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ARA-Sumudu Method for Solving Volterra Partial Integro-Differential Equations

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Abstract: The major goal of the study, is investigate solving various kinds of Volterra integral equations via via a new method. This article presents a novel double ARA- Sumudu (ARA-S) transformation . this novel approach is implemented to handle some integral equations and partial integro-differential equations. Fundamental characteristics and results related to double ARA-S transformation are investigated including the existence, inverse derivative and the convolution property. Furthermore, to prove the applicability of the presented transform, we discuss the solution of some examples on integral equations using double ARA-S transformation.

Keywords: Sumudu transformation, ARA transformation, Partial integral equations, Partial integro-differential equations, Double ARA– Sumudu transformation

1 Introduction

One of the most important techniques recently used to solve Volterra integral equations of various classes is the integral transformation approach. For this reason, many phenomena in the field of engineering, science, and mathematical physics can be introduced by integral equations of different types [1,2,3,4,5,6,7,8]. Using integral transformations, we can transformation integral equations into algebraic or differential equations and get the exact solution of the target integral equations. Developed through the hard work of many scientists and researchers, these techniques are used today to tackle challenging problems in contemporary arithmetic. For example, we mention ARA transform, Sumudu transformation, formable transformation, Elzaki transformation and others [9,10,11,12,13,14]. Recently, double transformations are extensively used to solve partial differential equations and partial integral equations, which gave good results in comparison other analytical techniques like decomposition method, power series method, variational iteration method and homotopy analysis method etc. [15,16,17,18,19,20]. In addition, there are other extensions of double transformations in the previous literature, such as double Laplace transformation, double Elzaki transformation, double Shehu transformation, double Sumudu transformation, double Laplace-Sumudu transformation and others[21,22,23,24,25,26]. In 2022, authors have presented a new transformation called the ARA-Sumudu transformation [25], which is a double transformation that combines ARA and Sumudu transformation, then it is implemented to solve differential equations of integer and fractional orders in [26]. In this research, we use double ARA-S transformation to solve integral equations of different kinds. We define the double ARA-S transformation to the integrable function $\Phi(x,y)$ as

$$\begin{aligned}\mathscr{G}_x S_y\left[\varphi\left(x,y\right)\right] &= \Phi\left(s,u\right)\\ &= \frac{s}{u}\int_0^{\infty}\int_0^{\infty} e^{-sx-\ \frac{y}{u}}\varphi\left(x,y\right)dxdy,\\ & s>0,\ \ u>0,\end{aligned}$$

provided the integrals exist. The kinds of this article is to investigate the solution of two types of equations: integral equations and integro-partial differential equations of Volterra kind.

We apply the double ARA-S transformation to solve the above integral equations by converting them to algebraic equations in the double ARA-S transformation space,

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then, the inverse double ARA-S transformation is applied to get the solution in the initial space. This study provides some fundamental properties of double ARA-S transformation to basic functions, derivatives properties and results related to the double convolution theorem. Furthermore, we establish new results related to the procedure of solving integral equations, and we utilize them to handle examples.

2 Basic Definitions and Properties

In this part, we spotlight the fundamental characteristics and definitions concerning.

Definition 1.[9] *Assume that $f(x)$ is a function in wich $|f(x)| \leq Me^{ax}$, $\forall x > 0$ and $M > 0$. Then*

$$S[f(x)] = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u}} f(x)\,dx,\ u > 0.$$

Definition 2.[12] *If the function $f(x)$ is of exponential order defined on $[0,\infty)$. Then ARA transformation of order one of the function $f(x)$ is defined and denoted by*

$$\mathcal{G}[f(x)] = s\int_0^\infty e^{-sx} f(x)\,dx,\ s > 0.$$

Definition 3.[25] *Let $\varphi(x,y)$ be a function expressed as a convergent infinite series. Then double ARA-S transformation definition to $\varphi(x,y)$ is given and denoted by*

$$\begin{aligned}\mathcal{G}_x S_y[\varphi(x,y)] &= \Phi(s,u) \\ &= \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\varphi(x,y)\,dxdy, \qquad (1)\\ & s>0,\ \ u>0.\end{aligned}$$

We define the inverse of the double ARA-S transformation by

$$\begin{aligned}\mathcal{G}_x^{-1}S_t^{-1}[\Phi(s,u)] &= \varphi(x,y) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}}{s}ds\,\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\frac{y}{u}}}{u}\Phi(s,u)\,du\,.\end{aligned}$$

Clearly, DARA-ST and its inverse are linear integral transformations as shown below:

$$\begin{aligned}&\mathcal{G}_x S_y[\gamma\varphi(x,y)+\eta\psi(x,y)]\\ &= \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}(\gamma\varphi(x,y)+\eta\psi(x,y))\,dxdy\\ &= \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\gamma\varphi(x,y)\,dxdy\\ &\quad+\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\eta\psi(x,y)\,dxdy\\ &= \frac{s\gamma}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\varphi(x,y)\,dxdy\\ &\quad+\frac{s\eta}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\psi(x,y)\,dxdy\\ &= \gamma\mathcal{G}_x S_y[\varphi(x,y)]+\eta\mathcal{G}_x S_y[\psi(x,y)]\\ &= \gamma\,\Phi(s,u)+\eta\,\Psi(s,u),\end{aligned}$$

where γ and η are constants, $\Phi(s,u) = \mathcal{G}_x S_y[\varphi(x,y)]$, $\Psi(s,u) = \mathcal{G}_x S_y[\psi(x,y)]$, φ and ψ are two continuous functions. Similarly, we can show the inverse double ARA-S transformation is linear, i.e

$$\mathcal{G}_x^{-1}S_y^{-1}[\gamma\Phi(s,u)+\eta\Psi(s,u)] = \gamma\,\varphi(x,y)+\eta\,\psi(x,y),$$

Definition 4.[25] *If $\varphi(x,y)$ defined on $[0,A]\times[0,B]$, and satisfies the condition*

$$|\varphi(x,y)| \leq Re^{\alpha x+\beta y},\ \ \exists R>0,\ \ \forall x>A \ \ \textit{and} \ \ y>B,$$

then, we call $\varphi(x,y)$ an exponential orders function with α and β as $x\to\infty$ and $y\to\infty$.

Theorem 1.[25,26] *The existence condition of DARA-ST of the continuous function $\varphi(x,y)$ defined on $[0,A]\times[0,B]$ is to be of exponential orders α and β, for $Re[s] > \alpha$ and $Re\left[\frac{1}{u}\right] > \beta$.*

Proof. The double ARA-S transformation of the function implies

$$\begin{aligned}|\Phi(s,u)| &= \left|\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\varphi(x,y)\,dx\,dy\right|\\ &\leq \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}|\varphi(x,y)|\,dx\,dy\\ &\leq \frac{R\,s}{u}\int_0^\infty e^{-(s-\alpha)x}dx\int_0^\infty e^{-\left(\frac{1}{u}-\beta\right)y}dy\\ &= \frac{R\,s}{u(s-\alpha)\left(\frac{1}{u}-\beta\right)}\\ &= \frac{R\,s}{(s-\alpha)(1-u\beta)},\end{aligned}$$

where $Re[s] > \alpha$ and $Re\left[\frac{1}{u}\right] > \beta$. □

Definition 5.*The convolution of $\varphi(x,y)$ and $\psi(x,y)$ is denoted by $(\varphi**\psi)(x,y)$ and defined by*

$$(\varphi**\psi)(x,y) = \int_0^x\int_0^y \varphi(x-\delta,y-\varepsilon)\,\psi(\delta,\varepsilon)\,d\delta d\varepsilon.$$

Theorem 2.[25] *Assume that $G(s,u) = \mathcal{G}_x S_y[g(x,y)]$ and then*

$$\mathcal{G}_x S_y[g(x-\delta,y-\varepsilon)H(x-\delta,y-\varepsilon)] = e^{-s\delta-\frac{\varepsilon}{y}}G(s,u),$$

where $H(x,y)$ is the Heaviside function defined by

$$H(x-\delta,y-\varepsilon) = \begin{cases}1, & x>\delta,\ y>\varepsilon\\ 0, & otherwise\end{cases} \qquad (2)$$

Theorem 3.*(Double Convolution Theorem) If $\mathcal{G}_x S_y[\varphi(x,y)] = \Phi(s,u)$ and $\mathcal{G}_x S_y[\psi(x,y)] = \Psi(s,u)$, then*

$$\mathcal{G}_x S_y[(\varphi**\psi)(x,y)] = \frac{u}{s}\Phi(s,u)\Psi(s,u)\,. \qquad (3)$$

Proof. The double ARA-S transformation definition implies

$$\begin{aligned}
&\mathscr{G}_x S_y\left[(\varphi**\psi)(x,y)\right]\\
&=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}(\varphi**\psi)(x,y)\,dxdy\\
&=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\\
&\left(\int_0^x\int_0^y\varphi(x-\delta,y-\varepsilon)\,\psi(\delta,\varepsilon)\,d\delta d\varepsilon\right)dxdy.
\end{aligned}$$

The definition of unit step function in Eq. (3) implies

$$\begin{aligned}
&\mathscr{G}_x S_y\left[(\varphi**\psi)(x,y)\right]=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\\
&\left(\int_0^\infty\int_0^\infty\varphi(x-\delta,y-\varepsilon)H(x-\delta,y-\varepsilon)\right.\\
&\left.\psi(\delta,\varepsilon)\,d\delta d\varepsilon\right)dxdy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathscr{G}_x S_y\left[(\varphi**\psi)(x,y)\right]=\int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)\,d\delta d\varepsilon\\
&\left(\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\varphi(x-\delta,y-\varepsilon)\right.\\
&\left.H(x-\delta,y-\varepsilon)\,dxdy\right)\\
&=\int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)\,d\delta d\varepsilon\left(e^{-sx-\frac{y}{u}}\Phi(s,u)\right)\\
&=\Phi(s,u)\int_0^\infty\int_0^\infty e^{-sx-\frac{y}{u}}\psi(\delta,\varepsilon)\,d\delta d\varepsilon\\
&=\frac{u}{s}\Phi(s,u)\Psi(s,u).
\end{aligned}$$

In Table 1 below, we introduce the values of double ARA-S transformation to several functions.

Table 1: double ARA-S transformation to basic functions [25].

$\varphi(x,y)$	$\mathscr{G}_x S_y[\varphi(x,y)]=\Phi(s,u)$
1	1
$x^\alpha y^\beta$	$s^{-a}\Gamma(a+1)u^b\Gamma(b+1)$
$e^{\alpha x+\beta y}$	$\frac{s}{(s-a)(1-bu)}$
$e^{i\,(\alpha x+\beta y)}$	$\frac{i\,s}{(s-i\,a)(bu+i)}$
$\sin(\alpha x+\beta y)$	$\frac{s\,(a+b\,s\,u)}{(a^2+s^2)(b^2\,u^2+1)}$
$\cos(\alpha x+\beta y)$	$\frac{s\,(s-ab\,u)}{(a^2+s^2)(b^2\,u^2+1)}$
$\sinh(\alpha x+\beta y)$	$\frac{s\,(a+b\,s\,u)}{(a^2-s^2)(b^2\,u^2-1)}$
$\cosh(\alpha x+by)$	$\frac{s\,(s+ab\,u)}{(a^2-s^2)(b^2\,u^2-1)}$
$J_0\left(c\sqrt{xy}\right)$,	
J_0 the zero order Bessel function	$\frac{4s}{4s+c^2u}$
$\varphi(x-\delta,y-\varepsilon)\;H(x-\delta,y-\varepsilon)$	$e^{-s\delta-\frac{\varepsilon}{u}}\Phi(s,u)$
$(\varphi**\psi)(x,y)$	$\left(\frac{u}{s}\right)\Phi(s,u)\Psi(s,u)$
$\phi(x)$	$\mathscr{G}_x[\phi(x)]=\Phi(s)$
$\psi(y)$	$S_y[\psi(y)]=\Psi(u)$
$\phi(x)\psi(y)$	$\Phi(s)\Psi(u)$

The following theorem, presents double ARA-S transformation for partial derivatives of order one and two.

Theorem 4.[25] *(Derivative properties) If* $\Phi(s,u)=\mathscr{G}_x S_y[\varphi(x,y)]$, *then*

1. $\mathscr{G}_x S_y\left[\frac{\partial\varphi(x,y)}{\partial x}\right]=s\Phi(s,u)-sS_y[\varphi(0,y)]$.

2. $\mathscr{G}_x S_y\left[\frac{\partial\varphi(x,y)}{\partial y}\right]=\frac{1}{u}\Phi(s,u)-\frac{1}{u}\mathscr{G}_x[\varphi(x,0)]$.

3. $\mathscr{G}_x S_y\left[\frac{\partial^2\varphi(x,y)}{\partial x^2}\right]=s^2\Phi(s,u)-s^2S_y[\varphi(0,y)]$

. $-sS_y[\varphi_x(0,y)]$.

4. $\mathscr{G}_x S_y[\frac{\partial^2\varphi(x,y)}{\partial y^2}]=\frac{1}{u^2}\Phi(s,u)-\frac{1}{u^2}\mathscr{G}_x[\varphi(x,0)]$

. $-\frac{1}{u}\mathscr{G}_x[\varphi_y(x,0)]$.

5. $\mathscr{G}_x S_y[\frac{\partial^2\varphi(x,y)}{\partial x\partial y}]=\frac{s}{u}(\Phi(s,u)-S_y[\varphi(0,y)]$

. $-\mathscr{G}_x[\varphi(x,0)]+\varphi(0,0))$.

Where S_y *and* $\mathscr{G}_x$ *denote the single Sumudu and ARA transformations respectively.*

3 Applications of double ARA-S transformation to Solve Integral Differential Equations

This part of the research presents the application of double ARA-S transformation to solve different classes of IEs. We also mention that the values of the transformed functions can be found in Appendix 1.

3.1 Two variables IEs

Consider the following IE o Volterra type

$$\varphi(x,y)=g(x,y)+a\int_0^x\int_0^y\varphi(x-\delta,y-\varepsilon)\,\psi(\delta,\varepsilon)\,d\delta d\varepsilon,\tag{4}$$

where $\varphi(x,y)$ is the function we need to find, a is any real number, $g(x,y)$ and $\psi(x,y)$ are two given functions. Operating double ARA-S transformation on Eq. (4) we get

$$\mathscr{G}_x S_y[\varphi(x,y)] = \mathscr{G}_x S_y[g(x,y)] + a\left(\int_0^x\int_0^y \varphi(x-\delta,y-\varepsilon)\psi(\delta,\varepsilon)\,d\delta d\varepsilon\right). \tag{5}$$

Theorem 3 implies that

$$\Phi(s,u) = G(s,u) + a\frac{u}{s}\Phi(s,u)\Psi(s,u), \tag{6}$$

where $\Phi(s,u) = \mathscr{G}_x S_y[\varphi(x,y)]$,$G(s,u) = \mathscr{G}_x S_y[g(x,y)]$ and $\Psi(s,u) = \mathscr{G}_x S_y[\psi(x,y)]$. Consequently,

$$\Phi(s,u) = \frac{s\,G(s,u)}{s - au\Psi(s,u)}. \tag{7}$$

Applying the inverse transformation $\mathscr{G}_x^{-1}S_y^{-1}[\Phi(s,u)]$ on (7), we get the exact value of $\varphi(x,y)$ in Eq. (4)

$$\varphi(x,y) = \mathscr{G}_x^{-1}S_y^{-1}\left[\frac{s\,G(s,u)}{s - au\Psi(s,u)}\right]. \tag{8}$$

The following examples applications on Eq. (8).
Example 1. Consider the following IE:

$$\varphi(x,y) = b - a\int_0^x\int_0^y \varphi(\delta,\varepsilon)\,d\delta d\varepsilon, \tag{9}$$

considering a and b are real number.
Solution. Applying double ARA-S transformation to Eq. (9) and depending on the linearity property and Theorem 3, we get

$$\Phi(s,u) = b - \frac{au}{s}\Phi(s,u). \tag{10}$$

As a result,

$$\Phi(s,u) = \frac{b\,s}{s+au}. \tag{11}$$

Running the inverse transformation $\mathscr{G}_x^{-1}S_y^{-1}$ on Eq. (11), we obtain the exact solution $\varphi(x,y)$ of Eq. (9) in the original space as

$$\varphi(x,y) = \mathscr{G}_x^{-1}S_y^{-1}\left[\frac{b\,s}{s+au}\right] = b\,J_0\left(2\sqrt{axy}\right).$$

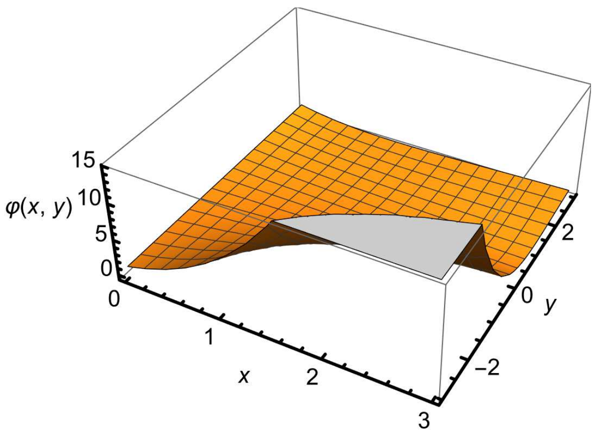

Fig. 1: The 3D plot of Example 1 with $a = b = 1$.

where J_0 is the Bessel function.
Example 2. Solve the following IE:

$$b^2 y = \int_0^x\int_0^y \varphi(x-\delta,y-\varepsilon)\varphi(\delta,\varepsilon)\,d\delta d\varepsilon, \tag{12}$$

where b real number.
Solution. Operating double ARA-S transformation to Eq. (12) and hiring the double convolution property to obtain

$$b^2 u = \frac{u}{s}\Phi^2(s,u). \tag{13}$$

Thus,

$$\Phi(s,u) = b\sqrt{s}. \tag{14}$$

Running the inverse transformation $\mathscr{G}_x^{-1}S_y^{-1}$ to Eq. (14), we get the exact solution $\varphi(x,y)$ of Eq. (14) as follows

$$\varphi(x,y) = \mathscr{G}_x^{-1}S_y^{-1}\left[b\sqrt{s}\right] = \frac{b}{\sqrt{\pi}}\frac{1}{\sqrt{x}}. \tag{15}$$

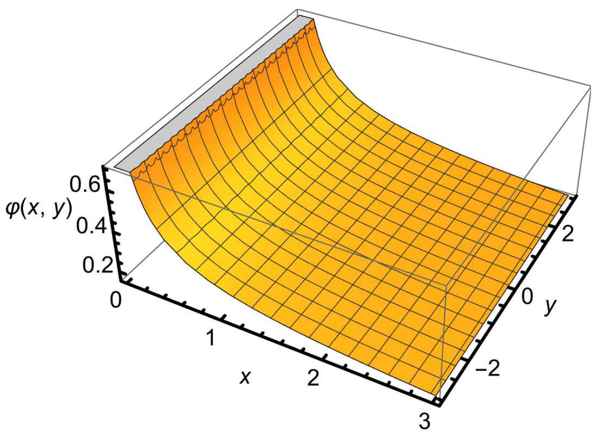

Fig. 2: The 3D plot of Example 2 with $b = 1$.

Example 3. Consider the following integral equation:

$$\int_0^x \int_0^y e^{\delta-\varepsilon} \varphi(x-\delta, y-\varepsilon) d\delta d\varepsilon = xe^{x-y} - xe^x. \tag{16}$$

Solution. Applying double ARA-S transformation on Eq. (16) and using convolution theorem, we get

$$\frac{u\,\Phi(s,u)}{(s-1)(1+u)} = \frac{s}{(s-1)^2(1+u)} - \frac{s}{(s-1)^2}. \tag{17}$$

After simple computations and applying the inverse transformation $\mathscr{G}_x^{-1} S_y^{-1}$ for Eq. (17), the solution of equation Eq. (16) becomes

$$\varphi(x,y) = \mathscr{G}_x^{-1} S_y^{-1} \left[\frac{-s}{s-1}\right] = -e^x. \tag{18}$$

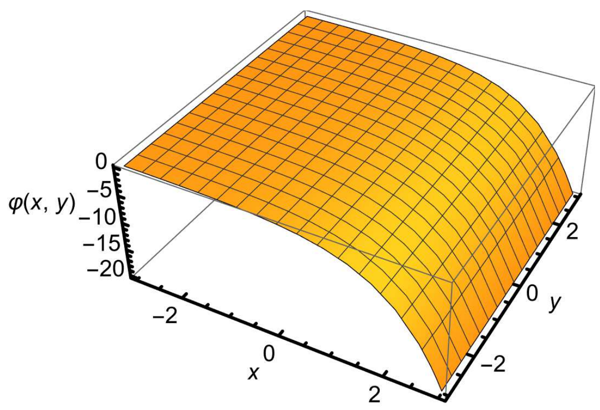

Fig. 3: The 3D plot of Example 3

3.2 First order partial integro–differential equations of two variables

Given the following Volterra partial integro–differential equation

$$\frac{\partial \varphi(x,y)}{\partial x} + \frac{\partial \varphi(x,y)}{\partial y} = \Psi(x,y) + a\int_0^x \int_0^y \varphi(x-\delta, y-\varepsilon)\,\psi(\delta,\varepsilon)\,d\delta d\varepsilon, \tag{19}$$

with the conditions

$$\varphi(x,0) = f_0(x), \quad \varphi(0,y) = h_0(y). \tag{20}$$

where $\varphi(x,y)$ is the unknown function, a is a real number, $g(x,y)$and $\psi(x,y)$ are given functions. Firstly, we operate double ARA-S transformation to Eq. (19), to get

$$s\Phi(s,u) - sS_y[\varphi(0,y)] + \frac{1}{u}\Phi(s,u) - \frac{1}{u}\mathscr{G}_x[\varphi(x,0)] = G(s,u) + a\frac{u}{s}\Phi(s,u)\Psi(s,u).$$

Substituting the values of the transformed condition Eq. (20)

$$\Phi(s,u) = \frac{sF_0(s) + s^2uH_0(u) + suG(s,u)}{s + s^2u - au^2\Psi(s,u)}, \tag{21}$$

where $F_0(s) = \mathscr{G}_x[\varphi(x,0)]$ and $H_0(u) = S_y[\varphi(0,y)]$ Running the inverse transform $\mathscr{G}_x^{-1} S_y^{-1}$ to Eq. (21),the solution of Eq. (19) is introduced by

$$\varphi(x,y) = \mathscr{G}_x^{-1} S_y^{-1} \left[\frac{sF_0(s) + s^2uH_0(u) + suG(s,u)}{s + s^2u - au^2\Psi(s,u)}\right]. \tag{22}$$

We implement the above technique to solve some examples.

Example 4. Solve the partial integro-differential:

$$\frac{\partial \varphi(x,y)}{\partial x} + \frac{\partial \varphi(x,y)}{\partial y} = -1 + e^x + e^y + e^{x+y} + \int_0^x \int_0^y \phi(x-\delta, y-\varepsilon)\,d\delta d\varepsilon, \tag{23}$$

with the conditions

$$\phi(x,0) = e^x = f_0(x), \quad \phi(0,y) = e^y = h_0(y). \tag{24}$$

Solution. Substituting the transformed values:

$$\begin{cases} F_0(s) = \mathscr{G}_x[e^x] = \frac{s}{s-1}, \\ H_0(u) = S_y[e^y] = \frac{1}{1-u}, \\ G(s,u) = \mathscr{G}_x S_y[-1 + e^x + e^y + e^{x+y}] \\ \quad = -1 + \dfrac{s}{s-1} + \dfrac{1}{1-u} + \dfrac{s}{(s-1)(1-u)}, \end{cases} \tag{25}$$

into Eq. (22) and after simple computations, we obtain the solution of Eq. (23)

$$\varphi(x,y) = \mathscr{G}_x^{-1} S_y^{-1} \left[\frac{s}{(s-1)(1-u)}\right] = e^{x+y}. \tag{26}$$

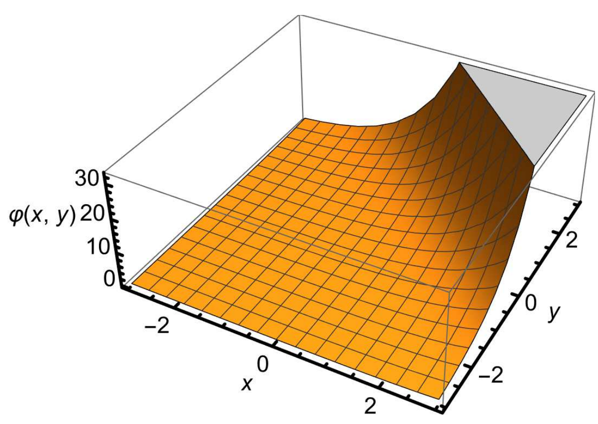

Fig. 4: The 3D plote of Example 4

3.3 Second order partial integro–differential equations of two variables

Given the following partial integro–differential equation

$$\frac{\partial^2\varphi(x,y)}{\partial y^2}-\frac{\partial^2\varphi(x,y)}{\partial x^2}+\varphi(x,y)+\int_0^x\int_0^y\psi(x-\delta,y-\varepsilon)\varphi(\delta,\varepsilon)\,d\delta d\varepsilon=g(x,y), \tag{27}$$

with the conditions

$$\begin{aligned}\varphi(x,0)&=f_0(x), & \frac{\partial\varphi(x,0)}{\partial y}&=f_1(x),\\ \varphi(0,y)&=h_0(y), & \frac{\partial\varphi(0,y)}{\partial x}&=h_1(y).\end{aligned} \tag{28}$$

Applying double ARA-S transformation on both sides of Eq. (27), we get

$$\begin{aligned}&\frac{1}{u^2}\Phi(s,u)-\frac{1}{u^2}\mathcal{G}_x[\varphi(x,0)]-\frac{1}{u}\mathcal{G}_x[\varphi_y(x,0)]\\&-\left(s^2\Phi(s,u)-s^2S_y[\varphi(0,y)]-sS_y[\varphi_x(0,y)]\right)\\&+\Phi(s,u)+\frac{u}{s}\Phi(s,u)\ \Psi(s,u)=G(s,u).\end{aligned} \tag{29}$$

After simple calculations, one can obtain

$$\begin{aligned}\Phi(s,u)=&\frac{sF_0(s)+suF_1(s)-s^3u^2H_0(u)}{s-s^3u^2+su^2+u^3\Psi(s,u)}\\&+\frac{-s^2u^2H_1(u)+su^2G(s,y)}{s-s^3u^2+su^2+u^3\Psi(s,u)},\end{aligned} \tag{30}$$

where $F_0(s)=\mathcal{G}_x[\varphi(x,0)]$, $F_1(s)=\mathcal{G}_x[\varphi_y(x,0)]$, $H_0(u)=S_y[\varphi(0,y)]$ and $H_1(u)=S_y[\varphi_x(0,y)]$. Running the inverse transform $\mathcal{G}_x^{-1}S_y^{-1}$ to Eq. (30), we obtain the solution of Eq.(27) as follows

$$\begin{aligned}\varphi(x,y)=\mathcal{G}_x^{-1}S_y^{-1}\Bigg[&\frac{sF_0(s)+suF_1(s)-s^3u^2H_0(u)}{s-s^3u^2+su^2+u^3\Psi(s,u)}\\&+\frac{-s^2u^2H_1(u)+su^2G(s,y)}{s-s^3u^2+su^2+u^3\Psi(s,u)}\Bigg].\end{aligned} \tag{31}$$

Example 5 below, is an application of Eq. (32).

Example 5. Consider the following partial integro - differential equation:

$$\frac{\partial^2\varphi(x,y)}{\partial y^2}-\frac{\partial^2\varphi(x,y)}{\partial x^2}+\varphi(x,y)+\int_0^x\int_0^y e^{x-\delta+y-\varepsilon}\varphi(\delta,\varepsilon)\,d\delta d\varepsilon=e^{x+y}+xye^{x+y}, \tag{32}$$

with conditions

$$\begin{aligned}\varphi(x,0)&=e^x=f_0(x), & \frac{\partial\varphi(x,0)}{\partial y}&=e^x=f_1(x),\\ \varphi(0,y)&=e^y=h_0(y), & \frac{\partial\varphi(0,y)}{\partial x}&=e^y=h_1(y).\end{aligned} \tag{33}$$

Solution. Firstly, compute double ARA-S transformation to the conditions in Eq. (33) and the source function $g(x,y)$, we get

$$\begin{cases}F_0(s)=F_1(s)=\frac{s}{s-1},\\ H_0(u)=H_1(u)=\frac{1}{1-u},\\ G(s,u)=\frac{s}{(s-1)(1-u)}+\frac{su}{(s-1)^2(1-u)^2}.\end{cases} \tag{34}$$

putting the values of Eq. (34) into Eq.(31) and simplifying, one can obtain the solution of Eq. (32) as follows

$$\varphi(x,y)=\mathcal{G}_x^{-1}S_y^{-1}\left[\frac{s}{(s-1)(1-u)}\right]=e^{x+y}. \tag{35}$$

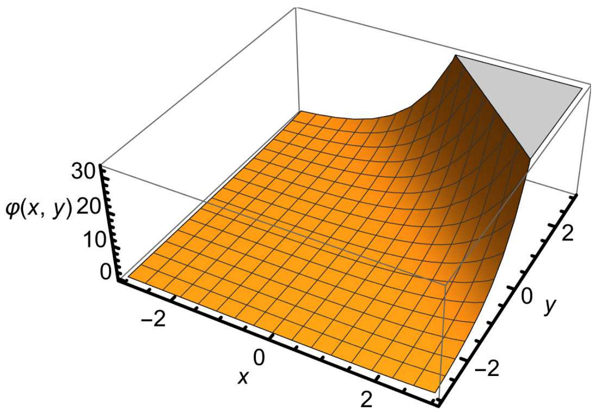

Fig. 5: The 3D plots of the solution of Example 4

4 Conclusion

In this research, a new method for solving different types of integral equations was developed. We apply the double ARA-S transformation transformation to resolve Volterra's partial integro-differential equations. To show the validity of the method, several examples were introduced and discussed. For possible future work, we are planing to solve nonlinear problems, and fractional differential equations. through the proposed transformation combination using one of the iteration methods. New results of DA-FT will be discussed in the future and implemented for solving fractional PDEs and nonlinear PDEs [27- 32].

Appendix 1. The values of ARA and Sumudu transformations of several functions.

Table 2: double ARA-S transformation to basic functions [25].

$\varphi(x)$	$\mathscr{G}[\varphi(x)] = \Phi(s)$	$S[\varphi(x)] = \Phi(u)$
1	1	1
x^{α}	$s^{-a}\Gamma(a+1)$	$u^{a}\Gamma(a+1)$
$e^{\alpha x}$	$\frac{s}{s-a}$	$\frac{1}{1-au}$
xe^{x}	$\frac{s}{(s-a)^2}$	$\frac{1}{(1-au)^2}$
$\sin(ax)$	$\frac{as}{a^2+s^2}$	$\frac{au}{1+a^2u^2}$
$\cos(ax)$	$\frac{s^2}{a^2+s^2}$	$\frac{1}{1+a^2u^2}$
$\sinh(ax)$	$\frac{as}{s^2-a^2}$	$\frac{au}{1-a^2u^2}$
$\cosh(ax)$	$\frac{s^2}{s^2-a^2}$	$\frac{1}{1-a^2u^2}$,
$(\varphi * \psi)(x)$	$\frac{\mathscr{G}_1[\varphi(x)]\mathscr{G}_1[\psi(x)]}{s}$	$u\,S(\varphi(x))S(\psi(x))$

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