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Oscillating Magnetohydrodynamic Stability Self-gravitating Fluid Cylinder Pervaded By General Varying Magnetic Field

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Abstract: The dispersion relation of magnetohydrodynamic stability of an oscillation self-gravitating flowing fluid cylindrical enclosed by varying magnetic field is studied. The problem is specified, investigated analytically and the data verified mathematically, the fundamental equations are resolved, non-singular solutions are found using the proper boundary circumstances, also, derived the total second order integro-differential equation of Mathieu equation type. Just the small axisymmetric region is destabilized by the capillary force but is stabilized by the otherwise perturbation modes. The constant magnetic field that permeates the fluid is strongly stabilized regardless of the perturbation type, while the varied magnetic field bound the fluid is completely destabilized in axisymmetric mode(m=0), but it is either so or not in the non-axisymmetric ($m \geq 1$) perturbation per the constraint. Moreover, the fluid's oscillating flow has a potent inclination to stabilize. Once the capillary force's destabilising effect on the model is diminished and subdued, stability develops.

Keywords: Varying Magnetic Field, Oscillating, Stability.

1 Introduction

The whole fluid jet's fundamental stability analysis has received substantial study, perceive (cf. Rayleigh [1], Robert [5], Drazin and Reid [7], Chandrasekhar [8], and Kendall [9]) and reported by Radwan [10]. The last researchers examined the hydrodynamic stability of an axisymmetric fluid jet perturbed by a uniform magnetic field. Also he has several works for studied the inertial force's effect on the capillary stability of this model and under the influence the other forces, view (Radwan, et.al [11],[12],[14]). Radwan and Hasan [13] have focused on the stability of several models under the operation of self-gravitational force in additional to other forces. (ELAZAB, Samia S. et.al [15-16]) are explored an oscillation hollow jet's hydromagnetic stability and the effect of the stability magnetohydrodynamic hollow jet under the acting other forces. Hasan [17] studied the instability magnetohydrodynamic of fluid jet permeated by transversely changing magnetic field, but its tenuous medium is pierced by a uniform magnetic field for all perturbations modes ($m = 0, m \geq 1$). Samia S. Elazab, and Zeinab M. Ismail [18-19] are investigated the magnetohydrodynamic of flowing resistance hollow jet under obliquely varied magnetic field. Moreover, are studied the gravity oscillator of fluid cylindrical under the effect of general varied electrical field and examined the compressible fluid steamer permeated by axial magnetic field and surrounds by various magnetic fields. [20] examined the influence of kinematics, concentrations and thermal fields for different materials on the unstable flowing of incompressible fluid above heating oscillation bottom. [21] studied the basic characteristic and behavior of metallic materials can be conducted in a number of ways. But the current work is varying to above that is examine the self-gravitating oscillating magnetohydrodynamic stability of fluid cylinder penetrated by a uniform magnetic field while the encircle medium is pervaded by the general varying magnetic field.

2 Formulation of the problem

Consider a perfectly conducting, incompressible, non-viscous fluid cylinder of radius $\mathbf{R_0}$ encircled by negligible motion medium. The model is assumed to be streaming with oscillating velocity

$$\underline{\mathbf{u}}_{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \mathbf{U}\cos\boldsymbol{\Omega}\mathbf{t}) \tag{1}$$

And permeated from inner and external with magnetic fields.

$$\underline{\mathbf{H}}_{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \mathbf{H_0}) \tag{2}$$

$$\underline{\mathbf{H}}_{\mathbf{0}}^{\mathbf{ex}} = \left(\mathbf{0}, \frac{\mathbf{R_0 H_0}}{\mathbf{r}}, \boldsymbol{\alpha}\mathbf{H_0}\right) \tag{3}$$

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Where $\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$ are parameters while $\mathbf{H_0}$ is the intensity of the magnetic field. $\mathbf{U}$ and $\boldsymbol{\Omega}$ are magnitude of the velocity $\underline{\mathbf{u}}_{\mathbf{0}}$ and oscillating frequencies of the fluid at time t respectively.

The basic equations in the fluid under the present circumstances are

$$\boldsymbol{\rho}\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\underline{\boldsymbol{u}}.\nabla)\underline{\boldsymbol{u}}\right]=-\nabla \boldsymbol{P}+\boldsymbol{\rho}\nabla \boldsymbol{V}+\frac{\mu}{4\pi}(\nabla\wedge\underline{\boldsymbol{H}})\wedge\underline{\boldsymbol{H}} \tag{4}$$

$$\nabla.\underline{\boldsymbol{u}}=\mathbf{0} \tag{5}$$

$$\frac{\partial \underline{\boldsymbol{H}}}{\partial t}=\nabla\wedge(\underline{\boldsymbol{u}}\wedge\underline{\boldsymbol{H}}) \tag{6}$$

$$\nabla.\underline{\boldsymbol{H}}=\mathbf{0} \tag{7}$$

$$\nabla^2\boldsymbol{V}=-4\boldsymbol{\pi}\boldsymbol{G}\boldsymbol{\rho} \tag{8}$$

In external the fluid cylinder

$$\nabla.\underline{\boldsymbol{H}}^{\boldsymbol{ex}}=\mathbf{0} \tag{9}$$

$$\nabla\wedge\underline{\boldsymbol{H}}^{\boldsymbol{ex}}=\mathbf{0} \tag{10}$$

$$\nabla^2\boldsymbol{V}^{\boldsymbol{ex}}=\mathbf{0} \tag{11}$$

Along the fluid interface

$$\boldsymbol{P_s}=\boldsymbol{T}(\nabla.\underline{\boldsymbol{N}}_{\boldsymbol{s}}) \tag{12}$$

$$\underline{\boldsymbol{N}}_{\boldsymbol{s}}=\nabla\boldsymbol{f}(\mathbf{r},\boldsymbol{\varphi},\mathbf{z};\mathbf{t})/\ |\nabla\boldsymbol{f}(\mathbf{r},\boldsymbol{\varphi},\mathbf{z};\mathbf{t})\ | \tag{13}$$

Where $f(r,\boldsymbol{\varphi}\ ,\ z\ ;t)=0$ (14)

Here $\underline{\boldsymbol{H}}$ ***and*** $\underline{\boldsymbol{H}}^{\boldsymbol{ex}}$ are the magnetic field inside and outside of the fluid. T is the surface tension coefficient, $\underline{\boldsymbol{N}}_{\boldsymbol{s}}$ is the surface's unit outward vector normal and $\boldsymbol{P_s}$ is the curvature pressure due to the capillary force. $\boldsymbol{u}$, $\boldsymbol{\rho}$, $\boldsymbol{p}$ are the velocity vector, density, and, static pressure respectively.

Equation (4) can be rewritten in the form

$$\rho\left[\frac{\partial u}{\partial t}+(\underline{u}.\nabla)\underline{u}\right]-\frac{\mu}{4\pi}(\underline{H}.\nabla)\underline{H}=-\nabla\Pi \tag{15}$$

with $\Pi=P-\rho V+\frac{\mu}{4\pi}(\underline{H}.\underline{H})$. (16)

where Π is the total magnetohydrodynamic pressure. The basic equations (4-16) are solved in the unperturbed state and applied boundary conditions at $r=R_0$. We get

$$\Pi_0=P_0-\rho V_0+\frac{\mu}{8\pi}(\underline{H}_0.\underline{H}_0)=constant. \tag{17}$$

The self-gravitational potentials V_0 ,V_0^{ex} in unperturbed state are

$$\nabla^2V_0=-4\pi G\rho \tag{18}$$

$$\nabla^2V_0^{ex}=0 \tag{19}$$

The non-singular solutions of equations (18),(19) in cylindrical system (r,φ,z) with cylindrical symmetries $(\frac{\partial}{\partial\varphi}=0,\frac{\partial}{\partial z}=0)$ are given

$$V_0=-\pi G\rho r^2+C_1 \tag{20}$$

$$V_0^{ex}=C_2\ \ln r\ +C_3 \tag{21}$$

Where C_1 , C_2 and C_3 are integrating constants can be found by application condition that.

(i) *At the unperturbed surface r=*$\boldsymbol{R_0}$ *,the self-gravitational potential V and its derivative must be continuous we get*

$$\boldsymbol{V_0}=-\boldsymbol{\pi}\boldsymbol{G}\boldsymbol{\rho}\boldsymbol{r}^2 \tag{22}$$

$$\boldsymbol{V_0^{ex}}=-\boldsymbol{\pi}\boldsymbol{G}\boldsymbol{\rho}\boldsymbol{R_0^2}[\mathbf{1}+\mathbf{2}\ \mathbf{ln}\frac{\boldsymbol{r}}{\boldsymbol{R_0}}] \tag{23}$$

The pressure due to the capillary force is given by

$$\boldsymbol{P_{0s}}=\frac{\boldsymbol{T}}{\boldsymbol{R_0}} \tag{24}$$

(ii) *At* $\boldsymbol{r}=\boldsymbol{R_0}$ *, the whole magnetohydrodynamic pressure must be identical , the distribution pressure in unperturbed state is obtain*

$$\boldsymbol{P_0}={}^{\boldsymbol{T}}/_{\boldsymbol{R_0}}+\boldsymbol{\pi}\boldsymbol{G}\boldsymbol{\rho}^2(\boldsymbol{R_0^2}-\boldsymbol{r}^2)+\frac{\boldsymbol{\mu}\boldsymbol{H_0^2}}{\mathbf{8}\boldsymbol{\pi}}(\boldsymbol{\beta}^2+\boldsymbol{\alpha}^2-\mathbf{1}) \tag{25}$$

3 Perturbation analysis.

Based on the normal mode analysis technique each variable quantity $Q(r,\varphi,z;t)$ as follows from slight deviation from unperturbed state $Q(r,\varphi,z;t)=Q_0(r)+\in(t)Q_1(r,\varphi,z;t)$ (26)

where $Q_1 = \in_0 q_1(r)\exp(\sigma t + i(kz + m\varphi))$ (27)

Due to the expression (28) the distortion in the cylindrical interface is provided by

$r = R_0 + R_1 + \ldots$ (28)

with $R_1 = \in(t)\exp(i(kz + m\varphi))$ (29)

where $\in(t) = \in_0 \exp(\sigma t)$

Here $Q(r,\varphi,z;t)$ denote to $\underline{u}$, P, $\underline{H}$, $\underline{H}^{ex}$, and $\underline{N}_s$, while $Q_0(r)$ is the value of $Q(r,\varphi,z;t)$ in the unperturbed state, also $\in_0$ is initial amplitude and σ is the growth rate. The applicable linearized perturbation equations for the fundamental equations ((4)–(14)) are presented as follows.

$$\rho\left[\frac{\partial \underline{u}_1}{\partial t} + (\underline{u}_0.\nabla)\underline{u}_1\right] - \frac{\mu}{4\pi}(\underline{H}_0 \cdot \nabla)\underline{H}_1 = -\nabla\Pi_1 \quad (30)$$

$$\Pi_1 = P_1 - \rho V_1 + \frac{\mu}{4\pi}\nabla(\underline{H}_0 \cdot \underline{H}_1) \quad (31)$$

$$\nabla \cdot \underline{u}_1 = 0 \quad (32)$$

$$\nabla \cdot \underline{H}_1 = 0 \quad (33)$$

$$\frac{\partial \underline{H}_1}{\partial t} = (\underline{H}_0 \cdot \nabla)\underline{u}_1 - (\underline{u}_0 \cdot \nabla)\underline{H}_1 \quad (34)$$

$$\nabla^2 V_1 = 0 \quad (35)$$

$$P_{1s} = \left(\frac{-T}{R_0^2}\right)\left(R_1 + \frac{\partial^2 R_1}{\partial\varphi^2} + R_0^2\left(\frac{\partial^2 R_1}{\partial z^2}\right)\right) \quad (36)$$

$$\nabla \cdot \underline{H}_1^{ex} = 0 \quad (37)$$

$$\nabla \wedge \underline{H}_1^{ex} = 0 \quad (38)$$

$$\nabla^2 V_1^{ex} = 0 \quad (39)$$

By using expansion (27), eqs. ((35), (39)) give the second-order ordinary differential equation.

$$\left(\frac{1}{r}\right)\frac{d}{dr}\left(r\frac{dq_1}{dr}\right) - \left(\frac{m^2}{r^2} + k^2\right)q_1(r) = 0 \quad (40)$$

Here q_1 refere to V_1 and V_1^{ex}, the solution of equation(40)is obtain in Bessel ordinary functions of imaginary argument. For the problem under the consideration the finally solution of eqs.((35),(39)) take the form

$$V_1 = A \in_0 I_m(kr)\exp(\sigma t + i(kz + m\varphi)) \quad (41)$$

$$V_1^{ex} = B \in_0 K_m(kr)\exp(\sigma t + i(kz + m\varphi)) \quad (42)$$

Where $I_m(kr)$ and $K_m(kr)$ are the first and second kind of Bessel function of order m, while A, B are constants or integration can be determined.

Equation (34) based on (27) is given by

$$\underline{H}_1 = \frac{i\,k\,H_0}{(\sigma + ikU\cos\Omega t)}\,\underline{u}_1 \quad (43)$$

By taking divergence to eq.(30) we get

$$\nabla^2\,\Pi_1 = 0 \quad (44)$$

The fluid is irrotational, so $\underline{u}_1 = \nabla\phi_1$ (45)

Combining equations (32), (45) we find

$$\nabla^2\phi_1 = 0 \quad (46)$$

Equation (38) means that the magnetic field $\underline{H}_1^{ex}$ could be derived from scalar function Ψ_1^{ex}(say).

$$\underline{H}_1^{ex} = \nabla\Psi_1^{ex} \quad (47)$$

Combining equation (37)with (47)we get

$$\nabla^2\Psi_1^{ex} = 0 \quad (48)$$

Based on expansion (27), the non-singular solutions of equations (44, 46, 48) are given in the form.

$$\phi_1 = C_4 \in_0 I_m(kr)\exp(\sigma t + i(kz + m\varphi)) \quad (49)$$

$$\Pi_1 = C_5 \in_0 I_m(kr)\exp(\sigma t + i(kz + m\varphi)) \quad (50)$$

$$\boldsymbol{\Psi_1^{ex} = C_6 \in_0 K_m(kr)\exp(\sigma t + i(kz + m\varphi))} \quad (51)$$

Here $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{C_6}$ are integral parameters. According to eqs.(29) ,(36) the surfaces pressure in the perturbed state caused by the capillary force along cylindrical fluid interface $\mathbf{r = R_0}$ is obtain.

$$\mathbf{P_{1s} = \left(\frac{-T}{R_0^2}\right)[1 - m^2 - x^2]R_1} \quad (52)$$

Where ($x=k\mathbf{R_0}$) is the longitudinal dimensionless wavenumber.

4 Boundary Conditions

The boundary conditions of the problem must be satisfied by the solutions of the basic equations (4-14) in the unperturbed obtain by eqs.(1-3),(17),(22-25), while in perturbed state given by(43), (52) , the following is a list of these boundaries conditions.

(i)*Magnetodynamic Condition*

the normal component or the magnetic field must be continuous across the fluid interface $r = R_0$. This condition is read

$$\underline{N}_s \cdot \underline{H} - \underline{N}_s \cdot \underline{H}^{ex} = 0 \quad . \text{At } r = R_0 \tag{53}$$

from which we get

$$C_6 = \frac{R_0}{xK_m'(x)}\left(ik\alpha H_0 + \frac{im\beta H_0}{R_o}\right) \tag{54}$$

(ii) *Kinematic conditions*

The normal component of the velocity vector u of the fluid must be compatible with the velocity of the perturbed boundary fluid interface at $r = R_0$, this condition state.

$$u_{1r} = \frac{\partial R_1}{\partial t} + U\cos\Omega t\,\frac{\partial R_1}{\partial z} \tag{55}$$

Combining eq.(55)with (56)

$$u_{1r} = \frac{\partial \phi_1}{\partial r} = C_4\, k\, \in_0 I_m'(kr)\exp\left(\sigma t + i(kz + m\varphi)\right) \tag{56}$$

we obtain

$$C_4 = \frac{1}{kI_m'(x)}(\sigma + ikU\cos\Omega t) \tag{57}$$

From eqs.(30) and (43)we have

$$\rho\left[\frac{\partial u_{1r}}{\partial t} + U\cos\Omega t\,\frac{\partial u_{1r}}{\partial z}\right] + \frac{\mu k^2 {H_0}^2}{4\pi(\sigma + ikU\cos\Omega t)}u_{1r} = \frac{-\partial \Pi_{1r}}{\partial r} \tag{58}$$

From which we get

$$C_5 = \left(\frac{-1}{kI_m'(x)}\right)\left[\rho(\sigma^2 + 2ik\sigma U\cos\Omega t - ikU\Omega\sin\Omega t - k^2U^2(\cos\Omega t)^2) + \frac{\mu k^2 {H_0}^2}{4\pi}\right] \tag{59}$$

(iii) *Self-gravitating conditions*

the gravitational potentials and it's derivative must be continuous across the perturbed fluid interface (30)at unperturbed surface $r = R_0$. These conditions are

$$V_1 + R_1\frac{\partial V_0}{\partial r} = V_1^{ex} + R_1\frac{\partial V_0^{ex}}{\partial r} \tag{60}$$

$$\frac{\partial V_1}{\partial r} + R_1\frac{\partial^2 V_0}{\partial r^2} = \frac{\partial V_1^{ex}}{\partial r} + R_1\frac{\partial^2 V_0^{ex}}{\partial r^2} \tag{61}$$

From which we find

$$A = 4\pi G\rho R_0 K_m(x) \tag{62}$$

$$B = 4\pi G\rho R_0 I_m(x) \tag{63}$$

5 Stability criterion

The jump of the normal component of the stresses in the fluid and surround medium must be discontinuous by surface pressure P_{1s} across the fluid cylindrical interface (28) at $r = R_0$.This condition is

$$P_1 + R_1\frac{\partial P_0}{\partial r} + \frac{\mu}{4\pi}\left(\underline{H}_0 \cdot \underline{H}_1\right) - \frac{\mu}{4\pi}\left(\underline{H}_0.\underline{H}_1\right)^{ex} = P_{1s} \tag{64}$$

From equation (31) the condition (64) takes the form

$$\Pi_1 = P_{1s} - \rho V_1 - R_1\frac{\partial P_0}{\partial r} + \frac{\mu}{4\pi}\left(\underline{H}_0.\underline{H}_1\right)^{ex} \tag{65}$$

Substituting for Π_1 , P_{1s} , R_1, P_0 , H_0^{ex} , *and*, H_1^{ex} .we find

$$\sigma^2 + 2(ik\sigma U\cos\Omega t) - ikU\Omega\sin\Omega t - k^2U^2(\cos\Omega t)^2 =$$

$$\frac{T}{\rho R_0^3}\left(\frac{xI'_m(x)}{I_m(x)}\right)(1-m^2-x^2)+4\pi G\rho\left(\frac{xI'_m(x)}{I_m(x)}\right)\left[K_m(x)I_m(x)-\frac{1}{2}\right]$$
$$+\frac{\mu H_0^2}{4\pi\rho R_0^2}\{-x^2+[-\beta^2+(\alpha x+m\beta)^2\frac{K_m(x)}{xK'_m(x)}]\left(\frac{xI'_m(x)}{I_m(x)}\right)\} \quad (66)$$

6 Stability Discussion

The formula (66), which describes the dispersion relation of oscillating in-compressible conducting fluid cylinder ambient with a uniform magnetic field and surround by general varying transversely magnetic field. It has connections to the problem's parameters $\rho, T, \mu, H_0, R_0, \alpha, G, and, \beta, U, \Omega, x, m$ and the modified Bessel functions$I_m(x) and\ K_m(x)$, and their derivatives, with the growth rate. This is contained $(\frac{T}{\rho R_0^3})$, $(\frac{\mu H_0^2}{4\pi\rho R_0^2})$, $4\pi G\rho$ as a unit of $(time)^{-2}$ and this truth is essential in creating the non-dimension form of equation (66). If the secular factor in eq. (66) can be ignored and the equation can be rewritten as a Mathieu-type equation (cf.Mclachan [2] and Kelly[4]), allowing us to indicate that the fluid's oscillating streaming has a stabilizing tendency.

If we assume $\Omega = 0$ and $\beta = 0$, the generalized relation (66) yields.

$$(\sigma+ikU)^2=\frac{T}{\rho R_0^3}(\frac{xI'_m(x)}{I_m(x)})(1-m^2-x^2)+4\pi G\rho\left(\frac{xI'_m(x)}{I_m(x)}\right)\left(K_m(x)I_m(x)-\frac{1}{2}\right)+\frac{\mu H_0^2}{4\pi\rho R_0^2}[-x^2+x^2\alpha^2\frac{K_m(x)I'_m(x)}{K'_m(x)I_m(x)}] \quad (67)$$

The explanation of this equation illustrates that the uniform fluid streaming has a destabilizing effect, and this effect holds true for both the axisymmetric (m=0) and non-axisymmetric mode ($m \geq 1$) perturbation.

If we put ($\Omega = 0, U = 0, G = 0, \beta = 0$ and $m \geq 0$) equation (66) is becomes

$$\sigma^2=\frac{T}{\rho R_0^3}(\frac{xI'_m(x)}{I_m(x)})(1-m^2-x^2)+\frac{\mu H_0^2}{4\pi\rho R_0^2}\left[-x^2+x^2\alpha^2\frac{K_m(x)I'_m(x)}{K'_m(x)I_m(x)}\right] \quad (68)$$

As a global, can be the relation (69) is stable if and only if

$$(\frac{H_0}{H_T})^2 \geq \frac{xK'_m(x)I'_m(x)(1-m^2-x^2)}{x^2K'_m(x)I_m(x)-x^2\alpha^2K_m(x)I'_m(x)} \quad (69)$$

Where $H_T = (\frac{4\pi T}{\mu R_0})^{\frac{1}{2}}$ has a unit of magnetic field. We must record some characteristics of the modified Bessel functions to analyses how the magneto-dynamic and capillary forces affect the stability of the current model.

Believe the recurrence relation (cf. Abramowitz and stegun [6])

$$2I'_m(x) = I_{m-1} + I_{m+1} \quad (70)$$

$$2K'_m(x) = -(K_{m-1} + K_{m+1}) \quad (71)$$

Where $I_m(x) > 0, and\ K_m(x) > 0$, for every non-zero value. And also, $I'_m(x) > 0,\ while\ K'_m(x) < 0$. Let ($\Omega = 0, U = 0, G = 0, H_0 = 0$ and $m \geq 0$), the relation (66) reduce to.

$$\sigma^2=\frac{T}{\rho R_0^3}(1-m^2-x^2)(\frac{xI'_m(x)}{I_m(x)}) \quad (72)$$

This formula is valid for all modes.This relation is derived by Drazin and Reid [6], which means that the capillary's fluid is un-stable only in $0 < x < 1$ in case (m=0), while the capillary is stable for otherwise perturbation mode.

If we take ($\Omega = 0, U = 0, G = 0, T = 0, \alpha = 0$ and $m \geq 0$), the dispersion relation (66) degenerates to

$$\sigma^2=\frac{\mu H_0^2}{4\pi\rho R_0^2}\{-x^2+\left[-\beta^2+m^2\beta^2\frac{K_m(x)}{xK'_m(x)}\right]\frac{xI'_m(x)}{I_m(x)}\} \quad (73)$$

This the dispersing relation actually found by (Radwanet.al.[10]). Here we find the different effects of the magnetic fields that is inside and outside the fluid regions separately. The phase ($-x^2$) after the natural amount $(\frac{\mu H_0^2}{4\pi\rho R_0^2})$ represents the influence of the axial magnetic field prevalent inner fluid in equation (66), it has a constant stabilising impact regardless of the type of perturbation. The azimuthally magnetic field exterior the fluid cylinder indicated by the phases containing β after the natural amount $(\frac{\mu H_0^2}{4\pi\rho R_0^2})$ in equation (66), it is mainly destabilising in the axisymmetric mode m = 0, but under certain conditions, it may stabilise or not in the non-axisymmetric mode (m>0).

If we let ($U = 0, T = 0, H_0 = 0, \Omega = 0$ and $m = 0$) the relation (66) yields

$$\sigma^2 = 4\pi G\rho\left(\frac{xI_1(x)}{I_0(x)}\right)[K_0(x)I_0(x)-\frac{1}{2}] \quad (74)$$

This formula is proved by Chandraseklar and Fermi [3]

In case ($H_0 = 0, U = 0, \Omega = 0$ *and* $m = 0$), equation (66) degenerates

$$\sigma^2 = \frac{T}{\rho R_0^3}\left(\frac{xI_1(x)}{I_0(x)}\right)(1 - x^2) + 4\pi G\rho\left(\frac{xI_1(x)}{I_0(x)}\right)\left[K_0(x)I_0(x) - \frac{1}{2}\right] \quad (75)$$

This relation is examining by Abromowiz and Stegun [6], that discussion the capillary gravito-dynamic stability of two fluids interface when the density of the outer fluid is vanished.

7 Numerical Analysis

The computational study has been done to define and investigate the influence of the magnetic field, capillary force, as well as the impact of streaming on the model stability. Additionally, for specific values of the magnetic field strength, the oscillating stages and the transitional points from these stages to those of in stability may also be calculated. This has been further developed by estimating the non-dimension of eigenvalue relation (66).

$$\sigma^* = U^* + \sqrt{\begin{array}{l}(W^* + N\left(\frac{xI'_m(x)}{I_m(x)}\right)(1 - m^2 - x^2) + \left(\frac{xI'_m(x)}{I_m(x)}\right)\left[K_m(x)I_m(x) - \frac{1}{2}\right] \\ \quad +h\{-x^2 + [-\beta^2 + (\alpha x + m\beta)^2 \frac{K_m(x)}{xK'_m(x)}](\frac{xI'_m(x)}{I_m(x)})\})\end{array}} \quad (76)$$

In the computer simulation for the most sever sausages modes $m \geq 0$ for the various values of h, N, U^*, W^* and range $0 \leq x \leq 3$, *where* $N = \frac{T}{4G\pi\rho^2 R_0^3}$, $h = \left(\frac{H_0}{H_s}\right)^2$, $H_s = \sqrt{\left(\frac{G}{\mu}\right)}\ (4\pi\rho R_0)$, $W^* = \frac{ikU\Omega \sin \Omega t}{4\pi G\rho}$, $U^* = \frac{-ikU \cos \Omega t}{(4\pi G\rho)^{\frac{1}{2}}}$.

The mathematical data related to $\omega / (4\pi G\rho)^{\frac{1}{2}}$, which corresponds to the stable states and those related to $\sigma / (4\pi G\rho)^{\frac{1}{2}}$, which corresponds to the unstable states. That are obtained, tabulated and graphically.

In case $m = 0$, with $I'_0(x) = I_1(x), K'_0(x) = -K_1(x)$, the relation (76) becomes.

$$\sigma^* = U^* + \sqrt{\begin{array}{l}(W^* + N\left(\frac{xI_1(x)}{I_0(x)}\right)(1 - x^2) + \left(\frac{xI_1(x)}{I_0(x)}\right)\left[K_0(x)I_0(x) - \frac{1}{2}\right] \\ \quad -h\{x^2 + [\beta^2 + \alpha^2 x \frac{K_0(x)}{K_1(x)}](\frac{xI_1(x)}{I_0(x)})\})\end{array}}$$

In case $m \geq 1$, and, by using recurrence relations (70), (71) equation (76) take the form

$$\sigma^* = U^* + \sqrt{\begin{array}{l}(W^* + N\left(\frac{x(I_{m-1}(x) + I_{m+1}(x))}{2I_m(x)}\right)(1 - m^2 - x^2) + \left(\frac{x(I_{m-1}(x) + I_{m+1}(x))}{I_m(x)}\right)\left[K_m(x)I_m(x) - \frac{1}{2}\right] \\ \quad -h\{x^2 + [\beta^2 + (\alpha x + m\beta)^2 \frac{2K_m(x)}{x(K_{m-1}(x) + K_{m+1}(x))}](\frac{x(I_{m-1}(x) + I_{m+1}(x))}{I_m(x)})\})\end{array}}$$

For (α, β) = (1, 0.5), $m = 0$ and N=0, 0.1, 0.2, 0.4 and 0.8 corresponding to $U^* = 0$, $W^* = 0$ and h=0.5. It's found the stable domain is $0 \leq x < \infty$, see figure (1).

For (α, β) = (1, 0.5), $m = 0$ and $N = 0,0.1,0.2,0.4$, *and* ,0.8 corresponding to $U^* = 1.4$, $W^* = 0.4$ and h=0.5. The unstable domains have been discovered to be 0 <x <0.7479, 0 <x <0.7487, 0 <x <0.8474, 0 <x <0.8469 and 0 <x <0.8444, while the stable domains neighboring are 0.7479≤x<∞, 0.7487≤x<∞, 0.8474≤x<∞, 0.8469≤x<∞, and 0.8444 ≤x<∞, show figure (2). Where $x_c = 0.7479, 0.7487, 0.8474, 0.8469, 0.8444$, this is the transition from unstable to stable domain and the equivalences that correspond to the limit states stability.

For (α, β) = (1, 0.5), $m = 1$ and $N = 0,0.1,0.2,0.4$, *and* ,0.8 corresponding to $U^* = 0, W^* = 0$ and h=0.5, of unstable domains are 0 <x <2.6351, 0 <x <2.8473, while the neighboring stable domains are 2.6351 ≤x < ∞ , 2.8473 ≤x < ∞, 0≤x<∞ et. see figure (3) Where x_c =2.6351,2.8473.

For (α, β) = (1, 0.5), $m = 1$ and $N = 0,0.1,0.2,0.4$, *and* ,0.8 corresponding to $U^* = 1.4, W^* = 0.4$ and h=0.5, the unstable domains are obtained 0 <x<2.6388, 0 <x <2.8454, 0 <x <0.7463, 0 <x <0.7466, 0 <x <0.6486, while the neighboring stable domains are 2.6388≤x < ∞ , 2.8454x < ∞, 0.7463≤x<∞, 0.7466≤x< ∞, and 0.6486≤x<∞, see figure (4) Where x_c = 2.6388, 2.8454, 0.7463,0.7466,0.6486.

For (α, β) = (1, 0.5), $m = 2$ and N=0, 0.1, 0.2, 0.4 and 0.8 corresponding to $U^* = 0$, $W^* = 0$ and h=2. The stable domain has been discovered to be $0 \leq x < \infty$, see figure (5).

For (α,β) = (1, 0.5), $m = 2$ and $N = 0,0.1,0.2,0.4\,,and\,,0.8$ corresponding to $U^* = 1.1, W^* = 0.1$ and h=0.5, the unstable domains are obtained 0 <x <0.4495, 0 <x <0.4494, 0 <x <0.4493, 0 <x <0.4490, 0 <x <0.4485, while the neighboring stable domains are 0.4495≤x < ∞ , 0.4494≤x < ∞, 0.4493≤x<∞, 0.4490≤x< ∞, and 0.4485≤x<∞, see figure (6) Where x_c =0.4495,0.4494, 0.4493,0.4490,0.4485.

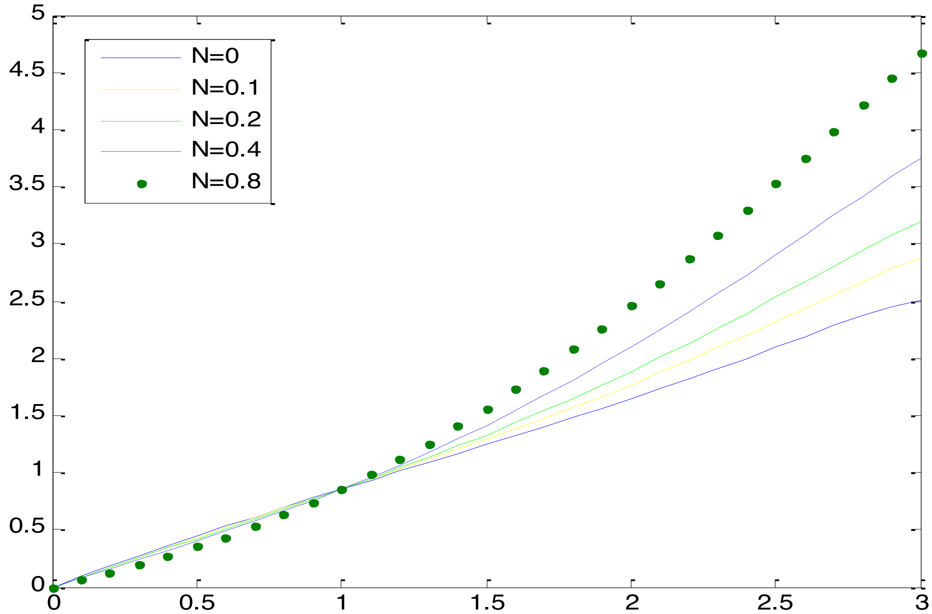

Fig. 1: MDH of oscillating fluid pervaded by varying magnetic field

For h=0.5, $U^* = 0, W^* = 0, m = 0$ and $(\alpha,\beta) = (1,0.5)$.

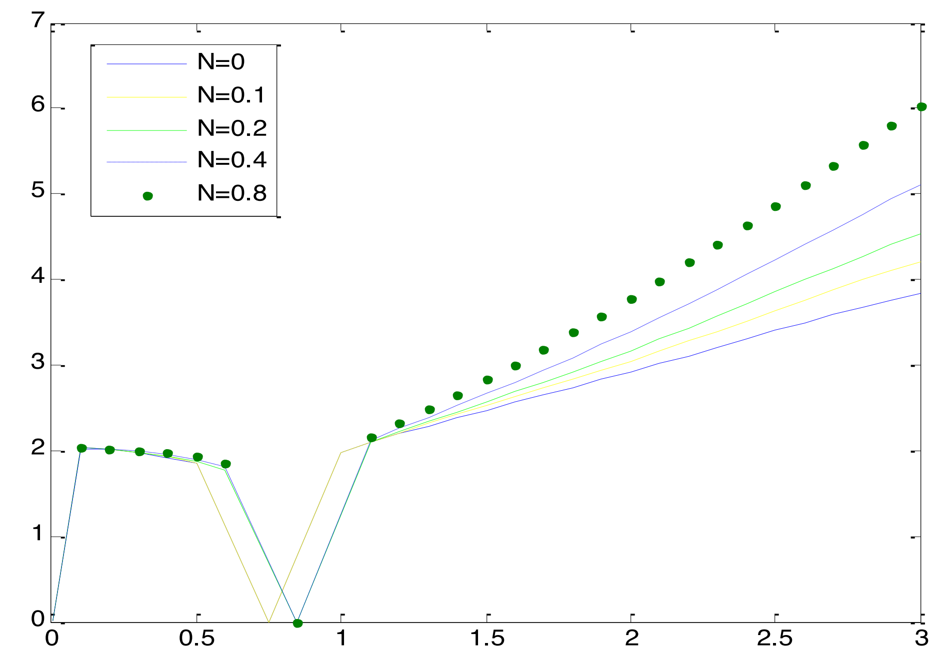

Fig. 2: MDH of oscillating fluid pervaded by varying magnetic field

For h=0.5, $U^* = 1.4, W^* = 0.4,$ $m = 0$ and $(\alpha,\beta) = (1,0.5)$.

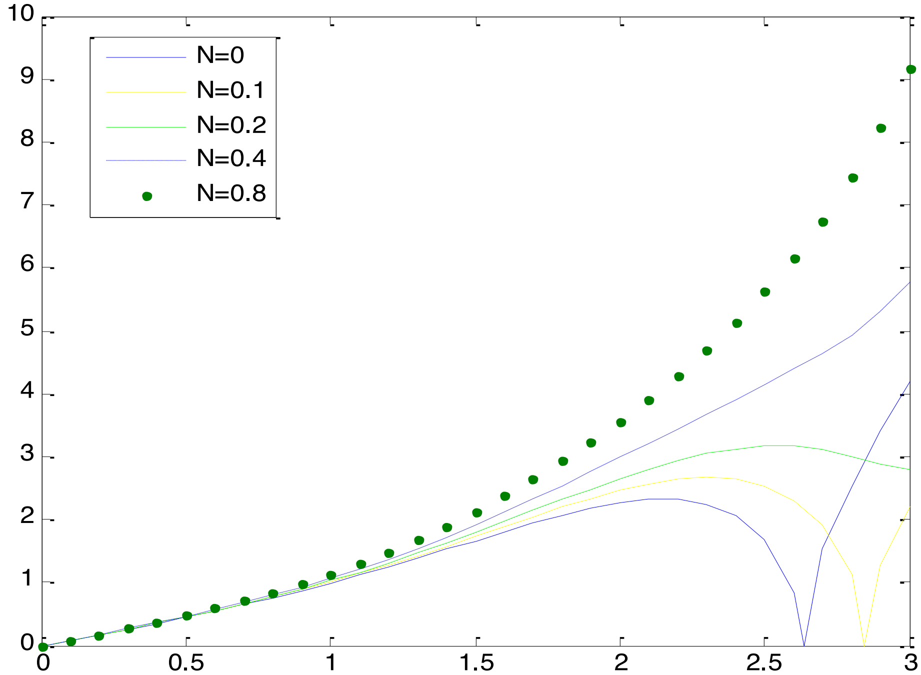

Fig. 3: MDH of oscillating fluid pervaded by varying magnetic field
For h=0.5, $U^* = 0, W^* = 0,\ m = 1\ and\ (\alpha, \beta) = (1{,}0.5)$.

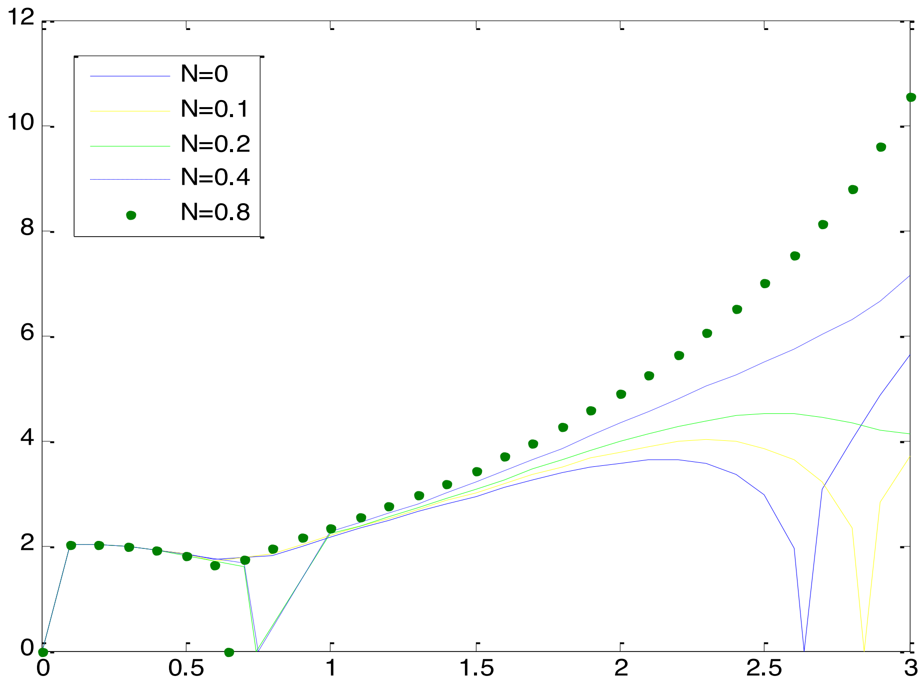

Fig. 4: MDH of oscillating fluid pervaded by varying magnetic field
for h=0.5, $U^* = 1.4, W^* = 0.4,$
$m = 1\ and\ (\alpha, \beta) = (1{,}0.5).$

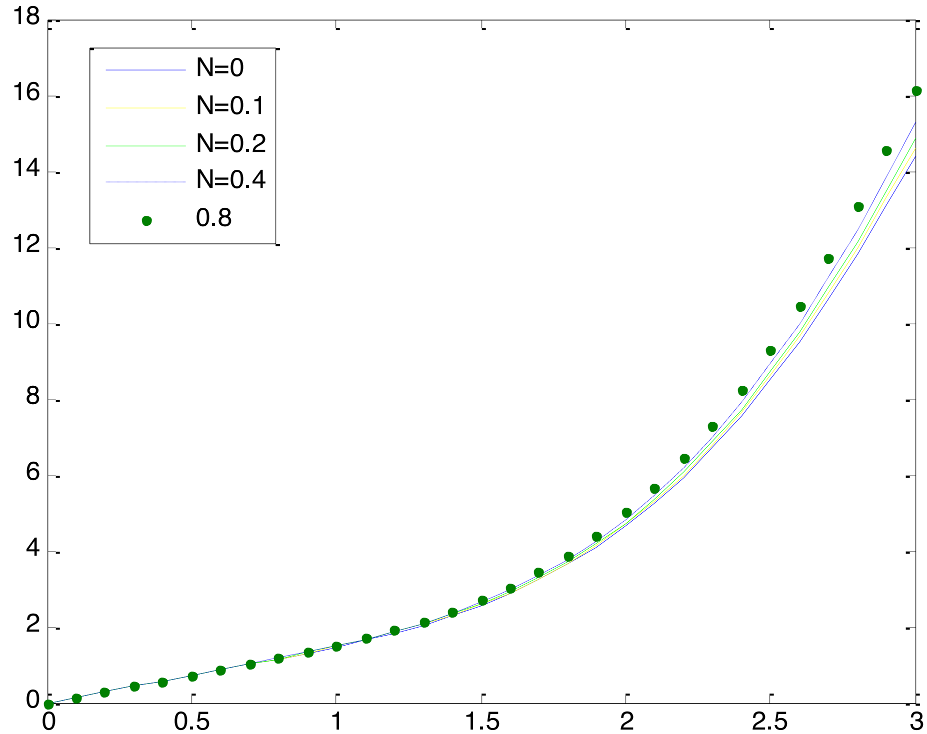

Fig. 5: MDH of oscillating fluid pervaded by varying magnetic field

For h=2, $U^* = 0, W^* = 0,\ m = 2\ and\ (\alpha, \beta) = (1,0.5)$.

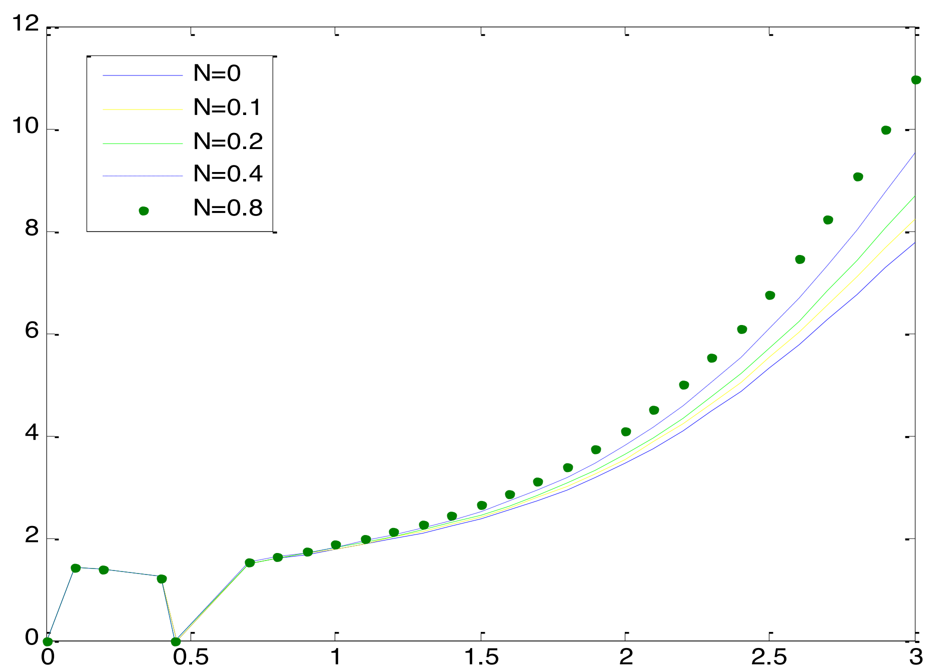

Fig. 6: MDH of oscillating fluid pervaded by varying magnetic field

For h=2, $U^* = 1.1, W^* = 0.1,\ m = 2\ and\ (\alpha, \beta) = (1,0.5)$.

8 Conclusions

Form the previous mathematical analysis we can be deduced that:

1 The capillary force is stabilizing for smaller wavelength, but it's destabilizing for long wavelength in case (m=0) axisymmetric perturbations. However, it's stabilizing in non-axisymmetric modes $m \geq 1$ for all short and long wavelength.

2 The effect of the magnetic field on the capillary instability of the proposed model is strong stabilized in axisymmetric and non-axisymmetric.

3 The streaming has largely destabilized effect on the current model.

4 The general varying magnetic field has strongly stabilized effect on the present model for all modes.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

References

[1] Rayleigh, JM. The Theory of Sound .Dover, *New York* (1945).

[2] N. McLachlan. Theory and Application of Mathieu Function. *Oxford University Press, N.Y* (1947).

[3] Chandrasekhar S, Fermi E. Problems of gravitational stability in the presence of a magnetic field. *Astrophs.J*, 118:116(1953).

[4] Kelly, R. E. The stability of an unsteady Kelvin–Helmholtz flow. Journal of Fluid Mechanics 22, no. 3: 547-560(1965).

[5] P. H. Robert, Longman. An Introduction to MHD. *London, UK* (1967).

[6] Abramowitz, M and Stegun, I. Handbook of Mathematical Functions. *Dover Publ, New York, USA* (1970).

[7] Drazin, PG and Reid, WH. Hydrodynamic Stability. *Cambridge Univ. Press, London, UK* (1980).

[8] Chandrasekhar S. Hydrodynamic and Hydromagnetic Stability. *Dover, New York, NY, USA* (1981).

[9] J. M. Kendall. Experiments on annular liquid jet instability and on the formation of liquid shells. *Physics of Fluids,* vol. 29, no. 7, pp. 2086–2094(1986).

[10] A.E. Radwan. Effect of magnetic fields on the capillary instability of an annular liquid jet. *Journal of Magnetism and Magnetic Materials*, vol. 72, no. 2, pp. 219–232(1988).

[11] Radwan, AE, El-Azab, SS and Hydia, WM. Magnetodynamic stability of a bounded hollow cylinder pervaded by a varying magnetic field. *Nuovo Cimento*, 117B, 257(2002).

[12] RADWAN, Ahmed E.; AL-KAABI, Aisha N. Hydromagnetic Capillary Stability of Streaming Finite Hollow Cylinder with Azimuthal Field. *Mechanics and Mechanical Engineering* , 9.2: 153-177(2005).

[13] A.E. Radwan and A. A. Hasan. Axisymmetric electrogravitational stability of fluid cylinder ambient with transverse varying oscillating field. *IAENG International Journal of Applied Mathematics* ,vol. 38, no. 3, pp. 113–120(2008).

[14] RADWAN, A. E.; HYDIA, W. M.; HASAN, A. A. Magnetohydrodynamic stability of an oscillating fluid cylinder. *the European Physical Journal Plus* , 126.1: 1-7(2011).

[15] ELAZAB, Samia S., et al. *Applied Mathematical Sciences*, (2011): Hydromagnetic stability of oscillating Hollow jet, 5.28: 1391-1400.

[16] ELAZAB, Samia S., et al.Axisymmetric hydromagnetic stability of a streaming resistive hollow jet under oblique varying magnetic field. *J. Math. Comput. Sci.* 2.3: 473-500 (2012).

[17] HASAN, Alfaisal A. Magneto-hydrodynamic stability of streaming jet pervaded internally by varying transverse magnetic field. *Mathematical Problems in Engineering*(2012).

[18] Samia S. Elazab, and Zeinab M. Ismail, The Gravitational Oscillation Of A Fluid Cylinder Under Varying Electric Field. *International Journal of Mathematical Archive EISSN* 2229-5046 4 (10) (2013).

[19] Samia S. Elazab, and Zeinab M. Ismail. Stability Of Streaming Compressible Fluid Cylinder Pervaded By Axial Magnetic Field And Surouunded By Different Magnetic Field. *Jokull*, 65 (1), 318-330(2015).

[20] BULINDA, Vincent M.; KANG'ETHE, Giterere P, KIOGORA, Phineas R. :Analysis of Magnetohydrodynamics Flow of Incompressible Fluids over Oscillating Bottom Surface with Heat and Mass Transfer. *International Journal of Mathematics and Mathematical Sciences*, (2020).

[21] Bracker, Gwendolyn P., and Robert W. Hyers. Modeling of Magnetohydrodynamic Flows in Electromagnetic Levitation. In *Metallurgy in Space: Recent Results from ISS*, pp. 281-297. Cham: Springer International Publishing, 2022.