+b+d-e-g-f-h|,0\right)$
$\preceq\left(|a+b-e-f|+|c+d-g-h|,0\right)$
$=\left(|a+b-e-f|,0\right)+\left(|c+d-g-h|,0\right)$
$\preceq\frac{1}{4}\left(\wp\left(S\left([a,b]\right),S\left([e,f]\right)\right)+\wp\left(S\left([c,d]\right),S\left([g,h]\right)\right)\right)$
$=\frac{1}{4}\left(\wp\left([8a,8b],[8e,8f]\right)+\wp\left([8c,8d],[8g,8h]\right)\right)$
$=\frac{1}{4}\left[\left(|8a+8b-8e-8f|,0\right)+\left(|8c+8d-8g-8h|,0\right)\right]$
$=\frac{1}{4}\left[\left(|8a+8b-8e-8f|+|8c+8d-8g-8h|,0\right)\right]$
$=2\left[\left(|a+b-e-f|+|c+d-g-h|,0\right)\right].$
Thus, the contractive inequality condition on T and S is satisfied. Now, we are going to solve the system (2) by means of the coupled Picard pair iterative scheme (5) for arriving at a near-coupled coincidence point for the given mappings. For this, let $[a_n,b_n]=\left[-\frac{1}{n},\frac{1}{n}\right]$ and $[c_n,d_n]=[-(n+1),n+1]$. We can easily see that $T\left(\left[-\frac{1}{n},\frac{1}{n}\right],[-(n+1),n+1]\right)=\left[-\frac{(2n^2+3n+2)}{2n},\frac{(2n^2+3n+2)}{2n}\right]\overset{\Omega}{=}S\left([a_{n+1},b_{n+1}]\right)=\left[-\frac{8}{n+1},\frac{8}{n+1}\right].$
If we map the sequence $\left\{\left([-(n+1),n+1],\left[-\frac{1}{n},\frac{1}{n}\right]\right)\right\}_{n\in\mathbb{N}}$ by T, we have
$T\left([-(n+1),n+1],\left[-\frac{1}{n},\frac{1}{n}\right]\right)=\left[-\frac{(2n^2+3n+2)}{2n},\frac{(2n^2+3n+2)}{2n}\right]\overset{\Omega}{=}S\left([c_{n+1},d_{n+1}]\right)=[-8(n+2),8(n+2)].$
Then, the sequence of the successive approximations, if convergent, converges to a near-coupled coincidence point of T and S. To analyze the convergence, consider the following numerical process:
$\left\|\wp\left(S\left(\left[-\frac{1}{n},\frac{1}{n}\right]\right),S\left([0,0]\right)\right)-\wp\left(S\left([0,0]\right),S\left([0,0]\right)\right)\right\|_\infty=\|(0,0)\|_\infty\underset{n\to\infty}{\to}0.$
Thus, $\wp\left(S\left(\left[-\frac{1}{n},\frac{1}{n}\right]\right),S\left([0,0]\right)\right)-\wp\left(S\left([0,0]\right),S\left([0,0]\right)\right)\ll c$ for any $c\in E$ with $c\gg\theta$ and for n enough large. Consequently, $S\left(\left[-\frac{1}{n},\frac{1}{n}\right]\right)\overset{\tau_\wp}{\to}\langle[0,0]\rangle$. In a similar manner, one can show that $S\left([-(n+1)1,n+1]\right)\overset{\tau_\wp}{\to}\langle[0,0]\rangle$. It insures that all the postulates on the considered mappings T and S are valid. Therefore, we can apply Theorem 22 and conclude that $\langle([0,0],[0,0])\rangle_{\Omega\times\Omega}:=\{([-k_1,k_1],[-k_2,k_2]):k_1,k_2\geq0\}$ defines the unique equivalence class of near-coupled coincidence points of the mappings T and S in $\mathscr{I}^2(\mathbb{R})$.

4 Conclusions

In the two-dimensional interval vector space $\mathscr{I}^2(\mathbb{R})$, which is not a (conventional) vector space because the concept of an inverse element is not available in general, we coined the concept of a null set $\Omega\times\Omega$ to play the vital role of a zero element in $\mathscr{I}^2(\mathbb{R})$. Several related terminologies are discussed. We presented the so-called near-coupled fixed-point and near-coupled coincidence point theorem and some interesting convergence results. The new results are formulated in the framework of partial cone-interval metric spaces and supported by relevant examples.

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