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Power–Linear Hazard Distribution Via k-th Record Values and Characterization

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Abstract: The power-linear hazard distribution was introduced by [1] as a mixture of power and linear hazard function. This distribution embodies the many lifetime distributions. This paper devotes to establish the recurrence relation based on k^{th} record values. Moreover, characterization results also derived via moment properties.

Keywords: k^{th} record values, life-time distribution, single- product moments and characterization.

1 Introduction

The power-linear hazard function (P-LHF) is addressed by [1]

$$h(p) = \alpha p^{\gamma} + \beta p, \quad \gamma > -1, \alpha, \beta \geq 0 \quad \text{and} \quad p > 0. \quad (1)$$

where $\gamma \neq 1$ and $\alpha + \beta > 0$.

The *P-LHF* is very simple and exhibits the bathtub-shaped, constant, decreasing and increasing hazard function. The *P-LHF* is considered more flexible among life-time distribution.

Suppose P denotes the random variable (R.V.) having *P-LHF* in (1). Then the cumulative density function (CDF) and probability density function (PDF) of P are outlined by

$$V(p) = 1 - e^{-\left\{\frac{\beta}{2}p^2 + \frac{\alpha}{\gamma+1}p^{\gamma+1}\right\}}, \quad p > 0 \quad (2)$$

$$v(p) = (\beta p + \alpha p^{\gamma})e^{-\left\{\frac{\beta}{2}p^2 + \frac{\alpha}{\gamma+1}p^{\gamma+1}\right\}}. \quad (3)$$

This distribution is currently widely used in different fields of studies, reliability, hydrology, engineering, insurance, and economics. *P-LH* distribution contains some important lifetime distributions (exponential, Rayleigh, Weibull, linear, quadratic, and power hazard distribution). For more details see [1].

From Equations (2) and (3), we note that.

$$v(p) = (\beta p + \alpha p^{\gamma})\bar{V}(p) \quad (4)$$

which will be considered for obtaining the required results from (3).

First time in literature, the idea of record values was incepted by [2] and studied many of its basic properties. It spans over several realms of applications, for example, sports, weather, economics, hydrology, future planning, and many more. For a more in-depth look at the record, see [3], [4], and [5].

The model of k^{th} record values is taken into consideration when record values itself taken as an outlier. The k^{th} record proposed by [6] and its applications in reliability theory was cited by [7] and [8] and many more.

Let $\{P_a, a \geq 1\}$ be a sequence of independent and identically distributed (iid) R.V. with CDF $V(p)$ and PDF $v(p)$. For a fixed positive integer k, we interpret the sequence $\{Z_a^{(k)},\ a \geq 1\}$ of k^{th} upper record times of $\{P_a, a \geq 1\}$ as:

$$Z_1^{(k)} = 1$$

$$Z_{(a+1)}^{(k)} = \min\left\{\varphi > Z_a^{(k)} : P_{\varphi:\varphi+k-1} > P_{Z_a^{(k)}:Z_{a+k-1}^{(k)}}\right\}.$$

The sequence $\{Q_a^{(k)}, a \geq 1\}$, where $Q_a^{(k)} = P_{Z_a^{(k)}}$ is termed the sequence of k^{th} upper record values of $\{P_a, a \geq 1\}$. The $\{Q_a^{(k)}, a \geq 1\}$ corresponds to the upper record values at $k = 1$.

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The PDF of $Q_a^{(k)}$ and the joint PDF of $Q_b^{(k)}$ and $Q_a^{(k)}$ are given in (5–6) by [6] and [9]

$$v_{Q_a^{(k)}}(p) = \frac{k^a}{(a-1)!}[-\ln\bar{V}(p)]^{a-1}[\bar{V}(p)]^{k-1}v(p), \quad (5)$$

$$v_{Q_b^{(k)},Q_a^{(k)}}(p,q) = \Psi[-\ln\bar{V}(p)]^{b-1}\frac{v(p)}{\bar{V}(p)} \times [\ln\bar{V}(p)-\ln\bar{V}(q)]^{a-b-1}[\bar{V}(q)]^{k-1}v(q), \quad p<q, \quad 1\le b<a, \quad a\ge 2 \quad (6)$$

where

$$\bar{V}(p) = 1-V(p) \quad \text{and} \quad \Psi = \frac{k^a}{(b-1)!(a-b-1)!}.$$

The moment properties of k^{th} record values for different distribution have been discussed by many authors. For a detailed see, [10,11,12,13,14,15,16] and cited therein. It has been seen that no attention paid to obtain the moments of k^{th} record from (3) in previous study.

This article outlined as follows. Moments of k^{th} record values are proved in Sections 2-3 via recurrence relations. Characterization results are concluded in Section 4. Conclusion is reported at the end.

2 Single Moments

The aim of this section is to attain single moments via recurrence relations.

Theorem 1. Fix a positive integer $k\ge 1, a\ge 1$ and $\varphi = 0,1,\cdots,$

$$E\left(Q_a^{(k)}\right)^{\varphi} = \frac{\beta k}{\varphi+2}\left\{E\left(Q_a^{(k)}\right)^{\varphi+2} - E\left(Q_{a-1}^{(k)}\right)^{\varphi+2}\right\} + \frac{\alpha k}{\varphi+\gamma+1}\left\{E\left(Q_a^{(k)}\right)^{\varphi+\gamma+1} - E\left(Q_{a-1}^{(k)}\right)^{\varphi+\gamma+1}\right\} \quad (7)$$

and

$$E\left(Q_1^{(k)}\right)^{\varphi} = \frac{\beta k}{\varphi+2}\left[E\left(Q_1^{(k)}\right)^{\varphi+2}\right] + \frac{\alpha k}{\varphi+\gamma+1}\left[E\left(Q_1^{(k)}\right)^{\varphi+\gamma+1}\right], \text{ for } a=1. \quad (8)$$

Proof. Using (5), we have

$$E\left(Q_a^{(k)}\right)^{\varphi} = \frac{k^a}{(a-1)!} \times \int_0^{\infty} p^{\varphi}[-\ln\bar{V}(p)]^{a-1}[\bar{V}(p)]^{k-1}v(p)dp. \quad (9)$$

On substituting (6) in (9), we get.

$$E\left(Q_a^{(k)}\right)^{\varphi} = \frac{k^n}{(a-1)!} \times \int_0^{\infty} p^{\varphi}[-\ln\bar{F}(p)]^{n-1}[\bar{F}(p)]^{k}(\beta p+\alpha p^{\gamma})dp$$
$$E\left(Q_a^{(k)}\right)^{\varphi} = \frac{\beta k^n}{(a-1)!}\int_0^{\infty} p^{\varphi+1}[-\ln\bar{V}(p)]^{a-1}[\bar{V}(p)]^{k}dp + \frac{\alpha k^n}{(a-1)!}\int_0^{\infty} p^{\varphi+\gamma}[-\ln\bar{V}(p)]^{n-1}[\bar{V}(p)]^{k}dp. \quad (10)$$

Integrating (10) by parts we obtain,

$$E\left(Q_a^{(k)}\right)^{\varphi} = \frac{\beta k^a}{(a-1)!(\varphi+2)} \times \left\{\int_0^{\infty} p^{\varphi+2}[-\ln\bar{V}(p)]^{a-1}[\bar{V}(p)]^{k-1}v(p)dp - (a-1)\int_0^{\infty} p^{\varphi+2}[-\ln\bar{V}(p)]^{a-2}[\bar{V}(p)]^{k-1}v(p)dp\right\}$$
$$+\frac{(\alpha k^n}{(a-1)!(\varphi+\gamma+1)} \times \left\{\int_0^{\infty} p^{\varphi+\gamma+1}[-\ln\bar{V}(p)]^{a-1}[\bar{V}(p)]^{k-1}v(p)dp - (a-1)\int_0^{\infty} p^{\varphi+\gamma+1}[-\ln\bar{V}(p)]^{a-2}[\bar{V}(p)]^{k-1}v(p)dp\right\}. \quad (11)$$

The relation (7) is established after rewriting the above expression.

Relation (8) follows from (11) by putting $a=1$.

Remark 1. Single moments of k^{th} record values for different parameters are listed in Table 1.

Table 1: Single moments of K-th record values.

S. No.	α	β	γ	Distribution	Author
1	α	0	0	exponential	[17]
2	a/θ	0	$a-1$	Weibull	[18]
3	0	β	γ	Rayleigh	[19]
4	α	0	γ	power hazard	[20]
5	α	β	0	linear hazard	[21]
6	α	β	2	quadratic hazard	[22]

Corollary 1. Relations given in (7) reduces at $k=1$, as follows.

$$E\left(P_{Z(a)}^{(\varphi)}\right) = \frac{\beta}{\varphi+2}\left\{E\left(P_{Z(a)}^{(\varphi+2)}\right) - E\left(P_{Z(a-1)}^{(\varphi+2)}\right)\right\} + \frac{\alpha}{\varphi+\gamma+1}\left\{E\left(P_{Z(a)}^{(\varphi+\gamma+1)}\right) - E\left(P_{Z(a-1)}^{(\varphi+\gamma+1)}\right)\right\}$$

Remark 2. Single moments of upper record values for different parameter are given in Table 2.

Table 2: Single moments of upper record values.

S. No.	α	β	γ	Distribution	Author
1	α	0	0	exponential	[17]
2	a/θ	0	$a-1$	Weibull	[18]
3	0	β	γ	Rayleigh	[19]
4	α	0	γ	power hazard	[20]
5	α	β	0	linear hazard	[21]
6	α	β	2	quadratic hazard	[22]

3 Product Moments

Relations for product moments of k^{th} record are provided in this section.

Theorem 2. For $b \geq 1$ and $\phi, \varphi = 0, 1, \cdots,$

$$E\left(Q_b^{(k)}\right)^{\phi}\left(Q_{b+1}^{(k)}\right)^{\varphi} = \frac{\beta k}{\varphi+2} \times \left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_{b+1}^{(k)}\right)^{\varphi+2}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi+\varphi+2}\right]\right\} + \frac{\alpha k}{\varphi+\gamma+1}\left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_{b+1}^{(k)}\right)^{\varphi+\gamma+1}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi+\varphi+\gamma+1}\right]\right\} \tag{12}$$

and for $1 \leq b \leq a-2$,

$$E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi} = \frac{\beta k}{\varphi+2} \times \left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi+2}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi} E\left(Q_{a-1}^{(k)}\right)^{\varphi+2}\right]\right\} + \frac{\alpha k}{\varphi+\gamma+1}\left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi+\gamma+1}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi} E\left(Q_{a-1}^{(k)}\right)^{\varphi+\gamma+1}\right]\right\}. \tag{13}$$

Proof. In view of (2) and (6), we have,

$$E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi} = \Psi \int_0^{\infty} q^{\varphi} D(q) \left[\bar{V}(q)\right]^{k-1} v(q) dq,$$

where

$$D(q) = \beta \int_0^q p^{\varphi+1}\left[-\ln \bar{V}(p)\right]^{b-1}\left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-1} dp + \alpha \int_0^q p^{\varphi+\gamma}\left[-\ln \bar{V}(p)\right]^{b-1}\left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-1} dp. \tag{14}$$

On integrating $D(q)$,we reach,

$$D(q) = \frac{\beta}{\varphi+2}\left\{(a-b-1)\int_0^q p^{\varphi+2}\left[-\ln \bar{V}(p)\right]^{b-1} \times \left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-2} \frac{v(p)}{\bar{V}(p)} dp - (b-1) \times \int_0^q p^{\varphi+2}\left[-\ln \bar{V}(p)\right]^{b-2}\left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-2} \frac{v(p)}{\bar{V}(p)} dp\right\} + \frac{\alpha}{\varphi+\gamma+1}\left\{(a-b-1)\int_0^q p^{\varphi+\gamma+1} \times \left[-\ln \bar{V}(p)\right]^{b-1}\left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-2} \frac{v(p)}{\bar{V}(p)} dp - (b-1)\int_0^q p^{\varphi+\gamma+1}\left[-\ln \bar{V}(p)\right]^{b-2} \left[-\ln \bar{V}(q) + \ln \bar{V}(p)\right]^{a-b-2} \frac{v(p)}{\bar{V}(p)} dp\right\}. \tag{15}$$

Substituting the above terms into (14) and simplifying it confirms to (13).
The relation (12) can easily be proved by taking $a = b+1$.
Equation (13) reduces to Equation (7) at $\phi = 0$.

Remarks 1 and 2 are also validates for product moments.
Corollary 2. From (13) the following expression

$$E\left(P_{Z(b)}^{(\phi)} P_{Z(a)}^{(\varphi)}\right) = \frac{\beta}{\varphi+2} \times \left\{E\left(P_{Z(b)}^{(\phi)} P_{Z(a)}^{(\varphi+2)}\right) - E\left(P_{Z(b)}^{(\phi)} P_{Z(a-1)}^{(\varphi+2)}\right)\right\} + \frac{\alpha}{\varphi+\gamma+1} \times \left\{E\left(P_{Z(b)}^{(\phi)} P_{Z(a)}^{(\varphi+\gamma+1)}\right) - E\left(P_{Z(b)}^{(\phi)} P_{Z(a-1)}^{(\varphi+\gamma+1)}\right)\right\}.$$

at $k = 1$.

4 Characterization

The following theorem expresses the characterization results for (3) via recurrence relations, which leads to an important role in mathematical statistics. There are several techniques are taken to characterize the distribution, one of them is recurrence relations.

Theorem 3. For a positive integer k and let $\phi, \varphi > 0$. A necessary and sufficient condition for R.V.P to be distributed with (3) is that

$$E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi} = \frac{\beta k}{\varphi+2}\left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi+2}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi} E\left(Q_{a-1}^{(k)}\right)^{\varphi+2}\right]\right\} + \frac{\alpha k}{\varphi+\gamma+1} \times \left\{\left[E\left(Q_b^{(k)}\right)^{\phi}\left(Q_a^{(k)}\right)^{\varphi+\gamma+1}\right] - \left[E\left(Q_b^{(k)}\right)^{\phi} E\left(Q_{a-1}^{(k)}\right)^{\varphi+\gamma+1}\right]\right\}. \tag{16}$$

Proof. From (14) necessary part follows. If (16) is satisfied, then on using (4) and (6), we have.

$$E\left(Q_m^{(k)}\right)^{\phi}\left(Q_n^{(k)}\right)^{\varphi}=\frac{\beta k}{\varphi+2}\Psi\int_0^{\infty}\int_p^{\infty}p^{\phi}q^{\varphi+2}\times [-\ln\bar{V}(p)]^{b-1}\frac{v(p)}{\bar{V}(p)}T'(q)dqdp+\frac{\alpha k}{\varphi+\gamma+1}\Psi\times \int_0^{\infty}\int_p^{\infty}p^{\phi}q^{\varphi+\gamma+1}[-\ln\bar{V}(p)]^{b-1}\frac{v(p)}{\bar{V}(p)}T'(q)dqdp, \quad (17)$$

where

$$T(q)=-\frac{[-\ln\bar{V}(q)+\ln\bar{V}(p)]^{a-b-1}[\bar{V}(q)]^{k}}{k}. \quad (18)$$

Now consider,

$$T(P)=\int_p^{\infty}q^{\varphi+2}T'(q)dq+\int_p^{\infty}q^{\varphi+\gamma+1}T'(q)dq. \quad (19)$$

Integrating (19) by parts and using (18). We have the following simplified expression.

$$\Psi\int_0^{\infty}\int_p^{\infty}p^{\phi}q^{\varphi}[-\ln\bar{V}(p)]^{b-1}\frac{v(p)}{\bar{V}(p)}\times [-\ln\bar{V}(q)+\ln\bar{V}(p)]^{a-b-1}[\bar{V}(q)]^{k-1}\{v(q)-(\beta q+\alpha q^{\gamma})\bar{V}(q)\}dqdp=0. \quad (20)$$

Applying the extension of [23] to (20). It confirms (3).

Corollary 3. Putting $\phi=0$ in (16), we can get the characterization result for single moment as follows.

$$E\left(Q_a^{(k)}\right)^{\varphi}=\frac{\beta k}{\varphi+2}\left\{E\left(Q_a^{(k)}\right)^{\varphi+2}-E\left(Q_{a-1}^{(k)}\right)^{\varphi+2}\right\}+\frac{\alpha k}{\varphi+\gamma+1}\left\{E\left(Q_a^{(k)}\right)^{\varphi+\gamma+1}-E\left(Q_{a-1}^{(k)}\right)^{\varphi+\gamma+1}\right\},$$

for $a=1,2,\cdots,$

5 Conclusion

In this paper, moment properties of k^{th} upper record values from power-linear hazard distribution based on recurrence relations are derived. These moments help to reduce the number of direct calculations to calculate the moments. Further, the characterization result is also presented.

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