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E-Bayesian and Hierarchical Bayesian Estimation of Hazard rate for Kumaraswamy Distribution

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Abstract: In this article, with the use of the Bayesian, Expected Bayesian, and H-Bayesian estimation methods, the shape parameter and hazard rate of the Kumaraswamy distribution are calculated. Three separate loss functions the Squared Error loss function(SELF), the Precautionary loss function(PLF), and the Asymmetry loss function(ALF) along with an informative prior the Gamma prior are used to provide the estimates . The definitions and properties of the proposed estimators are given. Monte Carlo simulation is used to compare all of the estimates in terms of mean square error (MSE). Taking data from the real world, different estimation methodologies' efficacy has also been studied. Numerical analyses show that the E-Bayesian estimates outperform the Bayesian and Hierarchical estimates.

Keywords: Bayesian estimation, Expected Bayesian estimation and H-Bayesian estimation, Kumaraswamy distribution.

1 Introduction

The Kumaraswamy distribution [1] is a probability distribution with two parameters, defined on the interval $(0,1)$. Its probability density function (PDF) can be expressed as follows:

$$f(x;\theta,\lambda) = \theta\lambda x^{\theta-1}(1-x^{\theta})^{\lambda-1}; x,\theta,\lambda > 0 \tag{1}$$

where θ, λ are the shape parameter respectively.

The CDF of the Kumaraswamy distribution is given as

$$F(x;\theta,\lambda) = 1-(1-x^{\theta})^{\lambda}; x,\theta,\lambda > 0 \tag{2}$$

The reliability function and hazard rate of the Kumaraswamy distribution are as follows respectively

$$R(t) = (1-t^{\theta})^{\lambda}; t,\theta,\lambda > 0 \tag{3}$$

$$H(t) = \frac{\theta\lambda t^{\theta-1}}{1-t^{\theta}}; t > 0 \tag{4}$$

The Kumaraswamy distribution and the beta distribution exhibit similar probabilistic characteristics. Depending on the chosen shape parameters, both distributions can display a range of features in their density functions, such as unimodal or multimodal shapes, increasing or decreasing patterns, or constant behavior. However, the Kumaraswamy distribution has an advantage over the beta distribution in terms of its distribution function. Unlike the beta distribution, the Kumaraswamy distribution's distribution function can be expressed in a closed form, allowing for easier computation of quantiles. In contrast, the distribution function of the beta distribution requires integration, which can be more computationally intensive. The Kumaraswamy distribution has demonstrated a strong correlation with various natural phenomena, including daily rainfall, water flows, and other relevant fields. This makes it particularly suitable for

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modeling variables with upper and lower bounds, such as human heights, test scores, air temperatures, and economic data. In recent years, scholars have increasingly recognized the significance of the Kumaraswamy distribution and have directed their attention towards studying and evaluating its properties. Works by Fletcher and Ponnambalam [2], Sundar and Subbiah [3], and Ponnambalam et al. [4] have contributed to advancing our understanding and practical application of this distribution in diverse domains. A variety of distributions, including the uniform, beta, exponential, and generalised beta distributions, can be created by changing the parameters alpha and beta of the Kumaraswamy distribution. According to Jones [5], Kumaraswamy distribution has valuable applications in simulating reliability data obtained from different life tests. It possesses several important distributional properties that make it a suitable choice for such analyses. The Kumaraswamy Weibull distribution, a versatile model for assessing failure time data, was introduced and some of its mathematical aspects were examined by Cordeiro, Ortega, and Nadarajah [6]. With a known alpha value, Reyad and Ahmed [7] were able to obtain Bayes estimates for the shape parameter beta. Both symmetric and asymmetric loss functions were used to get these estimates. Furthermore, they revealed significant comparative findings amongst the suggested methodologies utilising a Monte Carlo simulation analysis. Jones [5] added that despite the Kumaraswamy distribution's highly flexible probabilistic structure, researchers have not shown a sufficient level of interest in it. Ghosh and Nadarajah [8] investigated the use of two loss functions in Bayesian estimation with three distinct censoring schemes: left censoring, single type-II censoring, and double type-II censoring with a specified parameter. In order to gain insight into parameter estimation, issues for the same distribution, Sultana et al. [9] recently merged hybrid and progressive type I censoring techniques. With a hybrid censoring technique, Sultana et al. [10] analysed and calculated parameters for the kumaraswamy distribution. In addition to the Bayesian methodology, we have also studied Hierarchical Bayesian (H-Bayesian) approaches. The H-Bayesian approach is more resilient than Bayesian methods since it comprises two stages for constructing the prior distribution. The complicated integrals contained in the estimator formulation are one of the technique's drawbacks. It must be solved using a combination of numerical approximation methods or Matlab. As a result, the calculation takes longer and becomes more tedious.In response to the limitations of the Hierarchical Bayesian method, a different approach called the Expected Bayesian estimation method was introduced as an alternative. This method offers a simpler alternative compared to the H-Bayesian method.

Han and Ding [11] were the first to address the definition of E-Bayesian estimation. Since then, there has been a lot of work in the literature on inference using E-Bayesian estimation. Han [12], conducted a study on the estimation of failure rates in exponential distributions and proposed both Expected Bayesian and hierarchical Bayesian estimators. The research aimed to explore the characteristics and properties of these estimators. Han [13] also explored the estimators for Pareto distribution parameter under various loss functions. Basheer et.al [14] examined the estimation of parameter and reliability function in the inverse Weibull distribution, focusing on the Expected Bayesian and H-Bayesian estimation methods. The study aimed to analyze and discuss the properties and implications of these estimation techniques. Abdul-Sathar and Athirakrishnan [15] investigated the estimation of the scale parameter and inverted hazard rate in the inverse Rayleigh distribution using Expected Bayesian and H-Bayesian estimation approaches. The study aimed to explore the characteristics and performance of these estimation methods for the specified distribution. In this article, the parameter and hazard rate of the kumaraswamy distribution is estimated using Bayesian, Expected Bayesian, and H-Bayesian methods. We also present simulated data to evaluate the proposed estimators performance.

The information is arranged as follows: Section 2 deduces the Bayesian estimators of the parameter λ and hazard rate for various loss functions. Expected Bayesian estimators of the parameter and hazard rate under various loss functions are developed in section 3. The H-Bayesian estimation is covered in Section 4, and the Expected Bayesian and H-Bayesian estimators' properties are covered in Section 5. The performance of the estimators under various loss functions is investigated using a simulation study in section 6 using R. In section 7, the suggested study's conclusions are finally discussed.

2 Bayesian Estimation

In this part, we presented Bayesian estimators for the parameter λ and the hazard rate of the Kumaraswamy distribution. These estimators were formulated using three distinct loss functions: SELF, PLF and ALF.

Suppose we have a random sample of size n, denoted as $x_1, x_2, \ldots, x_n$, drawn from the Kumaraswamy distribution. In terms of the observed sample values, the likelihood function can be expressed as follows:

$$L(\lambda|x) = \prod_{i=1}^{n} f(x_i, \theta, \lambda)$$

$$L(\lambda|x) = n\log\theta + n\log\lambda + (\theta-1)\sum_{i=1}^{n}\log x_i + (\lambda-1)\sum_{i=1}^{n}\log(1-x_i^{\theta})$$

$$L(\lambda|x) \propto \lambda^n \exp(-\lambda T) \tag{5}$$

where $Q = \log(1 - x_i^\theta)^{-1}$.
We are using gamma distribution as prior distribution of λ. The gamma prior distribution with shape and scale parameter 'r' and 's' respectively has the following PDF

$$g_1(\lambda|r,s) = \frac{s^r \lambda^{r-1}}{\Gamma(r)} e^{-s\lambda}; \lambda, r, s > 0 \tag{6}$$

Using equation (5) and (6), the posterior distribution for the parameter λ becomes

$$P_1(\lambda|x) = \frac{L(\lambda|x) * g_1(\lambda|r,s)}{\int_0^\infty L(\lambda|x) * g_1(\lambda|r,s) d\lambda}$$

$$= \frac{\lambda^n \exp(-\lambda Q) \frac{s^r}{\Gamma r} \lambda^{r-1} \exp(-s\lambda)}{\int_0^\infty \lambda^n \exp(-\lambda Q) \frac{s^r}{\Gamma r} \lambda^{r-1} \exp(-s\lambda) d\lambda}$$

$$P_1(\lambda|x) = \frac{(Q+s)^{n+r}}{\Gamma(n+r)} \lambda^{n+r-1} \exp(-\lambda(Q+s))$$

2.1 Bayes estimate of λ with SELF

The Squared Error loss function (SELF) is defined as

$$L(\hat{\lambda}, \lambda) = c(\hat{\lambda} - \lambda)^2 \tag{7}$$

where $\hat{\lambda}$ is an estimator of λ. The Bayes estimator can be obtained as:

$$\hat{\lambda}_{BS} = E[\lambda|x] \tag{8}$$

Provided that the expectation $E[\lambda|x]$ exists and finite.

By using SELF, the Bayes estimator is given by

$$E[\lambda|x] = \int_0^\infty \lambda . P_1(\lambda|x)$$

$$= \int_0^\infty \frac{(Q+s)^{n+r} \lambda^{(n+r+1)-1} \exp -(\lambda(Q+s))}{\Gamma(n+r)} d\lambda$$

$$\implies \hat{\lambda}_{BS} = E[\lambda|x] = \frac{n+r}{Q+s} \tag{9}$$

The bayes estimator of Hazard rate is given as

$$E[H\hat{(t)}|x] = \int_0^\infty H(t) . P_1(\lambda|x)$$

$$= \int_0^\infty \frac{\theta \lambda t^{\theta-1}}{1-t^\theta} . \frac{(Q+s)^{n+r} \lambda^{(n+r)-1} \exp -(\lambda(Q+s))}{\Gamma(n+r)} d\lambda$$

$$= \frac{\theta t^{\theta-1}}{1-t^\theta} . \frac{(Q+s)^{n+r}}{\Gamma(n+r)} . \frac{\Gamma(n+r+1)}{(Q+r)^{n+r+1}}$$

$$\hat{H}_{BS} = E[\lambda|x] = \frac{\theta t^{\theta-1}}{1-t^\theta} . \frac{n+r}{Q+s} \tag{10}$$

2.2 Bayes estimate of λ with PLF

The Precautionary loss function (PLF) is defined as

$$L(\hat{\lambda},\lambda)=\frac{c(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \tag{11}$$

where $\hat{\lambda}$ is an estimator of λ. The Bayes estimator can be obtained as:

$$\hat{\lambda}_{BP}=E[\lambda^2|x] \tag{12}$$

Provided that the expectation $E[\lambda^2|x]$ exists and finite.

The Bayes estimator based on PLF, is given by

$$E[\lambda^2|x]=\int_0^\infty \frac{1}{\lambda^2}.P_1(\lambda|x)$$

$$=\int_0^\infty \frac{(Q+q)^{n+r}\lambda^{(n+r+2)-1}\exp-(\lambda(Q+s))}{\Gamma(n+r)}d\lambda$$

$$=\frac{(n+r+1)(n+r)}{(Q+s)^2}$$

$$\implies \hat{\lambda}_{BP}=E[\lambda^2|x]=\sqrt{\frac{(n+r+1)(n+r)}{(Q+s)^2}} \tag{13}$$

The bayes estimator of Hazard rate is given as

$$\hat{H}_{BP}=E[H(t)^2|x]=\int_0^\infty H(t)^2.P_1(\lambda|x)$$

$$=\left(\frac{\theta t^{\theta-1}}{1-t^\theta}\right)^2.\frac{(Q+s)^{n+r}}{\Gamma(n+s)}\int_0^\infty \lambda^{(n+r+2)-1}\exp(-\lambda(Q+s))d\lambda$$

$$\hat{H}_{BP}=\frac{\theta t^{\theta-1}}{1-t^\theta}.\sqrt{\frac{(n+r+1)(n+r)}{(Q+s)^2}} \tag{14}$$

2.3 Bayes estimate of λ with ALF

Wang and Wang [16] introduced the asymmetry loss function (ALF), defined as

$$L(\hat{\lambda},\lambda)=\left(\sqrt{\frac{\lambda}{\delta}}-\sqrt{\frac{\delta}{\lambda}}\right)^2=\frac{\lambda}{\delta}+\frac{\delta}{\lambda}-2 \tag{15}$$

where $\hat{\lambda}$ is an estimate of λ. And, the corresponding Bayes estimate of λ is given as

$$\hat{\lambda}_{BA}=\left[\frac{E[\lambda^{-1}|x]}{E[\lambda|x]}\right]^{-\frac{1}{2}} \tag{16}$$

$$E[\lambda^{-1}|x]=\int_0^\infty \frac{1}{\lambda}.P_1(\lambda|x)$$

$$=\int_0^\infty \frac{(Q+q)^{n+r}\lambda^{(n+r-1)-1}\exp-(\lambda(Q+s))}{\Gamma(n+r)}d\lambda$$

$$E[\lambda^{-1}|x]=\frac{Q+s}{n+r-1} \tag{17}$$

And, From equation (9)

$$E[\lambda|x] = \frac{n+r}{Q+s} \tag{18}$$

The Bayes estimate of λ is given by

$$\hat{\lambda}_{BA} = \frac{\sqrt{(n+s)(n+r-1)}}{(Q+s)} \tag{19}$$

Similarly, The Bayes estimate of Hazard Rate is given by

$$\hat{H}_{BA} = \left[\frac{E[H^{-1}(t)|x]}{E[H(t)|x]}\right]^{-\frac{1}{2}}$$

$$\hat{H}_{BA} = \frac{\theta t^{\theta-1}}{1-t^{\theta}}.\frac{\sqrt{(n+r)(n+r-1)}}{(Q+s)} \tag{20}$$

3 Expected-Bayesian Estimation

Han [17] mentioned that the prior parameters 'r' and 's' should be selected in such a manner that the prior given in (6) be a decreasing function of θ.

$$\frac{d}{d\lambda}g_1(\lambda|r,s) = \frac{(r^s)}{\Gamma(r)}\theta^{(s-2)}\exp(-\lambda r)((s-1)-r\lambda) \tag{21}$$

Thus, for $0 < r < 1$ and $s > 0$, our prior distribution (6) becomes a decreasing function of λ. The E-Bayesian estimate of λ is given by

$$\hat{\lambda}_{EB} = \int_0^1\int_0^v \hat{\lambda}_B * \pi(\lambda,r,s)drds \tag{22}$$

The domain of the first and second integrals in this case is the domain of the hyperparameters 'r' and 's', respectively, for which our prior density function is the decreasing function of λ. $\hat{\lambda}_B$ is the Bayesian estimate of λ obtained using three various loss functions. We now choose the prior distribution of the hyperparameters 'r' and 's' for the E-Bayesian estimates of λ. These distributions are primarily used to investigate the impact of different prior distributions on E-Bayesian estimates of λ. The prior distribution of the hyperparameters 'r' and 's' is given by

$$\pi(r,s) = \frac{1}{v}; 0 < r < 1, 0 < s < v. \tag{23}$$

3.1 Expected Bayesian estimation under SELF

E-Bayes estimate of λ based on (23) is given by

$$\hat{\lambda}_{E1} = \int_0^1\int_0^v \hat{\lambda}_{BS} * \pi(r,s)drds$$

$$= \int_0^1\int_0^v \frac{n+r}{Q+s}.\frac{1}{v}drds$$

$$\hat{\lambda}_{E1} = \frac{2n+1}{2v}\log\left(\frac{Q+s}{Q}\right) \tag{24}$$

The Expected-Bayesian estimate of hazard rate is given as

$$= \int_0^1\int_0^v \frac{\theta t^{\theta-1}}{1-t^{\theta}}.\frac{n+r}{Q+s}.\frac{1}{v}drds$$

$$H\hat{(t)}_{E_1} = \frac{\theta t^{\theta-1}}{(1-t^{\theta})}.\frac{2n+1}{2v}\log\left(\frac{Q+s}{Q}\right) \tag{25}$$

3.2 Expected Bayesian estimation under PLF

E-Bayes estimate of λ based on (23), is given by

$$\hat{\lambda}_{E2} = \int_0^1 \int_0^v \hat{\lambda}_{BP} * \pi(r,s) drds$$

$$= \int_0^1 \int_0^v \sqrt{\frac{(n+r+1)(n+r)}{(Q+s)^2}} . \frac{1}{v} drds$$

$$\hat{\lambda}_{E2} = \frac{1}{v} \log\left(\frac{Q+s}{Q}\right) \int_0^1 \sqrt{(n+r+1)(n+r)} dr \tag{26}$$

The Expected-Bayesian estimate of hazard rate is given as

$$= \int_0^1 \int_0^v \frac{\theta t^{\theta-1}}{1-t^\theta} . \sqrt{\frac{(n+r+1)(n+r)}{(Q+s)^2}} . \frac{1}{v} drds$$

$$\hat{H(t)}_{E_2} = \frac{\theta t^{\theta-1}}{v(1-t^\theta)} . \log\left(\frac{Q+s}{Q}\right) \int_0^1 \sqrt{(n+r+1)(n+r)} dr \tag{27}$$

3.3 Expected Bayesian estimation under ALF

E-Bayes estimate of λ based on (23), is given by

$$\hat{\lambda}_{E3} = \int_0^1 \int_0^v \hat{\lambda}_{BA} * \pi(r,s) dpdq$$

$$= \frac{\sqrt{(n+r)(n+r-1)}}{(Q+s)} . \frac{1}{v} dsdr$$

$$\hat{\lambda}_{E3} = \frac{1}{v} \log\left(\frac{Q+s}{Q}\right) \int_0^1 \sqrt{(n+r-1)(n+r)} dr \tag{28}$$

The Expected-Bayesian estimate of hazard rate is given as

$$= \int_0^1 \int_0^v \frac{\theta t^{\theta-1}}{1-t^\theta} . \frac{\sqrt{(n+r)(n+r-1)}}{(Q+s)} . \frac{1}{v} drds$$

$$\hat{H(t)}_{E_3} = \frac{\theta t^{\theta-1}}{v(1-t^\theta)} . \log\left(\frac{Q+s}{Q}\right) \int_0^1 \sqrt{(n+r-1)(n+r)} dr \tag{29}$$

4 Hierarchical Bayesian Estimation

In this section, the H-Bayesian estimates of shape parameter and Hazard rate of Kumaraswamy distribution based on SELF, PLF and ALF are derived.

Based on Lindley and Smith [18], if a and b are hyper parameters in the prior density function of λ i.e. $g_1(\lambda|r,s)$ given in Eq.(6) and the hyper prior distribution of r, s given in Equation (23), then the corresponding hierarchical prior distributions of λ have the following forms:

$$\pi_1(\lambda) = \int_0^1 \int_0^v g_1(\lambda|r,s)\pi(r,s) dsdr = \int_0^1 \int_0 \frac{s^r \lambda^{r-1}}{\Gamma(r)} \exp(-s\lambda) \frac{1}{v} dsdr \tag{30}$$

According to Bayesian theorem, the hierarchical posterior density for λ can be derived by combining Eqs. (5) and (30) to be

$$\pi_1(\lambda|r,s) = \frac{L(\lambda|x)\pi_1(\lambda)}{\int_0^\infty L(\lambda|x)\pi_1(\lambda) d\lambda}$$

$$= \frac{\frac{1}{v}\int_0^1\int_0^v \frac{s^r}{\Gamma r}\lambda^{r-1}\exp(-s\lambda).\lambda^n\exp(-\lambda Q)drds}{\frac{1}{v}\int_0^1\int_0^v\int_0^\infty \frac{s^r}{\Gamma r}\lambda^{r-1}\exp(-s\lambda).\lambda^n\exp(-\lambda Q)d\lambda drds}$$

$$\pi_1(\lambda|r,s) = \frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\lambda^{n+r-1}\exp(-\lambda(s+Q))dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr} \tag{31}$$

4.1 H-Bayesian estimation with SELF

Assuming Squared Error loss function defined in Eq. (8) and the hierarchical posterior density defined in Equation (31), the Hierarchical Bayes estimates $\hat{\lambda}_{H_1}$, of λ have the following expression

$$\hat{\lambda}_{H_1} = E[(\lambda|r,s)]$$

$$E[(\lambda|r,s)] = \int_0^\infty \lambda.\pi_1(\lambda|r,s)d\lambda$$

$$= \frac{\int_0^1\int_0^v\int_0^\infty \frac{s^r}{\Gamma(r)}\lambda^{(n+r+1)-1}\exp(-\lambda(s+Q))d\lambda dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(p)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr}$$

$$\implies \hat{\lambda}_{H_1} = \frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr} \tag{32}$$

Similarly, the Hierarchical Bayesian estimator of hazard rate is given as

$$\hat{H(t)}_{H_1} = E\left[\frac{\theta\lambda t^{\theta-1}}{1-t^\theta}|r,s\right]$$

$$= \int_0^\infty \frac{\theta\lambda t^{\theta-1}}{1-t^\theta}.\pi_1(\lambda|r,s)d\lambda$$

$$\hat{H(t)}_{H_1} = \frac{\frac{\theta t^{\theta-1}}{1-t^\theta}\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr} \tag{33}$$

4.2 H-Bayesian estimation with PLF

Assuming Precautionary loss function defined in Eq. (12) and the hierarchical posterior density defined in Equation (31), the hierarchical Bayes estimates $\hat{\lambda}_{H_2}$, of λ have the following expressions

$$\hat{\lambda}_{H_2} = \sqrt{E[(\lambda^2|r,s)]}$$

$$E[(\lambda^2|r,s)] = \int_0^\infty \lambda^2.\pi_1(\lambda|r,s)d\lambda$$

$$= \frac{\int_0^1\int_0^v\int_0^\infty \frac{s^r}{\Gamma(r)}\lambda^{(n+r+2)-1}\exp(-\lambda(s+Q))d\lambda dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr}$$

$$\implies \hat{\lambda}_{H_2} = \sqrt{\frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+2)}{(s+Q)^{n+r+2}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr}} \tag{34}$$

Similarly, the Hierarchical Bayesian estimator of hazard rate is given as

$$\hat{H(t)}_{H_2} = \sqrt{E[(H^2(t)|r,s)]}$$

$$E\left[\left(\frac{\theta\lambda t^{\theta-1}}{1-t^{\theta}}\right)^2 |r,s\right] = \int_0^{\infty}\left(\frac{\theta\lambda t^{\theta-1}}{1-t^{\theta}}\right)^2 .\pi_1(\lambda|r,s)d\lambda$$

$$\hat{H(t)}_{H_2} = \frac{\theta t^{\theta-1}}{1-t^{\theta}}\sqrt{\frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+2)}{(s+Q)^{n+r+2}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+p)}{(s+Q)^{n+r}}dsdr}} \tag{35}$$

4.3 H-Bayesian estimation with ALF

Assuming Asymmetry loss function as defined in Eq. (16) the Hierarchical posterior distribution defined in Equation (31), the hierarchical Bayes estimates $\hat{\lambda}_{H_3}$, of λ are

$$\hat{\lambda}_{H_3} = \left[\frac{E[(\lambda^{-1}|r,s)]}{E[(\lambda|r,s)]}\right]^{-\frac{1}{2}}$$

$$E[(\lambda^{-1}|r,s)] = \int_0^{\infty}\frac{1}{\lambda}\pi_1(\lambda|r,s)d\lambda$$

$$= \frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r-1)}{(s+Q)^{n+r-1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr} \tag{36}$$

From equation (32), we have

$$E[(\lambda|r,s)] = \frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr} \tag{37}$$

$$\Longrightarrow \hat{\lambda}_{H_3} = \left[\frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r-1)}{(s+Q)^{n+r-1}}dsdr}\right]^{\frac{1}{2}} \tag{38}$$

Similarly, the Hierarchical Bayesian estimator of hazard rate is given as

$$\hat{H(t)}_{H_3} = \left[\frac{E[(H^{-1}(t)|a,b)]}{E[(H(t)|a,b)]}\right]^{-\frac{1}{2}}$$

$$\Longrightarrow \hat{H(t)}_{H_3} = \frac{\theta t^{\theta-1}}{1-t^{\theta}}\left[\frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r-1)}{(s+Q)^{n+r-1}}dsdr}\right]^{\frac{1}{2}} \tag{39}$$

5 Properties of Estimators

Here, we covered through some of the key characteristics of E-Bayesian estimators, such as how they relate to H-Bayesian estimators.

Theorem 1. The relation among the Expected Bayesian estimators of λ based on SELF, PLF and ALF are given as
(i) $\hat{\lambda}_{E_2} < \hat{\lambda}_{E_1} < \hat{\lambda}_{E_3}$.

(ii) $\lim_{T\to\infty}\hat{\lambda}_{E_1} = \lim_{T\to\infty}\hat{\lambda}_{E_2} = \lim_{T\to\infty}\hat{\lambda}_{E_3} = 0$. Proof.

The relationship $\hat{\lambda}_{E_2} < \hat{\lambda}_{E_1} < \hat{\lambda}_{E_3}$ is same as

$$\int_0^1\sqrt{(n+r-1)(n+r)}dr < n+\frac{1}{2} < \int_0^1\sqrt{(n+r)(n+r+1)}dr \tag{40}$$

We prove the relation for n=1 by using mathematical induction, since $\int_0^1\sqrt{(n+r-1)(n+r)}dr = 0.8403$ and $\int_0^1\sqrt{(n+r)(n+r+1)}dr = 1.9349$. So, we have $0.8403 < 1.5 < 1.9349$. Hence the result is true for n=1.

for $r \in (0,1)$, $\sqrt{(n+r+1)(n+r)}$ and $\sqrt{(n+r-1)(n+r)}$ is continuous, through Generalized Mean Value theorem, At least one number can be found $r_1 \in (0,1)$ and $r_2 \in (0,1)$ s.t.
$\int_0^1 \sqrt{(n+r-1)(n+r)}dr = \sqrt{(n+r-1)(n+r)}\int_0^1 dp = \sqrt{(n+r_1-1)(n+r_1)}$
and $\int_0^1 \sqrt{(n+r+1)(n+r)}dr = \sqrt{(n+r+1)(n+r)}\int_0^1 dp = \sqrt{(n+r_2+1)(n+r_2)}$
Hence,(39) can be written as

$$\sqrt{(n+r_1-1)(n+r_1)} < n+0.5 < \sqrt{(n+r_2+1)(n+r_2)}$$

Squaring the above equation, we get

$$(n+r_1-1)(n+r_1) < (n+0.5)^2 < (n+r_2+1)(n+r_2)$$

Now, we assume that the result hold for n=k, i.e.

$$(k+r_1-1)(k+r_1) < (k+0.5)^2 < (k+r_2+1)(k+r_2) \tag{41}$$

Now, we prove that the result is true for n=k+1, we have

$$(k+1+r_1-1)(k+1+r_1) < (k+1+0.5)^2 < (k+1+r_2+1)(k+1+r_2) \tag{42}$$

$$(k+r_1)(k+1+r_1) < (k+1+0.5)^2 < (k+r_2+2)(k+1+r_2)$$

Now, $(k+r_2+2)(k+1+r_2) = (k+r_2+1)(k+r_2) + 2(k + r_2 + 1)$ and $(k+r_1)(k+1+r_1) = (k+r_1-1+2)(k+1+r_1) = (k+r_1)(k+r_1-1)+2(k+r_1)$.
Also, $(k+0.5+1)^2 = (k+0.5+1)^2+2(k+0.5)+1$ which is greater than $(k+r_1)(k+r_1-1)+2(k+r_1)$ as $r \in (0,1)$.
and, $(k+r_2+1)(k+r_2)+2(k+r_2+1) > (k+0.5+1)^2+2(k+0.5)+1$.
This proves the result.
(ii)
Using (23), we have

$$\hat{\lambda}_{E_1} = \frac{2n+1}{2v}\log\left(1+\frac{s}{Q}\right)$$

Taking the limit as $T \to \infty$

$$\lim_{T\to\infty} \hat{\lambda}_{E_1} = 0 \tag{43}$$

Using (25), we have

$$\hat{\lambda}_{E_2} = \frac{1}{v}\log\left(1+\frac{s}{Q}\right)\int_0^1 \sqrt{(n+r+1)(n+r)}dr$$

Taking the limit as $T \to \infty$

$$\lim_{T\to\infty} \hat{\lambda}_{E_2} = 0 \tag{44}$$

Using (27), we have

$$\hat{\lambda}_{E_3} = \frac{1}{v}\log\left(1+\frac{s}{Q}\right)\int_0^1 \sqrt{(n+r-1)(n+r)}dr$$

Taking the limit as $T \to \infty$

$$\lim_{T\to\infty} \hat{\lambda}_{E_3} = 0 \tag{45}$$

Theorem 2. The relation among Expected Bayesian estimators of H(t) based on SELF, PLF and ALF are given as
(i) $\hat{H(t)}_{E_2} < \hat{H(t)}_{E_1} < \hat{H(t)}_{E_3}$.
(ii) $\lim_{T\to\infty} \hat{H(t)}_{E_1} = \lim_{T\to\infty} \hat{H(t)}_{E_2} = \lim_{T\to\infty} \hat{H(t)}_{E_3} = 0$.

Theorem 3. The relationship between Expected Bayesian and Hierarchical Bayesian estimators of λ based on SELF, PLF and ALF are given as $\lim_{T\to\infty} \hat{\lambda}_{E_j} = \lim_{T\to\infty} \hat{\lambda}_{H_j} = 0, j = 1,2,3$

Proof.
Under SELF.

$$\hat{\lambda}_{E_1} = \frac{1}{v}\int_0^1\int_0^v \frac{n+r}{Q+s}drds$$

For $r\in(0,1)$, $s\in(0,v)$, $(n+r)(Q+s)^{-1}$ is continuous, so through Generalized Mean Value theorem, At least one number can be found $r_3\in(0,1)$ and $s_3\in(0,v)$ such that,

$$\frac{1}{v}\int_0^1\int_0^v \frac{n+r}{Q+s}drds = \frac{n+r_3}{Q+s_3}\int_0^1\int_0^v \frac{1}{v}drds$$

$$= \frac{n+r_3}{Q+s_3} \tag{46}$$

Now, taking the limit as $T\to\infty$, we have,

$$\lim_{T\to\infty}\hat{\lambda}_{E_1} = 0 \tag{47}$$

Using the result $\Gamma(n+r+1) = (n+r)\Gamma(n+r)$, we have,

$$\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(Q+s)^{n+r+1}}dsdr = \int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{n+r}{Q+s}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}dsdr$$

For $r\in(0,1)$, $s\in(0,v)$, $(n+r)(Q+s)^{-1}$ is continuous and $\frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}>0$ so through Generalized Mean value theorem, At least one number can be found $r_4\in(0,1)$ and $s_4\in(0,v)$ such that,

$$\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{n+r}{Q+s}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}dsdr = \frac{n+r_4}{T+s_4}\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}dsdr$$

Using (31), we have

$$\hat{\lambda}_{H_1}\frac{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r+1)}{(s+Q)^{n+r+1}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(s+Q)^{n+r}}dsdr}$$

$$= \frac{\frac{(n+r_4)}{(s_4+T)}\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}dsdr}{\int_0^1\int_0^v \frac{s^r}{\Gamma(r)}\frac{\Gamma(n+r)}{(Q+s)^{n+r}}dsdr}$$

$$\hat{\lambda}_{H_1} = \frac{(n+r_4)}{(s_4+Q)} \tag{48}$$

Taking limit as $T\to\infty$

$$\lim_{T\to\infty}\hat{\lambda}_{H_1} = 0 \tag{49}$$

Now, using eqs. (39) and (41), we have

$$\lim_{T\to\infty}\hat{\lambda}_{E_1} = \lim_{T\to\infty}\hat{\lambda}_{H_1} = 0. \tag{50}$$

Under PLF.

$$\hat{\lambda}_{E_2} = \frac{1}{v}\int_0^1\int_0^v \frac{\sqrt{(n+r)(n+r+1)}}{Q+s}drds$$

For $r\in(0,1)$, $s\in(0,v)$, $\frac{\sqrt{(n+r)(n+r+1)}}{Q+s}$ is continuous, so through Generalized Mean Value theorem, At least one number can be found $r_5\in(0,1)$ and $s_5\in(0,v)$ such that,

$$\frac{1}{v}\int_0^1\int_0^v \frac{\sqrt{(n+r)(n+r+1)}}{Q+s}drds = \frac{\sqrt{(n+r_5)(n+r_5+1)}}{Q+s_5}\int_0^1\int_0^v \frac{1}{v}drds$$

$$= \frac{\sqrt{(n+r_5)(n+r_5+1)}}{T+s_5} \tag{51}$$

Now, taking the limit as $T \to \infty$, we have,

$$\lim_{T\to\infty} \hat{\lambda}_{E_2} = 0 \tag{52}$$

Using the result $\Gamma(n+r+2) = (n+r+1)\Gamma(n+r+1) = (n+r+1)(n+r)\Gamma(n+r)$, we have,

$$\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r+2)}{(Q+s)^{n+r+2}} dsdr = \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{(n+r+1)(n+r)\Gamma(n+r)}{(Q+s)^{n+r+2}} dsdr$$

For $r \in (0,1)$, $s \in (0,v)$, $(n+r+1)(n+r)$ is continuous and $\frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r)}{(Q+s)^{n+r}} > 0$ so through Generalized Mean Value theorem, At least one number can be found $r_6 \in (0,1)$ and $s_6 \in (0,v)$ such that,

$$\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r+2)}{(Q+s)^{n+r+2}} dsdr = \frac{(n+r_6+1)(n+r_6)}{(Q+s_6)^2} \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r)}{(Q+s)^{n+r}} drds$$

Using (33), we have

$$\hat{\lambda}_{H_2} = \left[\frac{\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r+2)}{(s+Q)^{n+r+2}} dsdr}{\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r)}{(s+Q)^{n+r}} dsdr} \right]^{\frac{1}{2}}$$

$$= \frac{\frac{(n+r_6+1)(n+r_6)}{(Q+s_6)^2} \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r)}{(Q+s)^{n+r}} dsdr}{\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r)}{(Q+s)^{n+r}} dsdr}$$

$$\hat{\lambda}_{H_2} = \sqrt{\frac{(n+r_6+1)(n+r_6)}{(T+s_6)^2}}$$

Taking limit as $T \to \infty$

$$\lim_{T\to\infty} \hat{\lambda}_{H_2} = 0 \tag{53}$$

Now, using eqs. (49) and (50), we have

$$\lim_{T\to\infty} \hat{\lambda}_{E_2} = \lim_{T\to\infty} \hat{\lambda}_{H_2} = 0. \tag{54}$$

Under ALF.

$$\hat{\lambda}_{E_3} = \frac{1}{v} \int_0^1 \int_0^v \frac{\sqrt{(n+r)(n+r-1)}}{Q+s} drds$$

For $r \in (0,1)$, $s \in (0,v)$, $\frac{\sqrt{(n+r)(n+r-1)}}{Q+s}$ is continuous, so through Generalized Mean Value theorem, At least one number can be found $r_7 \in (0,1)$ and $s_7 \in (0,v)$ such that,

$$\frac{1}{v} \int_0^1 \int_0^v \frac{\sqrt{(n+r)(n+r-1)}}{Q+s} drds = \frac{\sqrt{(n+r_7)(n+r_7-1)}}{Q+s_7} \int_0^1 \int_0^v \frac{1}{v} drds$$

$$= \frac{\sqrt{(n+r_7)(n+r_7-1)}}{Q+s_7} \tag{55}$$

Now, taking the limit as $T \to \infty$, we have,

$$\lim_{T\to\infty} \hat{\lambda}_{E_3} = 0 \tag{56}$$

Using the result $\Gamma(n+r+1) = (n+r)\Gamma(n+r) = (n+r)(n+r-1)\Gamma(n+r-1)$, we have,

$$\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r+1)}{(Q+s)^{n+r+1}} dsdr = \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{(n+r)(n+r-1)\Gamma(n+r-1)}{(Q+s)^{n+r+1}} dsdr$$

For $r \in (0,1)$, $s \in (0,v)$, $(n+r)(n+r-1)$ is continuous and $\frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r-1)}{(Q+s)^{n+r-1}} > 0$ so through Generalized Mean Value theorem, At least one number can be found $r_8 \in (0,1)$ and $s_8 \in (0,v)$ such that,

$$\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r+1)}{(Q+s)^{n+r+1}} dsdr = \frac{(n+r_8)(n+r_8-1)}{(T+s_8)^2} \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r-1)}{(Q+s)^{n+r-1}} drds$$

Using (37), we have

$$\hat{\lambda}_{H_3} = \left[\frac{\int_0^1 \int_0^v \frac{q^p}{\Gamma(p)} \frac{\Gamma(n+p+1)}{(q+T)^{n+p+1}} dq dp}{\int_0^1 \int_0^v \frac{q^p}{\Gamma(p)} \frac{\Gamma(n+p-1)}{(q+T)^{n+p-1}} dq dp} \right]^{\frac{1}{2}}$$

$$= \frac{\frac{(n+r_8)(n+r_8-1)}{(T+s_8)^2} \int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r-1)}{(Q+s)^{n+r-1}} ds dr}{\int_0^1 \int_0^v \frac{s^r}{\Gamma(r)} \frac{\Gamma(n+r-1)}{(Q+s)^{n+r-1}} ds dr}$$

$$\hat{\lambda}_{H_3} = \sqrt{\frac{(n+r_8)(n+r_8-1)}{(Q+s_8)^2}}$$

Taking limit as $T \to \infty$

$$\lim_{T\to\infty} \hat{\lambda}_{H_3} = 0 \tag{57}$$

Now, using eqs. (45) and (46), we have

$$\lim_{T\to\infty} \hat{\lambda}_{E_3} = \lim_{T\to\infty} \hat{\lambda}_{H_3} = 0. \tag{58}$$

Theorem 4. Similarily, The relation between Expected Bayesian and Hierarchical Bayesian estimators of hazard rate based on SELF, PLF and ALF are given as $\lim_{T\to\infty} \hat{H(t)}_{E_j} = \lim_{T\to\infty} \hat{H(t)}_{H_j} = 0, j = 1,2,3$

6 Simulation Study

In this part, a Monto Carlo simulation has been performed to evaluate the effectiveness of the Bayesian, Expected Bayesian and H-Bayesian estimates associated to λ and Hazard rate of the Kumaraswamy distribution. The simulation process is outlined in the following steps:

1. Set the true value of θ, λ at $0.5, 1$ respectively.

2. Generate random sample of sizes $n = 30, 50, 75, 100$ from Kumaraswamy distribution.

3. Fix the value of $r = 0.5, v = 1.5, u = 0.3, s = 1$.

4. Bayes, Expected Bayesion and Hierarchical Bayesian estimates are obtained for the parameter λ and Hazard rate.

5. Iterate the above procedure 1000 times to figure out the value of the average estimates and their MSE.

All the results obtained through simulation are shown in Tables 1-2. The obtained estimators are compared based on the mean squared error. We observed that

- As n increases, the value of MSE decreases.
- Under any fixed loss function, the value of mean squared error under Expected Bayesian technique is less.
- The estimators performs better under PLF.
- As the value of n increases, the estimators approaches to their true value, shows the consistency of the estimators.

Table 1: Average estimates(AE) of λ and their MSE in parentheses.

n	Bayes			E-Bayes			H-Bayes		
	SELF	ALF	PLF	SELF	ALF	PLF	SELF	ALF	PLF
30	1.01493	1.00151	1.02117	1.02803	1.01808	1.03489	1.02036	1.01259	1.03559
	(0.03431)	(0.03558)	(0.03304)	(0.03372)	(0.03447)	(0.03163)	(0.03364)	(0.03470)	(0.03103)
50	1.01324	1.00958	1.01976	1.01479	1.00616	1.01739	1.01643	1.01175	1.03543
	(0.02103)	(0.02220)	(0.02007)	(0.02046)	(0.02108)	(0.01993)	(0.02071)	(0.02151)	(0.02032)
75	1.00635	1.00170	1.00877	1.01227	1.00247	1.01509	1.02657	1.0015	1.02183
	(0.01395)	(0.01426)	(0.01235)	(0.01299)	(0.01331)	(0.01151)	(0.01231)	(0.01373)	(0.01172)
100	1.00070	0.99914	1.00548	1.01125	0.99893	1.01570	1.00575	0.99513	1.01461
	(0.01083)	(0.01097)	(0.01011)	(0.01000)	(0.01006)	(0.00999)	(0.01019)	(0.01036)	(0.01007)

Table 2: Avaerage estimates(AE) of hazard rate and their MSE in parentheses.

n	Bayes			E-Bayes			H-Bayes		
	SELF	ALF	PLF	SELF	ALF	PLF	SELF	ALF	PLF
30	2.05857	2.07716	2.08579	2.07824	2.02772	2.06123	2.05367	2.03451	2.05045
	(0.14809)	(0.15559)	(0.13651)	(0.13945)	(0.14339)	(0.11650)	(0.13632)	(0.14034)	(0.11594)
50	2.03669	2.07087	2.05412	2.05280	2.03470	2.08167	2.05856	2.05853	2.08671
	(0.02103)	(0.02220)	(0.02007)	(0.02046)	(0.02108)	(0.01993)	(0.07940)	(0.08120)	(0.07557)
75	2.02959	2.04061	2.04261	2.04044	2.02043	2.04965	2.03529	2.02820	2.06037
	(0.05664)	(0.05728)	(0.05244)	(0.05129)	(0.05376)	(0.05070)	(0.05229)	(0.05446)	(0.05087)
100	2.03882	2.04134	2.02536	2.04222	2.02062	2.03689	2.03625	2.02805	2.03361
	(0.04062)	(0.04160)	(0.03979)	(0.03968)	(0.04062)	(0.03731)	(0.04003)	(0.04032)	(0.03812)

7 Data Analysis

This part showcases the application of the estimation methods presented using a real-world dataset. The dataset used consists of monthly water capacity measurements for the Shasta reservoir during the months of August and December from 1975 to 2016. These data have been previously utilized by statisticians, including Kohansal [19], for analysis. To ensure compatibility with the range defined for $K(\theta,\lambda)$, all the data points are normalized by dividing them by the total capacity of the Shasta reservoir, which is 4,552,000 acre-foot. The dataset includes the monthly water capacity values for the Shasta reservoir are as follows:
$0.667157, 0.287785, 0.126977, 0.768563, 0.703119, 0.729986, 0.767135, 0.463726, 0.371904, 0.291172, 0.414087, 0.650691, 0.538082, 0.744881, 0.722613, 0.561238, 0.813964, 0.709025, 0.668612, 0.524947, 0.605979, 0.715850, 0.529518, 0.824860, 0.742025, 0.468782, 0.345075, 0.425334, 0.767070, 0.679829, 0.613911, 0.461618, 0.294834, 0.392917, 0.688100.$

Table 3: AE of λ, hazard rate and their MSE in parentheses.

Estimates	Bayes			E-Bayes			H-Bayes		
	SELF	ALF	PLF	SELF	ALF	PLF	SELF	ALF	PLF
$\hat{\lambda}$	0.64997	0.64227	0.65757	0.65492	0.64576	0.66114	0.65352	0.64477	0.66012
	(0.12252)	(0.12796)	(0.11725)	(0.11903)	(0.12547)	(0.11482)	(0.12006)	(0.12618)	(0.11551)
$\hat{H(t)}$	1.31189	1.29636	1.33444	1.32202	1.30341	1.33239	1.31902	1.30140	1.32723
	(0.49913)	(0.52131)	(0.49777)	(0.48491)	(0.51118)	(0.47059)	(0.48911)	(0.51406)	(0.47768)

8 Conclusion

In this paper, different methods of estimation are used to estimate the shape parameter and hazard rate of Kumaraswamy distribution. The estimates were computed by performing the simulation using R. The findings and properties of the various estimates are also proved in the form of theorem 1-4. The results showed that the MSE of Bayesian estimates is more than the E-Bayesian and Hierarchical Bayesian so it is less efficient than the Expected Bayesian and H-Bayesian. Also, the H-Bayesian estimates are very close to E-Bayesian estimates, but the H-Bayesian estimates involve complicated integrals which cannot be solved explicitly, so E-Bayesian estimation technique is easy to use and preferable over Hierarchical Bayesian estimation. Also, the MSE of estimates under PLF is less than the SELF and ALF so it is better than the other two loss functions. Finally, a real life dataset has been examined to study the behavior of the proposed estimators.

Conflicts of Interest: The authors affirm that there are no conflicts of interest to disclose regarding the content presented in this paper.

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