Applied Mathematics and Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170417

The Double ARA-Formable Transform with Applications

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Received: 22 Mar. 2023, Revised: 2 May 2023, Accepted: 2 Jun. 2023
Published online: 1 Jul. 2023

Abstract: The main goal of this work is to introduce a new double transform named the double ARA-Formable transform (DARA-FT) we introduce the application of the new double transform to various fundamental functions and master properties. Additionally, we state and prove the convolution and existence theorems. Subsequently, by utilizing these findings, different kinds of partial differential equations (PDEs) are solved. We establish novel methods for resolving families of PDEs. With the latter, one can obtain precise answers to some well-known PDEs, including the telegraph equation, the advection-diffusion equation, the Klein-Gordon equation, and others. The results demonstrate the double ARA-Formable transform is effective and beneficial for handling these types of problems

Keywords: Single ARA transform; Single Formable transform; Partial differential equations; Integral equations.

1 Introduction

One of the most significant techniques used to solve PDEs is the use of integral transformations. As many scientific processes may be formally described in terms of PDEs [1,2,3,4,5], using integral transforms enables us to modify these equations and find the precise solution of PDEs[6,7,8,9,10]. Researchers and scientists have made great efforts to develop these techniques, and they use them to address contemporary scientific issues[11,12]. For instance, there are several transforms available, including the Laplace transform [13], Fourier transform [14], Sumudu transform [15], natural transform [16], Elzaki transform [17], Novel transform [18], M-transform [19], polynomial transform [20], Aboodh transform [21], and ARA transform [22,23,24]. For solving PDEs with unknown functions of two variables recently, the double Laplace transform has been employed successfully and often and has produced good results in comparison to numerical approaches [25,26,27,28,29]. Other double Laplace transform extensions include the double Shehu transform [30], the double Sumudu transform [31,32,33, 34,35,36], the double Elzaki transform [37], and the Laplace-Sumudu transform [38,39,40].
The main purpose of this article is to introduce a new double integral transform, double ARA-Formable transform. We describe some characteristics of the double ARA-Formable transform and show how to calculate it for various basic functions and operations.Some properties are discussed along with proofs such as the convolution theorem and the partial derivatives. In this article, we consider an inhomogeneous linear PDE of the form

$$A\frac{\partial^2\varnothing(z,t)}{\partial z^2}+B\frac{\partial^2\varnothing(z,t)}{\partial t^2}+C\frac{\partial\varnothing(z,t)}{\partial z}+D\frac{\partial\varnothing(z,t)}{\partial t}+E\varnothing(z,t)=u(z,t), \quad (1)$$

with the initial conditions (ICs)

$$\varnothing(z,0)=f_1(z),\quad \frac{\partial\varnothing(z,0)}{\partial t}=f_2(z),$$

and the boundary conditions (BCs)

$$\varnothing(0,t)=h_1(t),\quad \frac{\partial\varnothing(0,t)}{\partial z}=h_2(t),$$

where $\varnothing(z,t)$ is the unknown function, A,B,C,D and E are constants and $u(z,t)$ is the source term.
A simple formula for the solution of the above equation is established and employed to solve some applications.
This article presents the single ARA integral transform and some applications in Section 2, and we present the single Formable integral transform and some applications in Section 3. In Section 4 and 5 we find general formulas to solve a family of partial differential equations and integral equations. Finally, we utilize them to solve some equations such as heat equation and Laplace equation and others.

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2 Single ARA integral transform

In this section, we highlight to the definition and properties of a type of integral transform called ARA integral transform.

Definition 1. *The ARA integral transform (AIT) of order n of a continuous function $\varnothing(z)$ defined on the interval $[0,\infty)$ is given by*

$$\mathscr{G}_{n,z}[\varnothing(z)] = \Phi(n,v) = v\int_0^\infty z^{n-1}e^{-vz}\varnothing(z)dz, \quad v>0,\ n=1,2,\cdots. \tag{2}$$

Definition 2. *The single AIT of a function $\varnothing(z,t)$ of two variables with respect to z is given by*

$$\mathscr{G}_{n,z}[\varnothing(z,t)] = \Phi(v,t) = v\int_0^\infty z^{n-1}e^{-vz}\varnothing(z,t)dz, \quad v>0, n=1,2,\cdots. \tag{3}$$

Now, we are going to state some basic properties of the AIT.

Property 1. Let $\varnothing(z)$ and $\psi(z)$ be two continuous functions on $[0,\infty)$ in which the AIT exists. Then,

$$\mathscr{G}_{n,z}[\alpha\varnothing(z)+\beta\psi(z)] = \alpha\mathscr{G}_{n,z}[\varnothing(z)]+\beta\mathscr{G}_{n,z}[\psi(z)], \tag{4}$$

where α and β are nonzero constants.

$$\mathscr{G}_z[\varnothing'(z)] = v\,\mathscr{G}_z[\varnothing(z)] - v\,\varnothing(0), \tag{5}$$

$$\mathscr{G}_z[z^\alpha] = \frac{\Gamma(\alpha+1)}{v^\alpha}, \tag{6}$$

$$\mathscr{G}_z[\varnothing(z+t)] = e^{vz}\,\mathscr{G}_z[\varnothing(z)],\ v>0. \tag{7}$$

3 Single Formable integral transform

In this section, we highlight to the definition and properties of a type of integral transform called Formable integral transform.

Definition 3. *The Formable integral transform (FIT) of a function $\varnothing(t)$ of exponential order is defined as:*

$$R_t[\varnothing(t)] = \phi(s,u) = \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}\varnothing(t)dt, \quad u>0,\ s>0. \tag{8}$$

Definition 4. *The single FIT of a function $\varnothing(z,t)$ of two variables z and t with respect to t is given by*

$$R_t[\varnothing(z,t)] = \phi(z,s,u) = \frac{s}{u}\int_0^\infty e^{\frac{-st}{u}}\varnothing(z,t)dt, \quad u>0,\ s>0. \tag{9}$$

Property 2. Let $\varnothing(t)$ and $\psi(t)$ be two continuous functions on $[0,\infty)$ in which the FIT exists. Then

$$R_t[\alpha\varnothing(t)+\beta\psi(t)] = \alpha R_t[\varnothing(t)]+\beta R_t[\psi(t)], \tag{10}$$

where α and β are nonzero constants.

$$R_t\left[\varnothing'(t)\right] = \frac{s}{u}\phi(s,u) - \frac{s}{u}\varnothing(0), \tag{11}$$

$$R_t[t^n] = \frac{u^n\,n!}{s^n}. \tag{12}$$

4 Double ARA-Formable Transform

In this part of the study we build the double ARA-Formable definition and some fundamental characteristics of it.

Definition 5. *The double ARA-Formable Transform (DA-FT) of a continuous function $\varnothing(z,t)$ of two variables $z>0$ and $t>0$, is given by*

$$\begin{aligned}\mathscr{G}_{n,z}R_t[\varnothing(z,t)] &= \Phi_n(v,s,u)\\ &= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt,\\ &v>0, u>0, s>0.\end{aligned} \tag{13}$$

Whenever the integral exists for $n=1,2,3,\ldots$.
The inverse DA-FT of a function $\Phi_n(v,s,u)$ is given by

$$\mathscr{G}_{n,z}^{-1}R_t^{-1}[\Phi_n(v,s,u)] = \varnothing(z,t). \tag{14}$$

The DA-FT is a linear operator, because

$$\begin{aligned}&\mathscr{G}_{n,z}R_t[\alpha\,\varnothing(z,t)+\beta\,\psi(z,t)]\\ &= \alpha\mathscr{G}_{n,z}R_t[\varnothing(z,t)]+\beta\,\mathscr{G}_{n,z}R_t[\psi(z,t)],\end{aligned} \tag{15}$$

where α and β are real numbers.

4.1 DA-FT to some fundamental functions.

i. Let $\varnothing(z,t)=1,\ z>0,\ t>0$. Then

$$\begin{aligned}\mathscr{G}_{n,z}R_t[1] &= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}dzdt\\ &= v\Gamma(n)\left(\frac{1}{v}\right)^n\int_0^\infty\frac{z^{n-1}e^{-vz}}{v\Gamma(n)\left(\frac{1}{v}\right)^n}dz\,\frac{s}{u}\int_0^\infty e^{-\frac{st}{u}}dt\\ &= \frac{\Gamma(n)}{v^{n-1}},\quad Re(v)>0.\end{aligned}$$

ii. Let $\varnothing(z,t)=z^\alpha t^\beta,\ z>0,\ t>0$. Then,

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[z^\alpha t^\beta\right] &= \frac{sv}{u}\int_0^\infty\int_0^\infty z^\alpha t^\beta z^{n-1}e^{-vz-\frac{st}{u}}dzdt\\ &= v\int_0^\infty z^{\alpha+n-1}e^{-vz}dz\,\frac{s}{u}\int_0^\infty y^\beta e^{-\frac{st}{u}}dt\\ &= \mathscr{G}_{n,z}[z^\alpha]\ R_t\left[t^\beta\right] = \frac{\Gamma(\alpha+n)}{v^{\alpha+n-1}}\ \frac{u^\beta\Gamma(\beta+n)}{s^\beta}\\ &= \frac{u^\beta\Gamma(\beta+n)\Gamma(\alpha+n)}{s^\beta\ v^{\alpha+n-1}},\qquad Re(v)>0,\end{aligned}$$

where $\alpha > -1$ and $\beta > -1$ are constants.

iii. Let $\varnothing(z,t) = e^{\alpha z+\beta t}$. Then

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[e^{\alpha z+\beta t}\right] \\
&= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{\alpha z+\beta t}e^{-vz-\frac{st}{u}}dzdt \\
&= v\int_0^\infty z^{n-1}e^{\alpha z}e^{-vz}dz\frac{s}{u}\int_0^\infty e^{\beta t}e^{-\frac{st}{u}}dt \\
&= \frac{sv\Gamma(n)}{(v-\alpha)^n(s-u\beta)}, \quad v>\alpha\ ,\ \frac{s}{u}>\beta\ .
\end{aligned}$$

iv. Let $\varnothing(z,t) = \sin(\alpha z+\beta t)$. Then, using linearity, we get

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\frac{e^{i(\alpha z+\beta t)}-e^{-i(\alpha z+\beta t)}}{2i}\right] \\
&= \mathscr{G}_{n,z}\left[\frac{e^{i\alpha z}}{2i}\right]R_t\left[e^{i\beta t}\right]-\mathscr{G}_{n,z}\left[\frac{e^{-i\alpha z}}{2i}\right]R_t\left[e^{-i\beta t}\right],
\end{aligned}$$

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\sin(\alpha z+\beta t)\right] \\
&= \frac{v\Gamma(n)}{2i(v-\alpha i)^n}\frac{s}{(s-i\beta u)}-\frac{v\Gamma(n)}{2i(v+\alpha i)^n}\frac{s}{(s+i\beta u)} \\
&= \frac{sv\Gamma(n)}{2i(v-\alpha i)^n(s-iu\beta)}-\frac{sv\Gamma(n)}{2i(v+\alpha i)^n(s+i\beta u)}.
\end{aligned}$$

Using the facts

$$\begin{aligned}
\sin z &= \frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz}+e^{-iz}}{2}, \\
\sinh z &= \frac{e^{z}-e^{-z}}{2}, \quad \cosh z = \frac{e^{z}+e^{-z}}{2}.
\end{aligned}$$

And using linearity we can get DA-FT of the functions,

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\cos(\alpha z+\beta t)\right] \\
&= \frac{sv\Gamma(n)}{2(v-\alpha i)^n(s-iu\beta)}+\frac{sv\Gamma(n)}{2(v+\alpha i)^n(s+i\beta u)},
\end{aligned}$$

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\cosh(\alpha z+\beta t)\right] \\
&= \frac{sv\Gamma(n)}{2(v-\alpha)^n(s-u\beta)}+\frac{sv\Gamma(n)}{2(v+\alpha)^n(s+\beta u)},
\end{aligned}$$

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\sinh(\alpha z+\beta t)\right] \\
&= \frac{sv\Gamma(n)}{2(v-\alpha)^n(s-u\beta)}-\frac{sv\Gamma(n)}{2(v+\alpha)^n(s+\beta u)}.
\end{aligned}$$

v. Let $\varnothing(z,t) = \varnothing(z)\psi(t)$. Then

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[\ \varnothing(z,t)\right] = \mathscr{G}_{n,z}R_t\left[\varnothing(z)\psi(t)\right] \\
&= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z)\psi(t)dzdt \\
&= \left(v\int_0^\infty z^{n-1}e^{-vz}\varnothing(z)dz\right)\left(\frac{s}{u}\int_0^\infty e^{-\frac{st}{u}}\psi(t)dt\right) \\
&= \mathscr{G}_{n,z}\left[\varnothing(z)\right]R_t\left[\psi(t)\right].
\end{aligned}$$

vi. Let $\varnothing(z,t) = J_0\left(\alpha\sqrt{zt}\right)$.Then

$$\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[J_0\left(\alpha\sqrt{zt}\ \right)\right] \\
&= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}J_0(\alpha\sqrt{zt})dzdt \\
&= \frac{s}{u}\int_0^\infty e^{-\frac{st}{u}}dt\quad v\int_0^\infty z^{n-1}e^{-vz}J_0(\alpha\sqrt{zt})dz \\
&= R_t\left[\mathscr{G}_{n,z}\left[J_0(\alpha\sqrt{zt})\right]\right] = R_t\left[e^{-\frac{\alpha^2 t}{4v}}\right] \\
&= \frac{4vu}{4vs+\alpha^2 u}.
\end{aligned}$$

Here J_0 denotes the modified order zero Bessel function.

4.2 Existence conditions of DA-FT.

Let $\varnothing(z,t)$ be a function that satisfies the following condition. If there exists a positive constant K such that $\forall z > Z$ and $t > T$, we have

$$\left|z^{n-1}\varnothing(z,t)\right| \le Ke^{\alpha z+\beta t},$$

$\forall\alpha > 0$ and $\beta > 0$ as $z\to\infty$ and $t\to\infty$. Hence we say

$$z^{n-1}\varnothing(z,t) = O\left(e^{\alpha z+\beta t}\right), \tag{16}$$

as

$$\begin{aligned}
&z\to\infty \ \text{ and } \ t\to\infty, \\
&v>\alpha \ \text{ and } \ \frac{s}{u}>\beta.
\end{aligned}$$

Theorem 1. *Let $\varnothing(z,t)$ be a continuous function in a region $(0,Z)$ and $(0,T)$ that satisfies the condition in equation (16). Then DA-FT of $\varnothing(z,t)$ exists for all v and $\frac{s}{u}$ provided $Re[v] > \alpha$ and $Re\left[\frac{s}{u}\right] > \beta$.*

Proof of Theorem 1. From the definition of DA-FT , we have

$$\begin{aligned}
|\Phi_n(v,s,u)| &= \left|\frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,y)dz\,dt\right| \\
&\le \frac{sv}{u}\int_0^\infty\int_0^\infty e^{-vz-\frac{st}{u}}\left|z^{n-1}\varnothing(z,y)\right|dz\,dt \\
&\le \frac{K\,sv}{u}\int_0^\infty e^{-(v-\alpha)z}dz\int_0^\infty e^{-\left(\frac{s}{u}-\beta\right)t}dt \\
&= \frac{Ksv}{(v-\alpha)(s-u\beta)},
\end{aligned}$$

where $Re[v] > \alpha$ *and* $Re\left[\frac{s}{u}\right] > \beta$.

4.3 Basic Properties of DA-FT

In this section, we highlight to the Properties of derivatives, Periodicity Function and Convolution Theorem of DA-FT.

i. Shifting property

$$\mathscr{G}_{n,z}R_t\left[e^{\alpha z+\beta t}\varnothing(z,t)\right] = \frac{sv}{(v-\alpha)(s-u\beta)}\Phi_n\left(v-\alpha,\frac{s-u\beta}{u}\right), \tag{17}$$

where $\mathscr{G}_{n,z}R_t\left[\varnothing(z,t)\right]=\Phi_n(v,s,u)$.
Proof. From the definition of DA-FT, we get

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[e^{\alpha z+\beta t}\varnothing(z,t)\right] &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-(v-\alpha)z-\left(\frac{s-u\beta}{u}\right)t}\varnothing(z,t)dzdt\\ &= \frac{sv}{(v-\alpha)(s-u\beta)}\left(\frac{(v-\alpha)(s-u\beta)}{u}\right)\\ &\quad \int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-(v-\alpha)z-\left(\frac{s-u\beta}{u}\right)t}\varnothing(z,t)dzdt\\ &= \frac{sv}{(v-\alpha)(s-u\beta)}\Phi_n\left(v-\alpha,\frac{s-u\beta}{u}\right).\end{aligned}$$

ii. Properties of derivatives
Let $\Phi_n(v,s,u)=\mathscr{G}_{n,z}R_t\left[\varnothing(z,t)\right]$. Then, we have the following properties:

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v\Phi_1(v,s,u)-vR_t\left(\varnothing(0,t)\right)\right), \tag{18}$$

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial t}\right] = \frac{s}{u}\Phi_n(v,s,u)-\frac{s}{u}\mathscr{G}_{n,z}(\varnothing(z,0)), \tag{19}$$

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v^2\Phi_1(v,s,u) - v^2R_t\left(\varnothing(0,y)\right)-vR_t\left(\frac{\partial\varnothing(0,t)}{\partial z}\right)\right), \tag{20}$$

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] = \frac{s^2}{u^2}\Phi_n(v,s,u)-\frac{s^2}{u^2}\mathscr{G}_{n,z}(\varnothing(z,0)) - \frac{s}{u}\mathscr{G}_{n,z}\left(\frac{\partial\varnothing(z,0)}{\partial t}\right), \tag{21}$$

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z\partial t}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(\frac{vs}{u}\Phi_n(v,s,u) - \frac{vs}{u}R_t(\varnothing(0,t))\right)-\frac{sv^2}{u}\mathscr{G}_{n,z}(\varnothing(z,0)) + \frac{sv^2}{u}(\varnothing(0,0)). \tag{22}$$

Here, we state the proof of equations (18), (20) and (22). We get the proof of (19) and (21) in a same manner to (18) and (20), respectively.
Proof of (18). The definition of DA-FT implies

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\frac{\partial\varnothing(z,t)}{\partial z}dzdt\\ &= \frac{s}{u}\int_0^{\infty}e^{-\frac{st}{u}}\left(v\int_0^{\infty}z^{n-1}e^{-vz}\frac{\partial\varnothing(z,t)}{\partial z}\,dz\right)dt.\end{aligned}$$

But the integral inside the bracket is equal to $\mathscr{G}_{n,z}\left[\frac{\partial\varnothing(z,t)}{\partial z}\right]$, and

$$\mathscr{G}_{n,z}\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] = v(-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(\frac{\mathscr{G}_{1,z}\left[\frac{\partial\varnothing(z,t)}{\partial z}\right]}{v}\right), \tag{23}$$

but

$$\mathscr{G}_{1,z}\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] = v\mathscr{G}_{1,z}(\varnothing(z,t))-v(\varnothing(z,t)),$$

so equation (23) becomes

$$\mathscr{G}_{n,z}\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left[v\mathscr{G}_{1,z}(\varnothing(z,t))-v\ (\varnothing(z,t))\right].$$

By taking FIT with respect to t for equation (23), we get DA-FT in the form of

$$\mathscr{G}_{n,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial z}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left[v\Phi_1(v,s,u)-vR_t\left(\varnothing(0,t)\right)\right].$$

Proof of (20). The definition of DA-FT implies

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\frac{\partial\varnothing(z,t)}{\partial z}dzdt\\ &= \frac{s}{u}\int_0^{\infty}e^{-\frac{st}{u}}\left(v\int_0^{\infty}z^{n-1}e^{-vz}\frac{\partial^2\varnothing(z,t)}{\partial z^2}\,dz\right)dt.\end{aligned}$$

But the integral inside the bracket is equal to $\mathscr{G}_{n,z}\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right]$, and

$$\mathscr{G}_{n,z}\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] = v(-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left[\frac{\mathscr{G}_{1,z}\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right]}{v}\right]. \tag{24}$$

But

$$\mathscr{G}_{1,z}\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] = v^2\mathscr{G}_{1,z}(\varnothing(z,t)) - v^2\ (\varnothing(0,t)) - v\left(\frac{\partial\varnothing(0,t)}{\partial z}\right),$$

so equation (24) becomes

$$\mathscr{G}_{n,z}\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] = (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v^2\mathscr{G}_{1,z}(\varnothing(z,t)) - v^2\ (\varnothing(0,t)) - v\left(\frac{\partial\varnothing(0,t)}{\partial z}\right)\right).$$

By taking FIT with respect to t for equation (24), we get DA-FT in the form of

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right] &= (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v^2\mathscr{G}_{1,z}R_t(\varnothing(z,y)) - v^2R_t(\varnothing(0,y)) - v\,R_t\left(\frac{\partial\varnothing(0,t)}{\partial z}\right)\right)\\ &= (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v^2\Phi_1(v,s,u) - v^2\,R_t(\varnothing(0,t)) - v\,R_t\left(\frac{\partial\varnothing(0,t)}{\partial z}\right)\right).\end{aligned}$$

Proof of (22). The definition of DA-FT implies

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z\partial t}\right] &= \frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\frac{\partial\varnothing(z,t)}{\partial z}dzdt\\ &= v\int_0^\infty z^{n-1}e^{-vz}dz\ \left(\frac{s}{u}\int_0^\infty e^{-\frac{st}{u}}\frac{\partial^2\varnothing(z,t)}{\partial z\partial t}dt\right).\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z\partial t}\right] &= \frac{vs}{u}\int_0^\infty z^{n-1}e^{-vz}dz\left(-\frac{\partial\varnothing(z,0)}{\partial z} + \frac{s^2}{u^2}\int_0^\infty e^{-\frac{st}{u}}\frac{\partial\varnothing(z,t)}{\partial z}dt\right)\\ &= -\frac{vs}{u}\int_0^\infty z^{n-1}e^{-sz}\varnothing_z(z,0)dz + \frac{s^2v}{u^2}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\frac{\partial\varnothing(z,t)}{\partial z}dzdt\\ &= -\frac{vs}{u}\mathscr{G}_{n,z}\left(\frac{\partial\varnothing(z,0)}{\partial z}\right) + \frac{s}{u}\mathscr{G}_{n,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial z}\right].\end{aligned}$$

Using the fact in equation (4) and equation (17), we get

$$\begin{aligned}\mathscr{G}_{n,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z\partial t}\right] &= -\frac{vs}{u}\left(v\mathscr{G}_{n,z}[\varnothing(z,0)] - v(\varnothing(0,0))\right)\\ &\quad + (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(v\Phi_n(v,s,u) - vR_t(\varnothing(0,t))\right)\\ &= (-1)^{n-1}\frac{d^{n-1}}{dv^{n-1}}\left(\frac{vs}{u}\Phi_n(v,s,u) - \frac{vs}{u}R_t(\varnothing(0,t))\right)\\ &\quad - \frac{sv^2}{u}\mathscr{G}_{n,z}(\varnothing(z,0)) + \frac{sv^2}{u}(\varnothing(0,0)).\end{aligned}$$

Remark. Let $\Phi(v,s,u) = \mathscr{G}_{n,z}R_t[\varnothing(z,t)]$. Then DA-FT of the n-th partial derivatives with respect to z and t respectively, are

$$\mathscr{G}_{1,z}R_t\left[\frac{\partial^n\varnothing(z,t)}{\partial z^n}\right] = v^n\Phi(v,s,u) - \sum_{k=0}^{n-1}v^{n-k-1}R_y\left[\frac{\partial^k}{\partial z^k}\varnothing(0,t)\right]. \tag{25}$$

$$\mathscr{G}_{1,z}R_t\left[\frac{\partial^n\varnothing(z,t)}{\partial t^n}\right] = \left(\frac{s}{u}\right)^n\Phi(v,s,u) - \frac{s}{u}\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1}\mathscr{G}_{n,z}\left[\frac{\partial^k}{\partial t^k}\varnothing(z,0)\right]. \tag{26}$$

4.4 DA-FT of a Periodic Function

Theorem 2. *If $\varnothing(z,t)$ is a periodic function of periods α and β, such that $\varnothing(z+\alpha,t+\beta) = \varnothing(z,t)$ for all z and t.*

If DA-FT of $\varnothing(z,t)$ exists, then

$$\mathcal{G}_{n,z}R_t[\varnothing(z,t)] = \left(1-e^{-v\alpha-\frac{s\beta}{u}}\right)^{-1} \left(\frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt\right). \tag{27}$$

Proof of Theorem 2. From the definition of DA-FT, we have

$$\begin{aligned}\Phi_n(v,s,u) &= \mathcal{G}_{n,z}R_t[\varnothing(z,t)] = \\ &\frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt \\ &= \frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt + \\ &\frac{sv}{u}\int_{\alpha}^{\infty}\int_{\beta}^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt.\end{aligned}$$

Let $z=\alpha+p$ and $t=\beta+q$ on the second double integral, we get

$$\begin{aligned}\Phi_n(v,s,u) &= \frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt \\ &+\frac{sv}{u}\int_0^{\infty}\int_0^{\infty}(\alpha+p)^{n-1}e^{-v(\alpha+p)-\frac{s(q+\beta)}{u}} \\ &\varnothing(p+\alpha,q+\beta)dpdq.\end{aligned} \tag{28}$$

Using the periodicity of the function $\varnothing(z,t)$, equation (28) becomes

$$\begin{aligned}\Phi_n(v,s,u) &= \frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt \\ &+e^{-v\alpha-\frac{s\beta}{u}}\frac{sv}{u}\int_0^{\infty}\int_0^{\infty} p^{n-1}e^{-vp-\frac{sq}{u}}\varnothing(p,q)dpdq.\end{aligned}$$

From the definition of DA-FT, we get

$$\begin{aligned}\Phi_n(v,s,u) &= \frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt \\ &+e^{-v\alpha-\frac{s\beta}{u}}\Phi_n(v,s,u).\end{aligned} \tag{29}$$

Equation (29) can be simplified into

$$\Phi_n(v,s,u) = \left(1-e^{-v\alpha-\frac{s\beta}{u}}\right)^{-1} \left(\frac{sv}{u}\int_0^{\alpha}\int_0^{\beta} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z,t)dzdt\right).$$

4.5 Convolution theorem of DA-FT

Theorem 3.*Let*

$$\Phi_n(v,s,u) = \mathcal{G}_{n,z}R_t[\varnothing(z,t)].$$

Then

$$\mathcal{G}_{n,z}R_t\left[\varnothing(z-\delta,t-\varepsilon)\psi(z-\delta,t-\varepsilon)\right] = e^{-v\delta-\frac{s\varepsilon}{u}}\Phi_n(v,s,u), \tag{30}$$

where $\psi(z,t)$ is the Heaviside unit step function given by

$$\psi(z-\delta,t-\varepsilon) = \begin{cases}1, & z>\delta,\ t>\varepsilon \\ 0, & otherwise.\end{cases}$$

Proof of Theorem 3. The definition of DA-FT implies

$$\begin{aligned}&\mathcal{G}_{n,z}R_t[\varnothing(z-\delta,t-\varepsilon)\psi(z-\delta,t-\varepsilon)] \\ &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\big(\varnothing(z-\delta,t-\varepsilon) \\ &\qquad \psi(z-\delta,t-\varepsilon)\big)dzdt \\ &= \frac{sv}{u}\int_{\delta}^{\infty}\int_{\varepsilon}^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}\varnothing(z-\delta,t-\varepsilon)dzdt.\end{aligned} \tag{31}$$

Letting $z-\delta=p$ and $t-\varepsilon=q$ in equation (31), we get

$$\begin{aligned}&\mathcal{G}_{n,z}R_t[\varnothing(z-\delta,t-\varepsilon)\psi(z-\delta,t-\varepsilon)] \\ &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty}(\delta+p)^{n-1}e^{-v(\delta+p)-\frac{s(\varepsilon+q)}{u}}\varnothing(p,q)dpdq.\end{aligned} \tag{32}$$

Equation (32) can be simplified into

$$\begin{aligned}&\mathcal{G}_{n,z}R_t[\varnothing(z-\delta,t-\varepsilon)\psi(z-\delta,t-\varepsilon)] \\ &= e^{-v\delta-\frac{s\varepsilon}{u}}\left(\frac{sv}{u}\int_0^{\infty}\int_0^{\infty}p^{n-1}e^{-vp-\frac{sq}{u}}\varnothing(p,q)dpdq\right) \\ &= e^{-v\delta-\frac{s\varepsilon}{u}}\Phi_n(v,s,u).\end{aligned}$$

Theorem 4.(Convolution Theorem). *If $\mathcal{G}_{n,z}R_t[\varnothing(z,t)] = \Phi_n(v,s,u)$ and $\mathcal{G}_{n,z}R_t[\psi(z,t)] = \Psi_n(v,s,u)$, then*

$$\mathcal{G}_{n,z}R_t[(\varnothing**\psi)(z,t)] = \left(\frac{u}{sv}\right)\Phi_n(s,u,v)\Psi_n(v,s,u), \tag{33}$$

where

$$(\varnothing**\psi)(z,t) = \int_0^z\int_0^t\varnothing(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon.$$

Proof of Theorem 4. The definition of DA-FT implies

$$\begin{aligned}&\mathcal{G}_{n,z}R_t[(\varnothing**\psi)(z,t)] \\ &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}}(\varnothing**\psi)(z,t)dzdt \\ &= \frac{sv}{u}\int_0^{\infty}\int_0^{\infty} z^{n-1}e^{-vz-\frac{st}{u}} \\ &\left(\int_0^z\int_0^t\varnothing(z-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon\right)dzdt.\end{aligned} \tag{34}$$

From the definition of Heaviside function, equation (34) becomes

$$
\begin{aligned}
&\mathscr{G}_{n,z}R_t\left[(\varnothing**\psi)(z,t)\right]\\
&=\frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\left(\int_0^\infty\int_0^\infty \varnothing(z-\delta,t-\varepsilon)\right.\\
&\qquad\left.\Psi(z-\delta,t-\varepsilon)k(\delta,\varepsilon)d\delta d\varepsilon\right)dzdt\\
&=\int_0^\infty\int_0^\infty \psi(\delta,\varepsilon)d\delta d\varepsilon\left(\frac{sv}{u}\int_0^\infty\int_0^\infty z^{n-1}e^{-vz-\frac{st}{u}}\right.\\
&\qquad\left.\varnothing(z-\delta,t-\varepsilon)\Psi(z-\delta,t-\varepsilon)dzdt\right)\\
&=\int_0^\infty\int_0^\infty k(\delta,\varepsilon)d\delta d\varepsilon\left(e^{-v\delta-\frac{s\varepsilon}{u}}\Phi_n(v,s,u)\right)\\
&=\Phi_n(v,s,u)\int_0^\infty\int_0^\infty \psi(\delta,\varepsilon)\ e^{-vz-\frac{st}{u}}\ d\delta\ d\varepsilon\\
&=\left(\frac{u}{sv}\right)\Phi_n(v,s,u)\Psi_n(v,s,u).
\end{aligned}
$$

Now, we summarize all the above results in Table 1.

Table 1: DA-FT to some functions.

$\varnothing(z,t)$	$\mathscr{G}_{n,z}R_t\left[\varnothing(z,t)\right]=\Phi_n(v,s,u).$
1	$\frac{\Gamma(n)}{v^{n-1}}$
$\alpha\in\mathbb{R}$	$\frac{\alpha\Gamma(n)}{v^{n-1}}$
$z^{\alpha}t^{\beta}$	$\frac{u^{\beta}\Gamma(\beta+n)\Gamma(\alpha+n)}{s^{\beta}\ v^{\alpha+n-1}}$
$e^{\alpha z+\beta t}$	$\frac{sv\Gamma(v)}{(v-\alpha)^n(s-u\beta)}$
$\sin(\alpha z+\beta t)$	$\frac{sv\Gamma(n)}{2i(v-\alpha i)^n(s-iu\beta)}-\frac{sv\Gamma(n)}{2i(v+\alpha i)^n(s+i\beta u)}$
$\cos(\alpha z+\beta t)$	$\frac{sv\Gamma(n)}{2(v-\alpha i)^n(s-iu\beta)}+\frac{sv\Gamma(n)}{2(v+\alpha i)^n(s+i\beta u)}$
$\sinh(\alpha z+\beta t)$	$\frac{sv\Gamma(n)}{2(v-\alpha)^n(s-u\beta)}-\frac{sv\Gamma(n)}{2(v+\alpha)^n(s+\beta u)}$
$\cosh(\alpha z+\beta t)$	$\frac{sv\Gamma(n)}{2(v-\alpha)^n(s-u\beta)}+\frac{sv\Gamma(n)}{2(v+\alpha)^n(s+\beta u)}$
$J_0\left(\alpha\sqrt{zt}\right)$	$\frac{4vu}{4vs+\alpha^2u}$
$\varnothing(z-\delta,t-\varepsilon)$	
$\psi(z-\delta,t-\varepsilon)$	$e^{-v\delta-\frac{s\varepsilon}{u}}\Phi_n(v,s,u)$
$(\varnothing**\psi)(z,t)$	$(\frac{u}{sv})\Phi_n(v,s,u)\Psi_n(v,s,u)$

5 Applications on DA-FT

In this section, we apply DA-FT on a family of PDEs and get a simple formula for the general solution. For simplicity, we solve the applications on the DA-FT considering $n=1$ in single AIT.

5.1 DA-FT for Solving PDEs

Consider the nonhomogeneous PDE of the form

$$
\begin{aligned}
A\frac{\partial^2\varnothing(z,t)}{\partial z^2}+B\frac{\partial^2\varnothing(z,t)}{\partial t^2}+C\frac{\partial\varnothing(z,t)}{\partial z}+D\frac{\partial\varnothing(z,t)}{\partial t}\\
+E\varnothing(z,t)=u(z,t),
\end{aligned}
\tag{35}
$$

with the initial conditions (ICs)

$$
\varnothing(z,0)=f_1(z),\quad \frac{\partial\varnothing(z,0)}{\partial t}=f_2(z),
$$

and the boundary conditions (BCs)

$$
\varnothing(0,t)=h_1(t),\quad \frac{\partial\varnothing(0,t)}{\partial z}=h_2(t).
$$

Apply DA-FT on (35), we get

$$
\begin{aligned}
&A\mathscr{G}_{1,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial z^2}\right]+B\,\mathscr{G}_{1,z}R_t\left[\frac{\partial^2\varnothing(z,t)}{\partial t^2}\right]\\
&\quad+C\mathscr{G}_{1,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial z}\right]+D\mathscr{G}_{1,z}R_t\left[\frac{\partial\varnothing(z,t)}{\partial t}\right]\\
&\quad+E\mathscr{G}_{1,z}R_t\left[\varnothing(z,t)\right]=\mathscr{G}_{1,z}R_t\left[u(z,t)\right].
\end{aligned}
$$

The single AIT is applied on the initial conditions, to get

$$
\mathscr{G}_{1,z}\left[f_1(z)\right]=F_1(v),\quad \mathscr{G}_{1,z}\left[f_2(z)\right]=F_2(v),
\tag{36}
$$

and the single FIT is applied on the boundary conditions, to get

$$
R_t\left[h_1(t)\right]=H_1(s,u),\quad R_t\left[h_2(t)\right]=H_2(s,u).
\tag{37}
$$

From the properties of the derivatives in equations (18-22), and the transformed ICs (36) and BCs (37) yield that

$$
\begin{aligned}
&A\left(v^2\Phi(v,s,u)-v^2R_t[\varnothing(0,t)]-vR_t\left[\frac{\partial\varnothing(0,t)}{\partial z}\right]\right)\\
&\quad+B\left(\frac{s^2}{u^2}\Phi(v,s,u)-\frac{s^2}{u^2}\mathscr{G}_{1,z}[\varnothing(z,0)]\right.\\
&\qquad\left.-\frac{s}{u}\mathscr{G}_{1,z}\left[\frac{\partial\varnothing(z,0)}{\partial t}\right]\right)\\
&\quad+C\ \left(v\Phi(v,s,u)-vR_t[\varnothing(0,t)]\right)\\
&\quad+D\left(\frac{s}{u}\Phi(v,s,u)-\frac{s}{u}\mathscr{G}_{1,z}[\varnothing(z,0)]\right)\\
&\quad+E\Phi(v,s,u)=U(v,s,u)
\end{aligned}
\tag{38}
$$

Substitution (36), (37) in (38) we get.

$$
\begin{aligned}
&Av^2\Phi(v,s,u)Av^2H_1(s,u)-AvH_2(s,u)\\
&\quad+B\frac{s^2}{u^2}\Phi(v,s,u)-B\frac{s^2}{u^2}F_1(v)-B\frac{s}{u}F_2(v)\\
&\quad+CvH_1(s,u)+D\frac{s}{u}F_1(v)=U(v,s,u).
\end{aligned}
\tag{39}
$$

Following some quick calculations, equation (39) can be reduced to

$$\Phi(v,s,u)=\left[\frac{H_1(s,u)(Av^2+Cv)+H_2(s,u)(Av)}{Av^2+B\frac{s^2}{u^2}+Cv+D\frac{s}{u}+E}\right.$$
$$\left.+\frac{F_1(v)(B\frac{s^2}{u^2}+\frac{Ds}{u})+F_2(v)(\frac{Bs}{u})+U(v,s,u)}{Av^2+B\frac{s^2}{u^2}+Cv+D\frac{s}{u}+E}\right]. \quad (40)$$

To find the solution of equation (35) in the original space, apply the inverse DA-FT to both sides of equation (40), to get

$$\varnothing(z,t)=\mathscr{G}_{1,z}^{-1}R_t^{-1}\left[\frac{H_1(s,u)(Av^2+Cv)+H_2(s,u)(Av)}{Av^2+B\frac{s^2}{u^2}+Cv+D\frac{s}{u}+E}\right.$$
$$\left.+\frac{F_1(v)(\frac{Bs^2}{u^2}+\frac{Ds}{u})+F_2(v)(\frac{Bs}{u})+U(v,s,u)}{Av^2+B\frac{s^2}{u^2}+Cv+D\frac{s}{u}+E}\right]. \quad (41)$$

5.2 llustrative Examples

This section presents and resolves some problems using DA-FT. By resolving these problems and obtaining the exact results, the effectiveness and simplicity of the suggested approach are shown.

Problem 1. Take the wave equation

$$\frac{\partial^2\varnothing(z,t)}{\partial t^2}=\frac{\partial^2\varnothing(z,t)}{\partial z^2},\quad z\geq 0,\quad t\geq 0, \quad (42)$$

with the ICs

$$\varnothing(z,0)=\sin z,\quad \frac{\partial\varnothing(z,0)}{\partial t}=2, \quad (43)$$

and the BCs

$$\varnothing(0,t)=2t,\quad \frac{\partial\varnothing(0,t)}{\partial z}=\cos t. \quad (44)$$

Solution. Firstly, we have

$$f_1(z)=\sin z,\qquad f_2(z)=2,$$
$$h_1(t)=2t,\qquad h_2(t)=\cos t.$$

Applying the single AIT to $f_1(z)$ and $f_2(z)$, to get

$$F_1(v)=\frac{v}{v^2+1},\quad F_2(v)=2.$$

Applying the single FIT to $h_1(t)$ and $h_2(t)$, to get

$$H_1(s,u)=\frac{2u}{s},\quad H_2(s,u)=\frac{s^2}{u^2+s^2}.$$

Rewrite equation (42) in the form

$$\frac{\partial^2\varnothing(z,t)}{\partial z^2}-\frac{\partial^2\varnothing(z,t)}{\partial t^2}=0.$$

By comparing with the general formula in equation (35), we have

$$A=1,\ B=-1,\ C=D=E=0\text{and}U(v,s,u)=0.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 and above in the general formula in equation (40), we get

$$\Phi(v,s,u)=\frac{\frac{2uv^2}{s}+\frac{vs^2}{s^2+u^2}-\frac{vs^2}{u^2(v^2+1)}-\frac{2s}{u}}{v^2-\frac{s^2}{u^2}}$$
$$=\frac{2u}{s}+\frac{vs^2}{(v^2+1)(u^2+s^2)}. \quad (45)$$

Now, take DA-FT inverse to both sides of equation (45), we get

$$\varnothing(z,t)=\mathscr{G}_{1,z}^{-1}R_t^{-1}\left[\frac{2u}{s}+\frac{vs^2}{(v^2+1)(u^2+s^2)}\right]$$
$$=2t+\sin z\cos t.$$

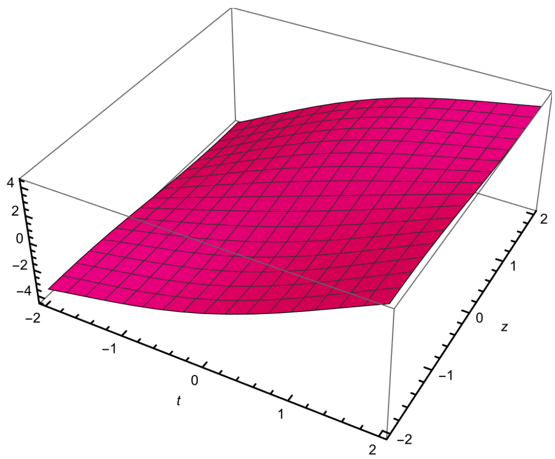

Fig. 1: 3D plot 1. The solution $\varnothing(z,t)$ of Problem 1.

Problem 2. Take the heat equation

$$\frac{\partial\varnothing(z,t)}{\partial t}=\frac{\partial^2\varnothing(z,t)}{\partial z^2}-3\varnothing(z,t)+3,\quad z\geq 0,\quad t\geq 0, \quad (46)$$

with the IC

$$\varnothing(z,0)=1+\sin z, \quad (47)$$

and the BCs

$$\varnothing(0,t)=1,\quad \frac{\partial\varnothing(0,t)}{\partial z}=e^{-4t}. \quad (48)$$

Solution. Firstly, we have

$$f_1(z)=1+\sin z,\ \ f_2(z)=0,\ \ h_1(t)=1,\ \ h_2(t)=e^{-4t}.$$

Applying the single AIT to $f_1(z)$ and $f_2(z)$, to get

$$F_1(v) = 1 + \frac{v}{v^2+1}, \quad F_2(s) = 0.$$

Applying the single FIT to $h_1(t)$ and $h_2(t)$, to get

$$H_1(s,u) = 1, \;\; H_2(s,u) = \frac{s}{s+4u}.$$

Rewrite equation (46) in the form

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} - \frac{\partial \varnothing(z,t)}{\partial t} - 3\varnothing(z,t) = -3.$$

By comparing with the general formula in equation (35), we have

$$A = 1, \; D = -1, \; E = -3, \; C = B = E = 0$$

and

$$U(v,s,u) = -3$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 and above in the general formula in equation (40), we get

$$\begin{aligned}\Phi(v,s,u) &= \frac{v^2 + \frac{sv}{s+4u} + \left(1+\frac{v}{v^2+1}\right)\left(-\frac{s}{u}\right) - 3}{v^2 - \frac{s}{u} - 3} \\ &= 1 + \frac{vs}{(v^2+1)(s+4u)}.\end{aligned} \tag{49}$$

Now, take DA-FT inverse to both sides of equation (49), we get

$$\varnothing(z,t) = \mathcal{G}^{-1}_{1,z} R^{-1}_t \left[1 + \frac{vs}{(v^2+1)(s+4u)}\right] = 1 + e^{-4t}\sin z.$$

3D plot Fog. 2, shows the answer of the heat equation (46) with the IC (47) and the BCs (48).

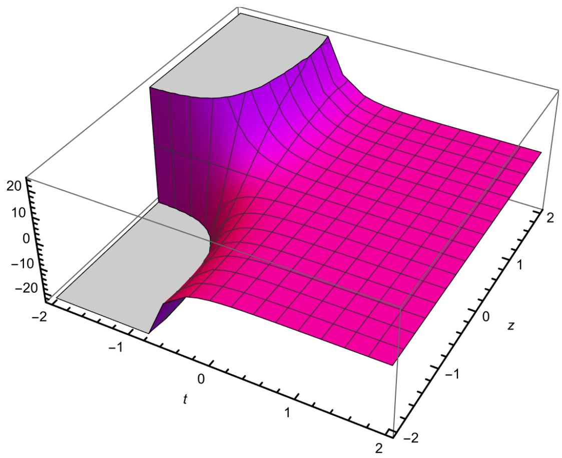

Fig. 2: 3D plot 2. The solution $\varnothing(z,t)$ of Problem 2.

Problem 3. Take the telegraph equation

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} = \frac{\partial^2 \varnothing(z,t)}{\partial t^2} + \frac{\partial \varnothing(z,t)}{\partial t} - \varnothing(z,t), \quad z \geq 0, \quad t \geq 0, \tag{50}$$

with the ICs

$$\varnothing(z,0) = e^z, \quad \frac{\partial \varnothing(z,0)}{\partial t} = -2e^z, \tag{51}$$

and the BCs

$$\varnothing(0,t) = e^{-2t}, \quad \frac{\partial \varnothing(0,t)}{\partial z} = e^{-2t}. \tag{52}$$

Solution. Firstly, we have

$$f_1(z) = e^z, \;\; f_2(z) = -2e^z, \;\; h_1(t) = e^{-2t}, \;\; h_2(t) = e^{-2t}$$

Applying the single ARA-T on $f_1(z)$ and $f_2(z)$, we get

$$F_1(v) = \frac{v}{v-1}, \;\; F_2(s) = \frac{-2v}{v-1}.$$

Applying the single FT on $h_1(t)$ and $h_2(t)$, we get

$$H_1(s,u) = \frac{s}{s+2u}, \;\; H_2(s,u) = \frac{s}{s+2u}.$$

Rewrite equation (50) in the form

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} - \frac{\partial^2 \varnothing(z,t)}{\partial t^2} - \frac{\partial \varnothing(z,t)}{\partial t} + \varnothing(z,t) = 0,$$
$$A = E = 1, \; B = -1, \; D = -1, \; c = 0 \;\text{ and } U(v,s,u) = 0.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 and above in the general formula in equation (40), we get

$$\begin{aligned} & Av^2 + B\frac{s^2}{u^2} + Cv + D\frac{s}{u} + E\Phi(v,s,u) \\ &= \frac{\frac{sv^2}{s+2u} + \frac{sv}{s+2u} + \left(\frac{v}{v-1}\right)\left(\frac{-s^2}{u^2} - \frac{Ds}{u}\right) + \frac{2vs}{u(v-1)}}{v^2 - \frac{s^2}{u^2} - \frac{s}{u} + 1} \\ &= \frac{vs}{(v-1)(s+2u)}. \end{aligned} \tag{53}$$

Now, take DA-FT inverse to both sides of equation (53), we get

$$\varnothing(z,t) = \mathcal{G}^{-1}_{1,z} R^{-1}_t \left[\frac{vs}{(v-1)(s+2u)}\right] = e^{z-2t}.$$

3D plot Fig. 3, shows the answer of the telegraph equation (50) with the ICs (51) and the BCs (52).

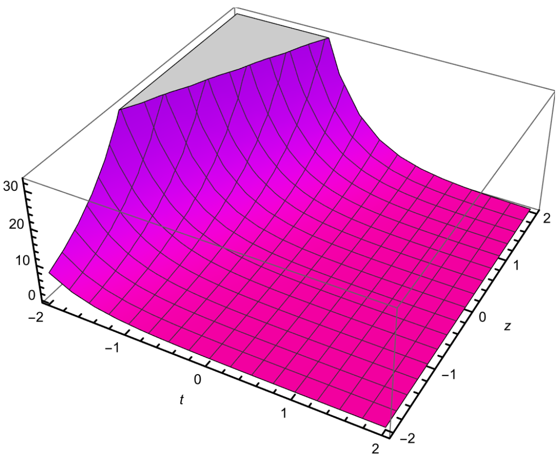

Fig. 3: 3D plot 3. The solution $\varnothing(z,t)$ of Problem 3.

Problem 4. Take the Klein-Gordon equation

$$\frac{\partial^2 \varnothing(z,t)}{\partial t^2} - \varnothing(z,t) = \frac{\partial^2 \varnothing(z,t)}{\partial z^2} - \cos(z)\cos(t), \quad (54)$$

with the ICs

$$\varnothing(z,0) = \cos(z), \quad \frac{\partial \varnothing(z,0)}{\partial t} = 0, \quad (55)$$

and the BCs

$$\varnothing(0,t) = \cos(t), \quad \frac{\partial \varnothing(0,t)}{\partial z} = 0. \quad (56)$$

Solution. Firstly, we have

$$f_1(z) = e^z, \;\; f_2(z) = 0, \;\; h_1(t) = e^{-2t}, \;\; h_2(t) = 0.$$

Applying the single ARA-T on $f_1(z)$ and $f_2(z)$, we get

$$F_1(v) = \frac{v^2}{v^2+1}, \;\; F_2(v) = 0.$$

Applying the single FT on $h_1(t)$ and $h_2(t)$, we get

$$H_1(s,u) = \frac{s^2}{s^2+u^2}\;, \;\; H_2(s,u) = 0.$$

Rewrite equation (54) in the form

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} - \frac{\partial^2 \varnothing(z,t)}{\partial t^2} + \varnothing(z,t) = \cos(z)\cos(t),$$

$$A = E = 1, \; B = -1, \; D = C = 0,$$

and

$$U(v,s,u) = \frac{v^2 s^2}{(v^2+1)(s^2+1)}.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 and above in the general formula in equation (40), we get

$$\begin{aligned}\Phi(v,s,u) &= \frac{\frac{s^2 v^2}{s^2+u^2} + \left(\frac{v^2}{v^2+1}\right)\left(\frac{-s^2}{u^2}\right)}{v^2 - \frac{s^2}{u^2} + 1} \\ &\quad - \frac{\frac{v^2 s^2}{(v^2+1)(s^2+1)}}{v^2 - \frac{s^2}{u^2} + 1} \\ &= \frac{v^2 s^2}{(v^2+1)(s^2+1)}.\end{aligned} \quad (57)$$

Now, take DA-FT inverse to both sides of equation (57), we get

$$\varnothing(z,t) = \mathscr{G}_{1,z}^{-1} R_t^{-1}\left[\frac{v^2 s^2}{(v^2+1)(s^2+1)}\right] = \cos(z)\cos(t).$$

3D plot Fig. 4, shows the answer of the Klein-Gordon equation (54) with the ICs (55) and the BCs (56).

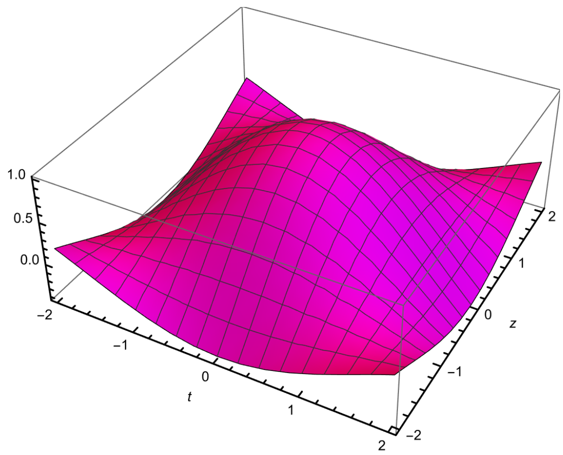

Fig. 4: 3D plot 4. The solution $\varnothing(z,t)$ of Problem 4.

Problem 5. Take the Homo-Laplace equation

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} = -\frac{\partial^2 \varnothing(z,t)}{\partial t^2}, \quad (58)$$

with the ICs

$$\varnothing(z,0) = 0, \quad \frac{\partial \varnothing(z,0)}{\partial t} = \cos(z), \quad (59)$$

and the BCs

$$\varnothing(0,y) = \sinh(t), \quad \frac{\partial \varnothing(0,t)}{\partial z} = 0. \quad (60)$$

Solution. Firstly, we have

$$\begin{aligned} f_1(z) &= 0, & f_2(z) &= \cos(z), \\ h_1(t) &= \sinh(t), & h_2(t) &= 0. \end{aligned}$$

Applying the single ARA-T on $f_1(z)$ and $f_2(z)$, we get

$$F_1(v) = 0,\ \ F_2(v) = \frac{v^2}{v^2+1}.$$

Applying the single FT on $h_1(t)$ and $h_2(t)$, we get

$$H_1(s,u) = \frac{su}{s^2-u^2}\ ,\ \ H_2(s,u) = 0.$$

Rewrite equation (58) in the form

$$\frac{\partial^2 \varnothing(z,t)}{\partial z^2} + \frac{\partial^2 \varnothing(z,t)}{\partial t^2} = 0,$$

$$A = B = 1,\ D = C = E = \ 0 \ and\ U(v,s,u) = 0,$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 and above in the general formula in equation (40), we get

$$\Phi(v,s,u) = \frac{\frac{suv^2}{s^2-u^2} + \frac{sv^2}{u(v^2+1)}}{v^2 + \frac{s^2}{u^2}} = \frac{v^2 su}{(v^2+1)(s^2-u^2)}. \tag{61}$$

Now, take DA-FT inverse to both sides of equation (61), we get

$$\varnothing(z,t) = \mathscr{G}_{1,z}^{-1} R_t^{-1}\left[\frac{v^2 su}{(v^2+1)(s^2-u^2)}\right] = \cos z \sinh t.$$

3D plot Fig 5, shows the answer of the Homo-Laplace equation (58) with the ICs (59) and the BCs (60).

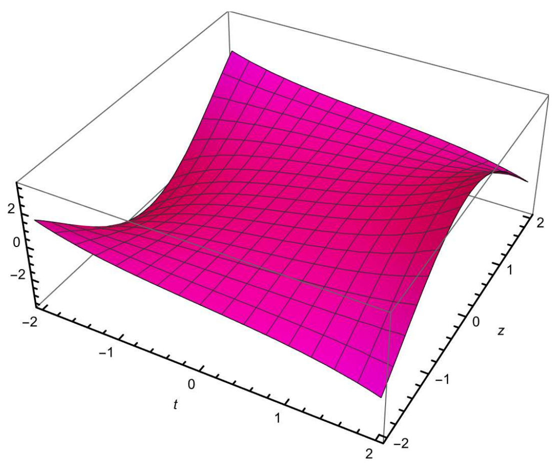

Fig. 5: 3D plot 5. The solution $\varnothing(z,t)$ of Problem 5.

6 Conclusion

In this research, a new approach in transform called DARA-FT was presented. Basic properties of the proposed double transform were introduced and implemented to get solutions of families of PDEs. The usage of the new double transform was illustrated in solving some interesting examples and getting the exact solution of them. New results of DA-FT will be discussed in the future and implemented for solving fractional PDEs and nonlinear PDEs [38- 45].

Funding: This research received no external funding.
Data Availability Statement: Not applicable.
Acknowledgments: The authors express their gratitude to the dear referees, who wish to remain anonymous, and the editor for their helpful suggestions, which improved the final version of this paper.
Conflicts of Interest: The authors declare no conflict of interest.

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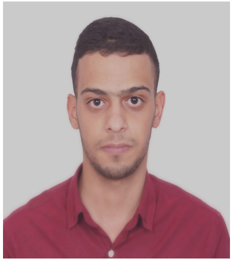

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