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Modelling and Optimal Control Strategies of Corruption Dynamics

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Abstract: In this study, we presented a non linear deterministic corruption transmission dynamics using optimal control analysis and cost effective strategies. To begin, we demonstrated that in a given set of initial conditions, the model solution is non-negative and bounded. A basic reproductive number is calculated using the corruption-free equilibrium point via the next generation matrix. The linearization and the Lyapunov function are then used to demonstrate how corruption-free equilibrium is both locally and globally stable. The corruption-free equilibrium point is asymptotically stable both locally and globally if the basic reproduction number is less than one; otherwise, an endemic corruption equilibrium emerges. Furthermore, the model's parameters were analyzed for sensitivity, and the model demonstrated forward bifurcation. Moreover, applying the Pontryagin minimum principle, the optimal corruption minimization interventions are determined using two control strategies, namely prevention and punishment. Lastly, based up on numerical prediction systems of optimality, prevention is the highest optimal and most cheapest corruption eradication strategy.

Keywords: Corruption dynamics; Optimal control; Cost-effectiveness strategies; Numerical simulation.

1 Introduction

Corrupt is derived from the Latin word ”corruptus,” which means ”to disturb or harm.” [1]. corruption is defined as an illegal activity committed for personal gain and benefit through the abuse of power by private [2]. Moreover, International Transparency defines corruption as ”the misuse of entrusted power for personal gain” [3]. Corruption can come from either the supply or demand side [4]. It is a major issue in all countries, but particularly in developing countries [5]. Indeed, the majority of countries have anti-corruption strategies in place, corruption remains a societal epidemic. In Ethiopia, it is one of the factors that contribute to tension and conflict [6].

Many researchers formulated a mathematical modeling for corruption dynamics to better understand the prevalence of corruption in populations. For example, Aychew Wondyfraw Tesfaye and Haileyesus Tessema Alemneh [8], developed a model of corruption transmission dynamics and then extended it to a stochastic model by incorporating stochastic factors. Furthermore, they concluded that the number of corrupted people decreases when people recover more through education or punishment. Zerihun Kinfe Birhanu and Abayineh Kebed Fantaye [8] investigated a model of the dynamics of corruption using mathematical with media coverage. Finally, they concluded that corruption is removed faster in the presence of media coverage, whereas less media news on the dynamics of corruption in the population. Adeyemi Olukayode Binuyo [9] presented a mathematical model of corruption transmission dynamics. According to the authors' findings, the corruption contact rate among the people has the most role on the corruption transmission dynamics. Finally, when the government takes the necessary and adequate measures, it is sufficient and expedient to minimize the dynamics of corruption among the people to the bare minimum. Nathan and Jackob [10] formulated deterministic modeling of corruption dynamics using the described prevention and disengagement strategies. The authors are used the model's parameters values to simulate prevention and disengagement strategies. Finally, they concluded that the most effective anti-corruption strategies are prevention and disengagement initiatives. Olanegan et al., [11], studied the dynamics of corruption among Nigerian tertiary students. The authors use an epidemiological

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compartment model to describe mathematical modeling of student corruption. The model's positivity and boundlessness were established. Furthermore, the numerical implementation of the model reveals that corruption will persist among Nigerian students if the root cause is not eradicated.

Many authors developed an optimal corruption control model to describe the role of control interventions on corruption dynamics. For examples, Haileyesus Tessema Alemneh [12] developed an optimal corruption control model with two controls variables, namely, media campaigning against corruption and exposing and punishing corrupted individuals. Furthermore, using Pontryagin's minimum principle, the necessary condition of an optimal control corruption dynamics is determined. Finally, the author concluded using the model's analysis that the combined control interventions is the best way to minimize corruption. Ebenezer Bonyah [13] developed a model for fractional optimal control of corruption dynamics. In order to optimize the best strategy for reducing corruption in society, the author incorporates variable controls into the model. Finally, the numerical results show that optimizing all controls simultaneously is the optimal strategy for reducing corruption. Akanni et al. [14] presented a mathematical modeling of dynamics of corruption that includes optimum control interventions. Then optimal control analysis is used to determine the role of control strategies such are preventive and corrective measures, on the corruption dynamics in a population. In addition, a cost effective analysis is investigated to determine the highest optimal and most cheapest strategies. Abayneh Kebede Fantaye and Zerihun Kinfe Birhanu [15] investigated the optimal control corruption dynamics. Finally the authors reveal that the combination of prevention and punishment intervention is the best strategy to reduce the dynamics of corruption. Saida et al. [16] proposed an optimal control model by incorporating two controls variables, namely: prevention corruption through the use media and effective anti-corruption policy; attempt to encourage the punishment of corrupt people. Lastly, authors suggested that the corruption dynamics can be minimized using media and punishing corrupted individuals.

However, all of these models didn't consider the corruption dynamics with optimal control analysis and cost-effective strategies. In this paper, the corruption dynamics model [7] is extended by introducing the exposed compartment and an optimal control via two controls interventions. Moreover, the effective strategies of cost is performed using the increasing cost effective ration method.

This paper is organized as follows: in section 2, we proposed a corruption dynamics model. The model's analytical analysis is shown in section 3. In section 4, we discuss the sensitive analysis of the model's parameters. In section 5, the corruption dynamics optimal control model is analytically performed via the Pontryagin minimum principle. In section 6, we determine the analytical quantity of the numerical. The effective strategies of cost is performed in section 7. The work's conclusion is mentioned in section 8.

2. Formulation of the Model

In this part, we considered the model formulation and its description. The all people of humans at time (t), represented by $\mathcal{N}(t)$, is categorized into five compartments based on disease status: susceptible people $\mathtt{S}(t)$, are human beings who are under risk to corruption, exposed humans $\mathtt{E}(t)$, are human beings who have exposed and those who are suffer from corruption but do not commit it are exposed, while those who are commit corruption are corrupted people $\mathtt{C}(t)$, recovered people $\mathtt{R}(t)$ involves all groups that got temporary immunity becomes recovered from the corruption and those who do not commit corruption always are grouped as honest human $H(t)$. Hence, the total human populations is given by

$$\mathcal{N}(t) = \mathtt{S}(t) + \mathtt{E}(t) + \mathtt{C}(t) + \mathtt{R}(t) + \mathtt{H}(t). \qquad (1)$$

Furthermore, people who are recruit (assumed to be under the risk) to the susceptible human population with rate of Ψ. Then α the rate of susceptible human become honesty that never engages in corruption. Besides, we assume the death rate naturally μ for all people at all times. Susceptible people become exposed after contact rate with corrupted people at a rate of β and then moved to exposed humans $\mathtt{E}(t)$. Also τ is the rate of exposed individuals become corrupted. Corrupted people learn the effects of corruption in prison and migrate to the recovered people at a rate of θ. A recovered group may either become honest or susceptible again. The rate ω is conversion rate of recovered human to honest or susceptible whereas γ is the rate of recovered human enters susceptible people. Moreover, we assumed that all system's parameters are non-negative. All of the parameter descriptions are listed in Table (1) and Figure (1) shows the diagram of corruption dynamics.

Based on a diagram depicted in Figure (1), the governs equation of corruption dynamics model is obtain as:

$$\begin{cases} \frac{dS}{dt} = \Psi - \beta SC - (\mu+\alpha)S + \gamma\omega \mathtt{R} \\ \frac{dE}{dt} = \beta SC - (\tau+\mu)E, \\ \frac{dC}{dt} = \tau E - (\theta+\mu)C, \\ \frac{dR}{dt} = \theta C - (\mu+\gamma)\mathtt{R}, \\ \frac{dH}{dt} = \alpha S + (1-\gamma)\omega R - \mu H, \end{cases} \qquad (2)$$

with initial condition

$$\mathtt{S}(0) = \mathtt{S}_0, \mathtt{E}(0) = \mathtt{E}_0, \mathtt{C}(0) = \mathtt{C}_0, \mathtt{R}(0) = \mathtt{R}_0, \mathtt{H}(0) = \mathtt{H}(0).$$

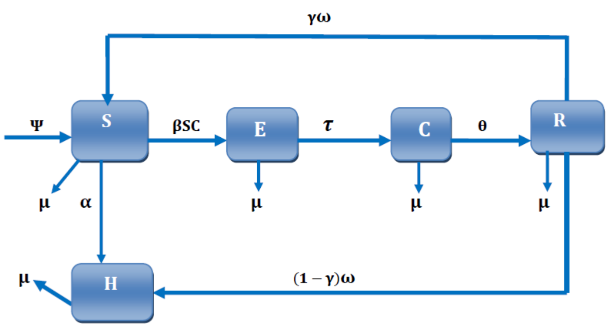


Fig. 1: The figure shows the schematic transmission of Corruption

Table 1: The model's parameters symbol and its description (1)

Parameters	Parameter's descriptions
Ψ	Recruitment rate of susceptible individuals
β	Contact of corrupted human with susceptible
γ	Proportion of recovered that becomes susceptible
μ	Human population natural death rate
ω	Conversion rate of recovered to honest or susceptible
τ	Rate of exposed human becomes corrupted
θ	Rate of corrupted human become recovered
α	Rate of susceptible human become honesty

3. Model Analysis

3.1 Invariant region

A model (1) has the total population of humans given by $\mathrm{N}(t) = S(t) + E(t) + C(t) + R(t) + H(t)$. Then differentiating $N(t)$ and sum up all equation of system (1), we obtain

$$\frac{d}{dt}(S+E+C+R+H) = \Psi - \mu \mathrm{N}. \tag{3}$$

Then Eq. (3) can re-arrange and becomes,

$$\frac{dN}{dt} = \Psi - \mu N. \tag{4}$$

Then to solve equation (4), taking the integration on both sides of the equation (4), we get that $N \leq \frac{\Psi}{\mu}$. Hence, the feasible bounded of the model (1) is obtain as

$$\Omega = \left\{ (S,E,C,R,H) \in \mathbb{R}^5_+ : S+E+C+R+H \leq \frac{\Psi}{\mu} \right\}. \tag{5}$$

3.2 Positivity Solutions

For the model (1) we can state that solutions with non-negative initial values can retain non-negative for future times $t \geq 0$.

Theorem 1. If $S(0)$, $E(0)$, $\mathrm{C}(0)$, $R(0)$ and $H(0)$ are all positive, then the solutions $S(t)$, $E(t)$, $\mathrm{C}(t)$, $R(t)$ and $H(t)$ of the model (1) are also positive for $t \geq 0$.

Proof. Considering from system (1) equation that is given by

$$\frac{dS_h}{dt} = \Psi - \beta SC - (\mu+\alpha)S + \gamma\omega \mathrm{R},$$
$$\frac{dS_h}{dt} \geq -(\beta C + \mu + \alpha)S. \tag{6}$$

Then Eq. (6) integrated and apply the initial conditions, we got

$$S(t) \geq S(0)e^{-(\beta C+\mu+\alpha)t} \geq 0. \tag{7}$$

By the same procedure, the other state variables $E(t), C(t), R(t)$ and $H(t)$ are positive for all time $t \geq 0$.

3.3 Corruption Free Equilibrium (CFE)

In this section, we find the corruption free equilibrium (CFE) with equating the model (1) to zero. Then consider the compartments $E = 0, C = 0$ and $R = 0$. Then the model's corruption-free equilibrium (1) is denoted by E_0 where

$$\mathrm{E}_0 = \left(\frac{\Psi}{\mu+\alpha}, 0, 0, 0, \frac{\alpha\Psi}{\mu(\mu+\alpha)} \right). \tag{8}$$

3.4 Basic reproductive number ($\mathfrak{R}_0$)

The basic reproductive number (R_0) is the average number of secondary cases caused by a single individual in an entirely susceptible environment [17,18]. It is computed using the next-generation matrix method [17]. For system (1) to get the $\mathfrak{R}_0$, we re-arrange the system (1) started with the infected human as:

$$\frac{dE}{dt} = \beta SC - (\tau+\mu)E, \tag{9}$$
$$\frac{dC}{dt} = \tau E - (\theta+\mu)C.$$

Then the equation (9) can be separated into two parts as the form of $f - v$, where

$$f = \begin{pmatrix} \beta SC \\ 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} (\tau+\mu)E \\ (\theta+\mu)C - \tau E \end{pmatrix}. \tag{10}$$

The Jacobian at CFE is yield the matrices $\mathbf{F}$ and ϑ respectively, where

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{\Psi\beta}{\mu+\alpha} \\ 0 & 0 \end{pmatrix} \text{ and } \vartheta = \begin{pmatrix} \tau+\mu & 0 \\ -\tau & \theta+\mu \end{pmatrix}. \quad (11)$$

As a result, the $\Re_0 = \rho(\mathtt{F}\vartheta^{-1})$, mean that ρ is the largest eigenvalue of $\mathtt{F}\vartheta^{-1}$. So that the $\Re_0$ at $\mathtt{E}_0$ is obtained as given by the equation (12) as follows

$$\Re_0 = \frac{\tau\Psi\beta}{(\tau+\mu)(\theta+\mu)(\mu+\alpha)}. \quad (12)$$

3.5 Local stability of corruption free equilibrium

Theorem 2. The corruption free equilibrium of the model (1) is locally asymptotically stable in Ω if $\Re_0 < 1$.

Proof. We begin by computing the linearization of model (1), we obtain:

$$\mathtt{J}(\mathtt{E}_*) = \begin{pmatrix} -\beta C-(\mu+\alpha) & 0 & -\beta S & \gamma\omega & 0 \\ \beta C & -(\mu+\tau) & \beta S & 0 & 0 \\ 0 & \tau & -(\theta+\mu) & 0 & 0 \\ 0 & 0 & 0 & -(\mu+\omega) & 0 \\ \alpha & 0 & 0 & (1-\gamma)\omega & -\mu \end{pmatrix}. \quad (13)$$

Then computing the Jacobian matrix of model (1) at CFE, we obtain:

$$\mathtt{J}(\mathtt{E}_*) = \begin{pmatrix} -(\mu+\alpha) & 0 & -\frac{\beta\Psi}{\mu+\alpha} & \gamma\omega & 0 \\ 0 & -(\mu+\tau) & \frac{\beta\Psi}{\mu+\alpha} & 0 & 0 \\ 0 & \tau & -(\theta+\mu) & 0 & 0 \\ 0 & 0 & 0 & -(\mu+\omega) & 0 \\ \alpha & 0 & 0 & (1-\gamma)\omega & -\mu \end{pmatrix}. \quad (14)$$

From Eq. (14) the linearization of the model is the polynomial function that given

$$(-\lambda-(\mu+\alpha))(-\lambda-\mu)(-\lambda-(\mu+\omega))(\lambda^2+d_1\lambda+d_2) = 0 \quad (15)$$

where,

$$d_1 = \tau+2\mu+\theta, \quad (16)$$
$$d_2 = (\mu+\tau)+(\mu+\theta)-\frac{\tau\beta\Psi}{(\mu+\alpha)}.$$

Then from equation (15), we obtain

$$\lambda_1 = -(\mu+\alpha) < 0,\ \lambda_2 = -\mu < 0,\ \lambda_3 = -(\mu+\omega) < 0, \quad (17)$$

Moreover, using the final characteristic equation (15) we got,

$$\lambda^2+d_1\lambda+d_2 = 0. \quad (18)$$

Lastly, using the stability criteria [18,19], the equation (18) has a non-positive solution if $d_1 > 0$ and $d_2 > 0$. As a result, we can observed that $d_1 > 0$ since expressed as all parameters are positive and d_2 is given by

$$d_2 = (\mu+\tau)+(\mu+\theta)-\frac{\tau\beta\Psi}{(\mu+\alpha)} = 1-\Re_0.$$

But d_2 to be positive, $1-\Re_0$ could be non-negative that shows to $\Re_0 < 1$. So that the CFE is locally asymptotically stable if $\Re_0 < 1$.

3.6 Global stability of corruption free equilibrium

Theorem 3. If $\Re_0 < 1$, then the CFE of the model (1) is globally asymptotically stable in Ω.
Proof. Using Lyapunov concept [20], begin consider the following function Lyapunov defined as

$$\mathtt{L} = z_1E+z_2C. \quad (19)$$

By differentiate the function of Lyapunov at time (t) gives,

$$\frac{d\mathtt{L}}{dt} = z_1\frac{dE}{dt}+z_2\frac{dC}{dt} \quad (20)$$

Substituting $\frac{dE}{dt}$ and $\frac{dC}{dt}$ from the model (1), we get

$$\begin{aligned}\frac{d\mathtt{L}}{dt} &= z_1\left[\beta SC-(\tau+\mu)E\right]+z_2[\tau E-(\theta+\mu)C],\\ &= z_1\beta SC-z_2(\theta+\mu)C-z_1(\tau+\mu)E+z_2\tau E,\\ &= z_1\beta SC-z_2(\theta+\mu)C-z_1(\tau+\mu)E+z_2\tau E,\\ &= \frac{\tau}{\tau+\mu}\beta SC-(\theta+\mu)C,\end{aligned}$$

By taking $z_1 = \frac{\tau}{\tau+\mu}z_2$ and $z_2 = 1$

$$\begin{aligned}\frac{d\mathtt{L}}{dt} &\le \left[\frac{\tau}{\tau+\mu}\frac{\beta\Psi}{(\mu+\alpha)}-(\theta+\mu)\right]C,\\ &= \left[(\theta+\mu)[\frac{\tau\beta\Psi}{(\tau+\mu)(\theta+\mu)(\mu+\alpha)}-1]\right]C,\\ &= [(\theta+\mu)[\Re_0-1]]C.\end{aligned} \quad (21)$$

Hence, we obtain $\frac{d\mathtt{L}}{dt} < 0$ if $\Re_0 < 1$ and $\frac{d\mathtt{L}}{dt} = 0$ iff $C = 0$. Thus, the DFE in Ω is the most powerful compact set in (S,E,C,R,H) : $\frac{dL}{dt} = 0$. Because of LaSalle invariant principle [21] if $\Re_0 < 1$, then the CFE is globally asymptotically stable in Ω.

3.7 Corruption present equilibrium

A equilibrium point said to be corruption endemic if the corruption is exist in the populations. The endemic corruption equilibrium is denoted by $E^* = (S^*, E^*, C^*, R^*, H^*)$ and can be computed setting the model (1) to zero. Hence, the corruption present equilibrium for the model (1), is given by

$$\begin{cases} S^* = \frac{\Psi+\omega\gamma R^*}{\beta C^*(\mu+\alpha)}, \\ E^* = \frac{\beta S^* C^*}{\tau+\mu}, \\ R^* = \frac{\theta C^*}{\mu+\gamma}, \\ H^* = \frac{\alpha S^*+(1-\gamma)\omega H^*}{\mu}. \end{cases} \quad (22)$$

The corruption endemic equilibrium can determined by the polynomial function from equation (22), and C^* is calculated from the equation:

$$D_1(C^*)^2 + D_2(C^*) = 0 \quad (23)$$

where,

$$D_1 = \tau\beta\Psi(\mu+\alpha),$$
$$D_2 = (\tau+\mu)(\theta+\mu)(\alpha+\mu)(1-\Re_0)]. \quad (24)$$

Hence, $D_1 > 0$ and $D_2 \geq 0$ if $\Re_{01} \geq 1$. Solving for C^* , we got $C^* = -\frac{D_2}{D_1} \leq 0$. As a result, whenever $\Re_0 < 1$, the model has no positive corruption present equilibrium. This lends support to the forward bifurcation depicted in Figure (5).

3.8 Global stability of corruption endemic equilibrium

Theorem 4. If $\Re_0 > 1$, then the corruption endemic equilibrium of system (1) is globally asymptotically stable in Ω.

Proof. In order to prove the theorem, we consider the Lyapunov function constructed as follows:
Let $A_1 = (S-S^*)$,$A_2 = (E-E^*)$,$A_3 = (C-C^*)$,$A_4 = (R-R^*)$ and $A_5 = (H-H^*)$. Then

$$Q = \frac{1}{2}\left[(A_1+A_2+A_3+A_4+A_5)\right]^2. \quad (25)$$

By computing the derivative of Eq. (25) with respect to time, we obtain as:

$$\frac{dQ}{dt} = [(A_1+A_2+A_3+A_4+A_5)]\frac{d}{dt}[S+E+C+R+H],$$

$$= [(A_1+A_2+A_3+A_4+A_5)]\frac{dN}{dt}. \quad (26)$$

Moreover, from equations (4), we have,

$$\frac{dN}{dt} = \frac{d}{dt}[S+E+C+R+H] \leq \Psi - \mu N, \quad (27)$$

By substituting Eq. (27) into the Eq. (26)and simplify the expression, we get

$$\begin{aligned} \frac{dQ}{dt} &= [(A_1+A_2+A_3+A_4+A_5)]\frac{dN}{dt}, \\ &\leq [(A_1+A_2+A_3+A_4+A_5)]\frac{dN}{dt}[\Psi-\mu N], \end{aligned} \quad (28)$$

$$\leq \left[N - \frac{\Psi}{\mu}\right][\Psi-\mu N]. \quad (29)$$

By rearranging and simplifying the equation (28), we get the following result:

$$\frac{dQ}{dt} \leq -\frac{1}{\mu}[\Psi-\mu N]^2 \quad (30)$$

Therefore, $(\frac{dQ}{dt})(S,E,C,R,H) \leq 0$ and $\frac{dQ}{dt} = 0$, iff $S = S^*, E = E^*, C = C^*, R = R^*, H = H^*$. As a result, the dominant invariant was established in $\Omega : \frac{dQ}{dt} = 0$ is one set E^*. Using the LaSalle's principle [21], the corruption endemic equilibrium E^* is globally asymptotically stable in Ω.

4. Sensitivity analysis

A purpose of the model's parameter sensitivity analysis is to show which parameters affect the corruption dynamics. To identify the most effective corruption-control strategies, we must first understand the parameters that influence the basic reproductive number ($\Re_0$). The sensitivity analysis was carried out using the method described in [22].

Definition 4.1 (see [22]). The forward sensitivity index of $\Re_0$ with respect to a given basic parameter Q is defined as

$$\Pi_Q^{\Re_0} = \frac{\partial \Re_0}{\partial Q} \times \frac{Q}{\Re_0}. \quad (31)$$

As example, the sensitivity of $\Re_0$ in relation to the parameter β is calculated as

$$\begin{aligned} \Pi_\beta^{\Re_0} &= \frac{\partial \Re_0}{\partial \beta} \times \frac{\beta}{\Re_0} = \\ &\frac{\Psi}{(\tau+\mu)(\theta+\mu)(\mu+\alpha)} \times \frac{\beta}{\Re_0} = 1 > 0. \end{aligned} \quad (32)$$

Taking the same approach as with rest parameters,

$$\Pi_\Psi^{\Re_0}, \ \Pi_\alpha^{\Re_0}, \ \Pi_\mu^{\Re_0}, \quad \Pi_\tau^{\Re_0}, \ \Pi_\theta^{\Re_0}$$

are mentioned as written in Table 2:

Table 2: Sensitivity index of the parameters

Symbol of parameters	Index of sensitivity
Ψ	1
β	1
α	-0.005
τ	-0.045
μ	-1
θ	-0.028

The sensitivity index of the basic reproductive number ($\Re_0$) regarding to seven parameters were shown in Table (2). The results depicted that some parameters having a positive sensitivity index increased the value of ($\Re_0$) as their values added, whereas increasing the values of the parameters having negative indices will reduces the value of ($\Re_0$) while keeping the values of the other parameters remain not changed.

Table 3: Parameter symbol and its descriptions for model (1)

Parameter	Descriptions of Parameter	Values	References
Ψ	Recruitment rate of susceptible individuals	85.000	[12]
β	Contact of corrupted human with susceptible	0.024	[12]
γ	Rate of recovered human become honesty	0.350	[6]
τ	Rate of exposed people become corrupted	0.007	[12]
θ	Rate of corrupted human become recovered	0.010	[6]
μ	Human population natural death rate	0.0160	[15]
ω	Rate at recovered human become susceptible	0.0021	[15]
α	The rate at which susceptible human becomes honesty	0.030	[12]

5. Optimal control model

An optimal control model, which includes a mathematical model of biological situations, is used to design control strategy decisions [23]. Here, the corruption dynamics model (1) was extended to an optimal control model. The state equations obtained by included the controls variables into the corruption dynamics model (1) is given as:

$$\begin{cases} \frac{dS}{dt} = \Psi - (1-u_1)\beta SC - (\mu+\alpha)S + \gamma\omega \mathtt{R}, \\ \frac{dE}{dt} = (1-u_1)\beta SC - (\tau+\mu)E, \\ \frac{dC}{dt} = \tau E - (\theta+\mu+u_2)C, \\ \frac{dR}{dt} = (\theta+u_2)C - (\mu+\omega)\mathtt{R}, \\ \frac{dH}{dt} = \alpha S + (1-\gamma)\omega R - \mu H, \end{cases} \tag{33}$$

where the control functions $u_1(t)$ represents prevent individuals away from the corrupted populations, $u_2(t)$ deals develop powerful laws regarding the corruption. The optimal control model's objective functional (33) is given as

$$\mathtt{J}(u_1,u_2) = \underbrace{\min}_{u_1,u_2,} \int_0^{t_f} \left(\mathtt{A}E + \mathtt{B}C + \frac{1}{2}(\mathtt{C}u_1^2 + \mathtt{D}u_2^2) \right) dt, \tag{34}$$

where t_f is the terminal time, the expression $\frac{1}{2}\mathtt{C}_i u_i^2$ represents the cost of functions for the controls $u_i(t)$ [24, 25]. The objective functional (34) is to minimize the corrupted people and control costs $u_i(t)$. The target is to find an optimum controls u_1^* and u_2^* satisfies

$$\mathtt{J}(u_1^*,u_2^*) = \min\{\mathtt{J}(u_1,u_2) : u_1,u_2 \in \vartheta\} \tag{35}$$

where $\vartheta = (u_1,u_2) : u_i(t)$ are Lebesgue integrable on $t \in [0,t_f]$.
The Hamiltonian function of an optimal control model is the combination of equations (33) and (34) is given as

$$\mathscr{H} = [\mathtt{A}E + \mathtt{B}C + \frac{1}{2}(\mathtt{C}u_1^2 + \mathtt{D}u_2^2] + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dE}{dt} + \lambda_3 \frac{d\mathtt{C}}{dt} + \lambda_4 \frac{d\mathtt{R}}{dt} + \lambda_5 \frac{dH_1}{dt}. \tag{36}$$

From equation (36) the minimize Hamiltonian regarding the controls to u_1, u_2 is given by

$$\begin{aligned} \mathscr{H} = {} & [\mathtt{A}E + \mathtt{B}C + \frac{1}{2}(\mathtt{C}u_1^2 + \mathtt{D}u_2^2)] \\ & + \lambda_1[\Psi - (1-u_1)\beta SC - (\mu+\alpha)S + \gamma\omega\mathtt{R}] \\ & + \lambda_2[(1-u_1)\beta SC - (\tau+\mu)E] \\ & + \lambda_3[\tau E - (\theta+\mu+u_2)C] \\ & + \lambda_4[(\theta+u_2)C - (\mu+\omega)\mathtt{R}] \\ & + \lambda_5[\alpha S + (1-\gamma)\omega R - \mu H] \end{aligned} \tag{37}$$

where λ_i, for $i = 1,2,3,4,5$ are adjoint variables. The adjoint equations are computed via Pontryagin's minimum principle [25], with the evidence of [26], we stated the theorem as follow:

Theorem 5. Suppose that the control variables u_1^*, u_2^* and a solution $\mathtt{S}^*, \mathtt{E}^*, \mathtt{C}^*, \mathtt{R}^*, \mathtt{H}^*$ of the corresponding state equations that minimize $\mathtt{J}(u_1,u_2)$ over ϑ subject to the equation (33), then there exist co-state variables λ_i, for $i=1,2,3,4,5$ a hold the co-state systems

$$\begin{cases} \frac{d\lambda_1}{dt} = (1-u_1)(\lambda_1-\lambda_2)\beta C+(\mu_h+\alpha)\lambda_1-\alpha\lambda_5, \\ \frac{d\lambda_2}{dt} = (\lambda_2-\lambda_3)\tau+\mu\lambda_2-\mathtt{A}, \\ \frac{d\lambda_3}{dt} = (1-u_1)(\lambda_1-\lambda_2)\beta S+(\lambda_3-\lambda_4)(\theta+u_2)+ \\ \lambda_3\mu-\mathtt{B}, \\ \frac{d\lambda_4}{dt} = \lambda_4(\omega+\mu)-\lambda_5(1-\gamma)\omega-\lambda_1\gamma\omega, \\ \frac{d\lambda_5}{dt} = \lambda_5\mu, \end{cases} \quad (38)$$

with transversality conditions

$$\lambda_i(t_f)=0, \quad for \quad i=1,2,3,4,5. \quad (39)$$

Furthermore, the optimal controls u_1^* and u_2^* are denoted as

$$u_1^* = \min\left\{1,\max\left\{0,\frac{(\lambda_2-\lambda_1)\beta SC}{\mathtt{C}}\right\}\right\}, \quad (40)$$
$$u_2^* = \min\left\{1,\max\left\{0,\frac{(\lambda_3-\lambda_4)C}{\mathtt{D}}\right\}\right\}.$$

Proof. The co-state equations can be computed by the derivative of the Hamiltonian Eq. (36), respectively, with $\mathtt{S}^*,\mathtt{E}^*,\mathtt{C}^*,\mathtt{R}^*$ and $\mathtt{H}^*$. Hence, the co-state equations obtained are given by

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\partial\mathscr{H}}{\partial S} = (1-u_1)(\lambda_1-\lambda_2)\beta C+(\mu+\alpha)\lambda_1-\alpha\lambda_5, \\ \frac{d\lambda_2}{dt} = -\frac{\partial\mathscr{H}}{\partial E} = (\lambda_2-\lambda_3)\tau+\mu\lambda_2-\mathtt{A}, \\ \frac{d\lambda_3}{dt} = -\frac{\partial\mathscr{H}}{\partial C} = (1-u_1)(\lambda_1-\lambda_2)\beta S+(\lambda_3-\lambda_4)(\theta+u_2) \\ +\lambda_3\mu-\mathtt{B}, \\ \frac{d\lambda_4}{dt} = -\frac{\partial\mathscr{H}}{\partial R} = \lambda_4(\omega+\mu)-\lambda_5(1-\gamma)\omega-\lambda_1\gamma\omega, \\ \frac{d\lambda_5}{dt} = -\frac{\partial\mathscr{H}}{\partial H} = \lambda_5\mu, \end{cases} \quad (41)$$

with transversality conditions

$$\lambda_i(t_f)=0, \quad for \ i=1,2,3,4,5. \quad (42)$$

Then using the optimality conditions, we obtain the values of the control controls, as given as:

$$u_1^* = \frac{(\lambda_2-\lambda_1)\beta SC}{\mathtt{C}},$$
$$u_2^* = \frac{(\lambda_3-\lambda_4)C}{\mathtt{D}}. \quad (43)$$

Using the boundary conditions of the controls and rearranging the solution of (43), we got:

$$u_1^* = \min\left\{1,\max\left\{0,\frac{(\lambda_2-\lambda_1)\beta SC}{\mathtt{C}}\right\}\right\}, \quad (44)$$
$$u_2^* = \min\left\{1,\max\left\{0,\frac{(\lambda_3-\lambda_4)C}{\mathtt{D}}\right\}\right\}.$$

The simulation of an optimality system will then be used to determine the best intervention for minimizing corruption dynamics.

6. Numerical simulation

In this part, to find the best strategy, we applied the iterative method to solve the states and co-state variables. Because of the initial values of the state variables, via the forward fourth-order Runge-Kutta method, we computed the state equations. Besides, via backward fourth-order Runge Kutta method the adjoint equations were obtained. In the optimality system, the initial values for the state variables are $\mathtt{S}(0)=300,\mathtt{E}(0)=60,\mathtt{C}(0)=40,\mathtt{R}(0)=20,\mathtt{H}(0)=40$. Moreover, we assigned the weight constant values as follows: $\mathtt{A}=60,\mathtt{B}=80,\mathtt{C}=100$ and $\mathtt{D}=90$. We used the following three strategies to design the intervention for the minimization of corruption dynamics.

6.1 Strategy A: using prevent individuals away from the corrupted human (u_1)

In this subsection, we minimized the Eq. (34) using the prevent of corruption u_1 and without developing laws to punish corrupted humans (u_2). In Figure 2(**a**), if the controls are applied, the exposed people $\mathtt{E}$ reduces while the number of exposed people increases if the controls are omitted. The corrupted humans $\mathtt{C}$ decreases as the control strategy is included in Figure 2(**b**), whereas it rapidly growths if no controls are applied. The control u_1 remained its maximum level (100%) for 120 days as in Figure 2(**c**).

6.2 Strategy B: using powerful laws to punish the corrupted human (u_2).

This strategy used the develop laws to punish the corrupted human u_2 when using the prevent of corruption u_1 was not included. Figure 3(**a**) shows that exposed human $\mathtt{E}$ with controls is minimized whereas if there are no controls the exposed human increases. Figure 3(**b**) indicates that corrupted human $\mathtt{C}$ decreases in the control strategy whereas corrupted human growth rapidly if no control. Figure 3(**c**), the punishment u_2 kept its high (100%) with 25 days.

6.3 Strategy C: Integrating of prevention (u_1) and punishment of corrupted human (u_2).

In this study, we apply a combination prevention u_1 and punishment u_2 to minimize all infected populations. Figure 4(**a**) indicates exposed human E minimized if we apply the control strategy, whereas if there are no controls the exposed human increases. Figure 4(**b**) indicates the corrupted human C decreases if the control strategy is used, whereas in the no controls, the corrupted human increases. Figure 4(**c**) shows that prevention controls u_1

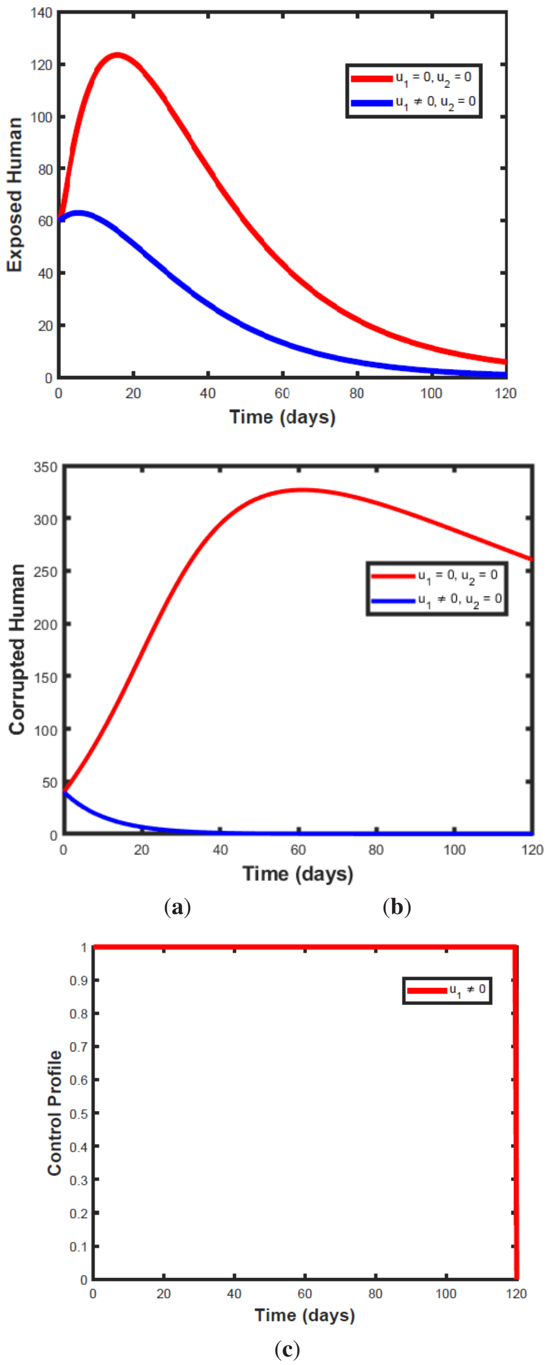


Fig. 2: Numerical simulation with prevent individuals away from the corrupted human (u_1).

maintain a high level (100 %) for 120 days, while punishment controls u_2 maintain a high level for 25 days.

Figure (5) shows that the bifurcation diagram for corruption dynamics model that shows forward bifurcation. This implies that if $\Re_0 < 1$ then

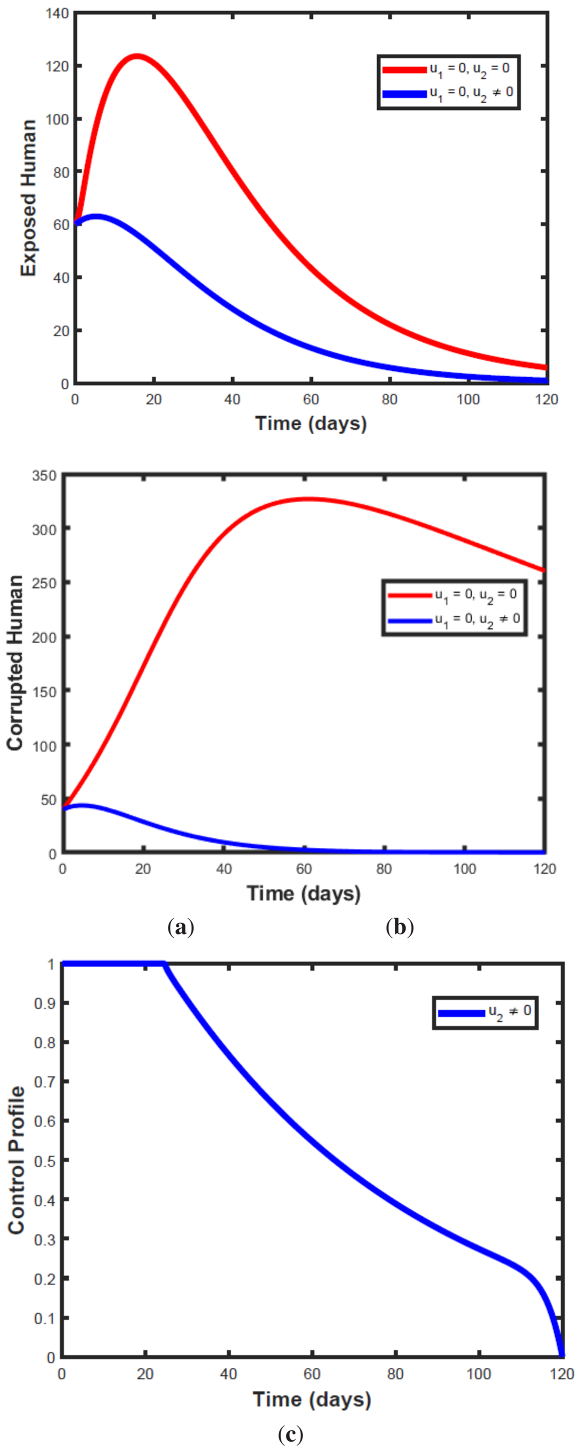
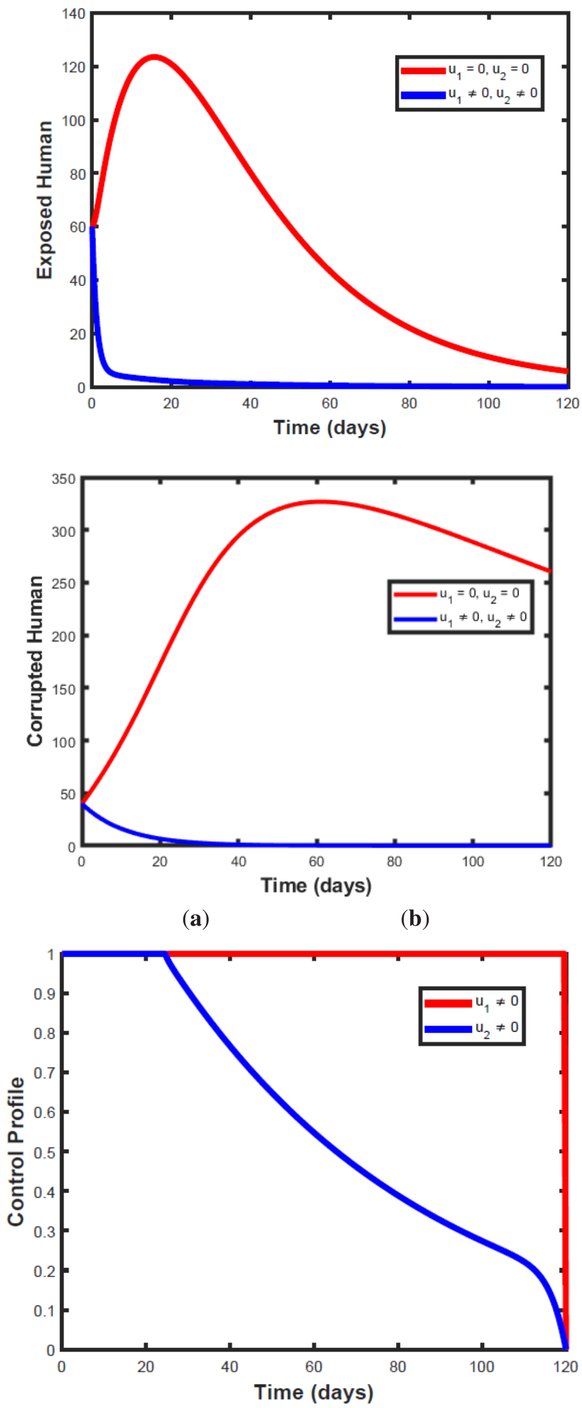


Fig. 3: Numerical simulation using powerful laws to punish the corrupted human.

automatically implies that DFE exists and stable whereas if $\Re_0 > 1$ endemic equilibrium is exist and stable.



(**a**) (**b**)

(**c**)

Fig. 4: Numerical simulations with prevention (u_1) and punishment of corrupted (u_2).

7. Cost effective strategies

In this subsection, we should devise the best optimal and most cost-effective strategies for minimizing the

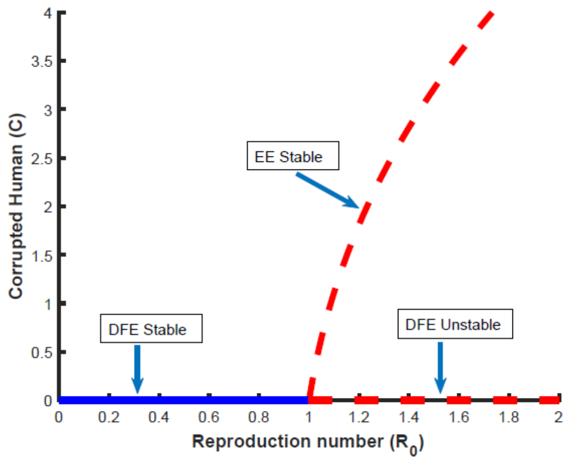


Fig. 5: Figure shows bifurcation diagram corruption dynamics model (1).

corruption dynamics. This strategy was created via the incremental cost effective technique(ICER). Furthermore, ICER can be expressed as the division of the change in difference costs between two interventions to the change in the number with the corrupted averted [27,28,29,30]. Total infected saved and cost avoided for three strategy is shown in Table (4).

Table 4: Total infected saved and cost avoided for three strategy

Strategy	Description	Total of infected people	Dollars ($)
A	Prevention individuals	4744.74	2421.32
B	Develop powerful laws	4892.56	3258.42
C	Prevention and laws	6216.48	4276.64

Based on the data obtained in Table (4) we can compute the ICRE and obtain as:

$$\mathrm{ICER}(\mathbf{A}) = \frac{2421.32}{4744.74} = 0.51$$

$$\mathrm{ICER}(\mathbf{B}) = \frac{3258.42 - 2421.32}{4892.56 - 4744.74} = 5.66$$

$$\mathrm{ICER}(\mathbf{C}) = \frac{4276.64 - 3258.42}{6216.48 - 4892.56} = 0.76$$

Table (5) shows the number of infected averted with ICER for all strategy based on the results in Table (4).

Table 5: Total infected averts and costs incurred as a result of using ICER

Strategy	Total of infected people	Dollars ($)	ICER
A	4744.74	2421.32	0.51
B	4892.56	3258.42	5.66
C	6216.48	4276.64	0.76

The table (5) compares the interventions **A** and **B**. The table shows that ICER(**B**) is higher than ICER(**A**). It implies strategy **B** is expensive and unlikely to save lives. Thus, strategies **A** averts people than **B**. Then, we removed **B** among the interventions. The ICER of strategy **A** and **C** is then calculated, as shown in Table (6).

Table 6: Total infected averted and costs incurred as a result of using ICER

Strategies	Total of infected people	Dollars ($)	ICER
A	4744.74	2421.32	0.51
C	6216.48	4276.64	0.76

The table (6) calculates the interventions **A** and **C**. The table shows ICER(**C**) is higher than ICER(**A**). It shows intervention **C** is higher expensive. Consequently, we suggested that strategy **A**, which involves educating individuals to keep them away from the corrupted population, is the optimal and highest cheap intervention to minimize the corruption dynamics.

8. Conclusion

In this study, to describe the corruption dynamics, we used nonlinear differential equations. The solution of the model is bounded and non-negative, via the analytical analysis of the model. The basic reproductive number in relation to the corrupted free equilibrium is calculated via the next generation matrix technique. Then to depict the local and global stability of corruption equilibriums, the linearization and the Lyapunov method are used. Besides, if the basic reproductive number is smaller than unity, the corruption free-equilibrium is asymptotically stable both locally and globally; otherwise, a corruption endemic equilibrium exists. The model's parameter sensitivity was described, and also forward bifurcation has been observed. Moreover, the corruption dynamics model is extended to an optimal control model by incorporating two time-dependent controls, namely the corrupted individual's personal deterrence and the development of strong anti-corruption laws. To determine the optimal control conditions, the Pontraygin minimum principle is applied, and a cost-effective intervention is described to design the strategies with the lowest cost. As a result of the analysis, we came to the conclusion that personal prevention away from the corrupted individual is the most effective strategy for reducing corruption dynamics.

Data Availability

The manuscript included all of the data.

Conflicts of Interests

The authors have no particular conflicts of interest on the manuscript.

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