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Parameters and Reliability Estimation of Left Truncated Gumbel Distribution under Progressive Type II Censored with Repairable Mechanical Equipment Data

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Abstract: The estimation of two parameters of the left truncated Gumbel distribution using the progressive type II censoring scheme is discussed. We first derived the maximum likelihood estimators of the unknown parameters. The approximate asymptotic variance-covariance matrix and approximate confidence intervals based on the asymptotic normality of the classical estimators are calculated. Also, the survival and hazard functions are derived. Further, the delta method is used to construct approximate confidence intervals for survival and hazard functions. Using the left truncated normal prior for the location parameter and an inverted gamma prior for the scale parameter, several Bayes estimates based on squared error and general entropy loss functions are computed. Bayes estimators of the unknown parameters cannot be calculated in closed forms. Markov chain Monte Carlo method, namely Metropolis-Hastings algorithm, has been used to derive the approximate Bayes estimates. Also, the credible intervals are constructed by using Markov chain Monte Carlo samples. Finally, The Monte Carlo simulation study compares the performances among various estimates in terms of their root mean squared errors, mean absolute biased, average confidence lengths, and coverage probabilities under different sets of values of sample sizes, number of failures and censoring schemes. Moreover, a numerical example with a real data set and Markov chain Monte Carlo data sets are tackled to highlight the importance of the proposed methods. Bayes Markov chain Monte Carlo estimates have performed better than those obtained based on the likelihood function.

Keywords: Bayesian estimation, Censoring schemes, Metropolis–Hastings algorithm, Truncated Distribution.

1 Introduction

In many Industrial and clinical experiments, items are lost or removed from the experiment before the event of interest occurs. Complete information about the event of interest for all experimental items may be lacked. Data resulting from such experiments are called censored data. Pre-planned censoring helps saving the total test time and reducing experiment cost. The most common censoring schemes are type I and type II censoring but the conventional type I and type II are not flexible enough to allow removal of items at points other than the terminal point of the experiment. Therefore, a more general censoring scheme; progressive censoring is proposed. Progressive type II censoring is a beneficial and a more general scheme where a specific fraction of individuals at risk can be removed from the study at each of the several ordered failures. Progressive type II censoring scheme is introduced by [1] who explained it as follows: suppose n independent items are put in a life testing experiment and $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ denote the vector of surviving items. When the first failure time happens, R_1 of surviving items are randomly selected and removed from the remaining $(n-1)$ surviving items, thereby leaving $(n-1-R_1)$ surviving items. Similarly, when the second failure time is happens, R_2 of surviving items are randomly selected and removed from the remaining $(n-2-R_1)$ surviving items, thereby leaving $(n-2-R_1-R_2)$ surviving items and so on. Finally, this process continues until the m^{th} failure time happans. Then, the life testing experiment terminates and all the remaining R_m surviving items are removed at random from $(n-m-R_1-R_2-\ldots-R_{m-1})$ survival items. The likelihood function based on progressive type II censored sample is indicated by [13] as follows

$$L(\underline{\nu}) = C\prod_{i=1}^{m} f(x_{(i)};\underline{\nu})(1-F(x_{(i)};\underline{\nu}))^{R_i} \quad (1)$$

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where $\underline{v}$ denotes the vector of unknown parameters. C is normalizing constant and $n = \sum_{i=1}^{m} R_i + m$. Recently, progressive type II censoring has been used in truncated distribution; [7] have initiated deriving the maximum likelihood estimates (MLEs) and Bayes estimates from left truncated normal distribution. This paper aims at using the maximum likelihood and Bayesian estimation to derive point and interval estimates of the unknown parameters of left truncated Gumbel (LTG) distribution under a progressive type II censored scheme. Based on the squared error (SE) and general entropy (GE) loss functions, Bayes estimates are obtained when both parameters are unknown. It is expected that the maximum likelihood and Bayes estimators cannot be obtained in a closed form, but they can be evaluated numerically. Based on Markov chain Monte Carlo (MCMC) technique, namely Metropolis-Hastings (M-H) algorithm, Bayes estimates of unknown parameters are obtained. Monte Carlo simulation study compares the performances among various estimates in terms of their root mean squared errors (RMSEs), mean absolute biased (MAB), average confidence lengths (ACLs), and coverage probabilities (CPs). Moreover, a numerical example with a real data and MCMC data sets are tackled to highlight the importance of the proposed methods. Lastly, since no attempt has been made to estimate the unknown parameters of LTG distribution under the censoring scheme, this paper aims at filling this gap using a progressive type II censored sample. The paper is organized as follows: In Section 2, we present the model description that is utilized throughout this paper. In Section 3, the maximum likelihood estimates for unknown parameters under the progressive type II censored scheme are derived. In Section 4, Bayes estimates of the unknown parameters under squared error and the general entropy loss function are derived using MCMC. In Section 5, a simulation study is implemented. In Section 6, computational results are presented to illustrate all the developed methods. Concluding remarks are discussed in Section 7.

2 Model description

The truncated distribution for a continuous random variable is a pivotal research topic in both theory and application. It happens in many problems of probabilistic modeling in engineering, insurance forecasting, lifetime data analysis, reliability analysis, etc., as indicated by [10]. The probability density function (PDF) of the LTG distribution is denoted by, $LTG(\mu,\sigma)$, and is defined by [12] as follows

$$f(x;\mu,\sigma) = \frac{e^{-\psi^*(\xi_x)-\xi_x}}{\sigma(1-e^{-\psi(\xi)})}, \quad \mu,\sigma > 0, \quad x > 0 \tag{2}$$

where $\xi_x = \frac{x-\mu}{\sigma}$, $\xi = \frac{\mu}{\sigma}$, $\psi^*(\xi_x) = e^{-\xi_x}$ and $\psi(\xi) = e^{\xi}$. μ and σ are the location and scale parameters respectively. The corresponding cumulative distribution function (CDF) is given by

$$F(x;\mu,\sigma) = \frac{e^{-\psi^*(\xi_x)} - e^{-\psi(\xi)}}{1-e^{-\psi(\xi)}}, \quad \mu,\sigma > 0, \quad x > 0. \tag{3}$$

The survival and hazard functions of LTG distribution are defined by

$$S(t) = \frac{1-e^{-\psi^*(\xi_t)}}{1-e^{-\psi(\xi)}}, \quad \mu,\sigma > 0, \quad t > 0, \tag{4}$$

and

$$h(t) = \frac{e^{-\psi^*(\xi_t)-\xi_t}}{\sigma(1-e^{-\psi^*(\xi_t)})}, \quad \mu,\sigma > 0, \quad t > 0, \tag{5}$$

respectively, where, $\xi_t = \frac{t-\mu}{\sigma}$. Introducing the LTG distribution, it is worth noting that [12] derived many statistical properties of LTG distribution specially moments and skewness and kurtosis coefficients. They investigated the MLEs and percentiles estimates of the unknown parameters and their asymptotic confidence intervals (ACIs) under a complete sample case.

3 Maximum likelihood estimation

In this section, the MLEs of the unknown parameters μ and σ of the LTG distribution based on a progressive Type II censored scheme are derived. Point and interval estimation of the unknown parameters are obtained. Suppose that

$\mathbf{x} = (x_{(1)}, x_{(2)}, \dots, x_{(m)})$ is a progressive Type II censored sample drawn from the LTG distribution, whose PDF and CDF are given by (2) and (3), respectively, and a censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$ is applied. Then, the likelihood function can be obtained from (2), (3) and (4), as follows:

$$L(\mu,\sigma) = C\frac{1}{\sigma^m}\frac{e^{-\sum_{i=1}^{m}\psi^*(\xi_{x_{(i)}})-\sum_{i=1}^{m}\xi_{x_{(i)}}}}{(1-e^{-\psi(\xi)})^n}\prod_{i=1}^{m}(1-e^{-\psi^*(\xi_{x_{(i)}})})^{R_i} \tag{6}$$

where, $\xi_{x_{(i)}} = \frac{x_{(i)}-\mu}{\sigma}$ and $n = \sum_{i=1}^{m} R_i + m$. The natural logarithm of likelihood function for the unknown parameters μ and σ, without normalizing constant, is obtained from (6) as

$$\ln L(\mu,\sigma) \propto -m\ln\sigma - n\ln(1-e^{-\psi(\xi)}) - \sum_{i=1}^{m}\psi^*(\xi_{x_{(i)}}) - \sum_{i=1}^{m}\xi_{x_{(i)}} + \sum_{i=1}^{m}R_i\ln(1-e^{\psi^*(\xi_{x_{(i)}})}) \tag{7}$$

By applying the partial derivatives of the natural logarithm of likelihood function (7) with respect to μ and σ, and equating each of them to zero, the results are:

$$\frac{n\psi(\hat{\xi})e^{-\psi(\hat{\xi})}}{(1-e^{-\psi(\hat{\xi})})} + \sum_{i=1}^{m}\psi^*(\hat{\xi}_{x_{(i)}}) + m - \sum_{i=1}^{m}R_i\frac{\psi^*(\hat{\xi}_{x_{(i)}})e^{-\psi^*(\hat{\xi}_{x_{(i)}})}}{(1-e^{-\psi^*(\hat{\xi}_{x_{(i)}})})} = 0, \tag{8}$$

and

$$m - \frac{n\hat{\xi}\psi(\hat{\xi})e^{-\psi(\hat{\xi})}}{(1-e^{-\psi(\hat{\xi})})} + \sum_{i=1}^{m}\hat{\xi}_{x_{(i)}}\psi^*(\hat{\xi}_{x_{(i)}}) + \sum_{i=1}^{m}\hat{\xi}_{x_{(i)}} + \sum_{i=1}^{m}R_i\frac{\hat{\xi}_{x_{(i)}}\psi^*(\hat{\xi}_{x_{(i)}})e^{-\psi^*(\hat{\xi}_{x_{(i)}})}}{(1-e^{-\psi^*(\hat{\xi}_{x_{(i)}})})} = 0 \tag{9}$$

Since (8) and (9) do not have closed form, they can be solved numerically to get the MLEs of unknown parameters μ and σ .Obtaining the estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ and using the invariance property of the MLEs, the MLEs $\hat{S}(t)$ and $\hat{h}(t)$ of $S(t)$ and $h(t)$, as in (3) and (4) respectively, can be found out replacing μ and σ by their the MLEs $\hat{\mu}$ and $\hat{\sigma}$ as follows

$$\hat{S}(t) = \frac{(1-e^{-\hat{\psi}^*(\hat{\xi}_t)})}{(1-e^{-\hat{\psi}(\hat{\xi})})}, \quad \mu,\sigma > 0, \quad t > 0$$

$$\hat{h}(t) = \frac{e^{-\hat{\psi}^*(\hat{\xi}_t)-\hat{\xi}_t}}{\sigma(1-e^{-\hat{\psi}^*(\hat{\xi}_t)})}, \quad \mu,\sigma > 0, \quad t > 0$$

respectively, where, $\hat{\xi}_t = \frac{t-\hat{\mu}}{\hat{\sigma}}$. The variances and covariances of the MLEs are obtained by the elements of the inverse Fisher information matrix. Fisher information matrix, $I(\mu,\sigma)$, is given by:

$$I(\mu,\sigma) = -E\begin{bmatrix}\frac{\partial^2\ln L(\mu,\sigma)}{\partial\mu^2} & \frac{\partial^2\ln L(\mu,\sigma)}{\partial\mu\partial\sigma} \\ \frac{\partial^2\ln L(\mu,\sigma)}{\partial\sigma\partial\mu} & \frac{\partial^2\ln L(\mu,\sigma)}{\partial\sigma^2},\end{bmatrix} \tag{10}$$

where,

$$\frac{\partial^2\ln(L(\mu,\sigma))}{\partial\mu^2} = \frac{n}{\sigma^2}\Big(e^{-\psi(\xi)}\psi(\xi)(\frac{(1-\psi(\xi))}{(1-e^{-\psi(\xi)})} + \frac{e^{-\psi(\xi)}\psi(\xi)}{(1-e^{-\psi(\xi)})^2})\Big) + \frac{1}{\sigma^2}\sum_{i=1}^{m}\psi(\xi_{x_{(i)}})$$
$$-\frac{1}{\sigma^2}\sum_{i=1}^{m}R_i\psi^*(\xi_{x_{(i)}})e^{-\psi^*(\xi_{x_{(i)}})}\Big(\frac{(1-\psi^*(\xi_{x_{(i)}}))}{(1-e^{-\psi^*(\xi_{x_{(i)}})})} - \frac{\psi^*(\xi_{x_{(i)}})e^{-\psi^*(\xi_{x_{(i)}})}}{(1-e^{-\psi^*(\xi_{x_{(i)}})})^2}\Big),$$

$$\frac{\partial^2\ln(L(\mu,\sigma))}{\partial\sigma^2} = \frac{m}{\sigma^2} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\xi_{x_{(i)}}\psi^*(\xi_{x_{(i)}})(\frac{1}{\sigma}\xi_{x_{(i)}}+1) - \frac{n\xi\psi(\xi)e^{-\psi(\xi)}}{\sigma^2}\Big(\frac{(1+\sigma^{-1}\xi^2\psi(\xi))-1}{(1-e^{-\psi(\xi)})}$$
$$-\frac{e^{-\psi(\xi)}\xi\psi(\xi)}{(1-e^{-\psi(\xi)})^2}\Big) + \sum_{i=1}^{m}R_i\xi_{x_{(i)}}\psi^*(\xi_{x_{(i)}})e^{-\psi^*(\xi_{x_{(i)}})}\Big(\frac{(1-\psi^*(\xi_{x_{(i)}}))}{\sigma^3(1-e^{-\psi^*(\xi_{x_{(i)}})})} - \frac{\xi_{x_{(i)}}\psi^*(\xi_{x_{(i)}})e^{-\psi^*(\xi_{x_{(i)}})}}{\sigma(1-e^{-\psi^*(\xi_{x_{(i)}})})^2}\Big),$$

and

$$\frac{\partial^2 \ln(L(\mu,\sigma))}{\partial\sigma\partial\mu} = \frac{m}{\sigma^2} + \frac{n}{\sigma^2}\left[\frac{\xi\psi(\xi)(1+\psi(\xi))e^{-\psi(\xi)}}{(1-e^{-\psi(\xi)})} - \frac{\xi(\psi(\xi))^2 e^{-2\psi(\xi)}}{(1-e^{-\psi(\xi)})^2}\right] - \frac{1}{\sigma^2}\sum_{i=1}^{m}\psi^*(\xi_{x_{(i)}})(1-\xi_{x_{(i)}})$$
$$-\sum_{i=1}^{m} R_i\Big(\frac{\xi_{x_{(i)}}\psi^*(\xi_{x_{(i)}})e^{-\psi^*(\xi_{x_{(i)}})}(1-\psi^*(\xi_{x_{(i)}}))}{\sigma^2(1-e^{-\psi^*(\xi_{x_{(i)}})})} - \frac{\xi_{x_{(i)}}(\psi^*(\xi_{x_{(i)}}))^2 e^{-2\psi^*(\xi_{x_{(i)}})}}{\sigma^4(1-e^{-\psi^*(\xi_{x_{(i)}})})^2} - \frac{(\psi^*(\xi_{x_{(i)}}))^2 e^{-2\psi^*(\xi_{x_{(i)}})}}{\sigma^4(1-e^{-\psi^*(\xi_{x_{(i)}})})}\Big).$$

Since the mathematical expectation of $I(\mu,\sigma)$ cannot be easily obtained, the observed Fisher information matrix, $I(\hat{\mu},\hat{\sigma})$, is used. It is obtained by replacing the expected value of the second derivatives of the logarithm of likelihood function by their MLEs. Thus, the approximate variance-covariance $[\hat{V}]$ matrix for the MLEs is obtained by inverting the observed information matrix $I(\hat{\mu},\hat{\sigma})$ as becomes

$$[\hat{V}] \cong I^{-1}(\hat{\mu},\hat{\sigma}) \cong \begin{bmatrix} V(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ Cov(\hat{\mu},\hat{\sigma}) & V(\hat{\sigma}) \end{bmatrix}.$$

According to [6], the asymptotic distribution of the MLEs $\hat{\mu}$ and $\hat{\sigma}$ is approximately distributed as the multivariate normal distribution, i.e. $\hat{\mu} \sim N(\mu, V(\hat{\mu}))$ and $\hat{\sigma} \sim N(\sigma, V(\hat{\sigma}))$. Hence, the $(1-p)100\%$ two-sided approximate confidence intervals for μ and σ are obtained respectively as:

$$\hat{\mu} \mp Z_{\frac{p}{2}}\sqrt{V(\hat{\mu})} \quad \text{and} \quad \hat{\sigma} \mp Z_{\frac{p}{2}}\sqrt{V(\hat{\sigma})},$$

respectively, where $V(\hat{\mu})$ and $V(\hat{\sigma})$ are the main diagonal elements of the approximate variance-covariance matrix and $Z_{p/2}$ is the percentile of the standard normal distribution with right tail probability $p/2$. To find out the variances of the survival and hazard functions, needed to construct their approximate confidence intervals, the delta method (developed by [17]) is used. Thereby, the variance of $\hat{S}(t)$ and $\hat{h}(t)$ can be obtained respectively as:

$$\hat{V}(\hat{S}(t)) = [\nabla\hat{S}(t)]^T[\hat{V}][\nabla\hat{S}(t)]$$

and

$$\hat{V}(\hat{h}(t)) = [\nabla\hat{h}(t)]^T[\hat{V}][\nabla\hat{h}(t)]$$

where T denotes the transpose symbol. $\Delta\hat{S}(t)$ and $\Delta\hat{h}(t)$ are the gradient (vector of first partial derivatives) of $S(t)$ and $h(t)$ with respect to μ and σ, obtained at $\mu = \hat{\mu}$ and $\sigma = \hat{\sigma}$, defined as:

$$[\Delta\hat{S}(t)]^T = \begin{bmatrix} \frac{\partial\Delta\hat{S}(t)}{\partial\mu} & \frac{\partial\Delta\hat{S}(t)}{\partial\sigma} \end{bmatrix}_{(\mu=\hat{\mu},\sigma=\hat{\sigma})},$$

and

$$[\Delta\hat{h}(t)]^T = \begin{bmatrix} \frac{\partial\Delta\hat{h}(t)}{\partial\mu} & \frac{\partial\Delta\hat{h}(t)}{\partial\sigma} \end{bmatrix}_{(\mu=\hat{\mu},\sigma=\hat{\sigma})}$$

where,

$$\frac{\partial\Delta S(t)}{\partial\mu} = \frac{1}{\sigma}\Big(\frac{e^{-\psi^*(\xi_t)-\xi_t}}{(1-e^{-\psi(\xi_t)})} - \frac{e^{-\psi(\xi)-\xi}(1-e^{\psi^*(\xi_t)})}{(1-e^{-\psi(\xi)})^2}\Big),$$
$$\frac{\partial\Delta S(t)}{\partial\sigma} = \frac{1}{\sigma}\Big(\frac{\xi_t e^{-\psi^*(\xi_t)}(1-e^{-\xi_t})}{(1-e^{-\psi^*(\xi_t)})} - \frac{\xi e^{-\psi(\xi)}(1-e^{-\xi_t})}{(1-e^{-\psi(\xi)})^2}\Big)$$
$$\frac{\partial\Delta h(t)}{\partial\mu} = \frac{1}{\sigma^2}\Big(\frac{(1+\psi^*(\xi_t))e^{-\psi^*(\xi_t)-\xi_t}}{(1-e^{-\psi^*(\xi_t)})} - \frac{\psi^*(\xi_t)e^{-2\psi^*(\xi_t)-\xi_t}}{(1-e^{-\psi^*(\xi_t)})^2}\Big),$$

and

$$\frac{\partial\Delta h(t)}{\partial\sigma} = \frac{1}{\sigma^2}\Big(\frac{\xi_t(1+\psi^*(\xi_t))e^{-\psi(\xi_t)}}{(1-e^{-\psi^*(\xi_t)})} - \frac{e^{-\psi^*(\xi_t)-\xi_t}(1-e^{-\psi^*(\xi_t)}(1+\xi_t\psi^*(\xi_t)))}{(1-e^{-\psi^*(\xi_t)})^2}\Big).$$

Thus, the $(1-p)100\%$ ACLs for $S(t)$ and $h(t)$ are found out by:

$$\hat{S}(t) \mp Z_{\frac{p}{2}}\sqrt{V(\hat{S}(t))} \quad \text{and} \quad \hat{h}(t) \mp Z_{\frac{p}{2}}\sqrt{V(\hat{h}(t)}$$

respectively.

4 Bayesian estimation

In this section, Bayes estimates of unknown parameters μ and σ together with the survival and hazard functions $S(t)$ and $h(t)$ are obtained in the light of a progressive type II censored scheme. It is assumed that the suitable prior distribution for μ is truncated normal distribution and for σ is the inverted gamma distribution. Therefore, the PDF of the prior distribution for μ and σ are:

$$\pi(\mu|\sigma) \propto \frac{e^{-\frac{b_1}{2\sigma}(\mu-a_1)^2}}{\sqrt{\sigma}\Phi(\frac{a_1\sqrt{b_1}}{\sqrt{\sigma}})}, \quad ,a_1,b_1>0,\mu,\sigma>0$$

and

$$\pi(\sigma) \propto \sigma^{-(a_2+1)}e^{-\frac{b_2}{2\sigma}}, \qquad \sigma>0,b_2>0. \tag{11}$$

respectively, where $\Phi(\cdot)$ denotes the CDF of standard normal distribution. a_1, a_2, b_1 and b_2 are the hyper-parameters assumed to be known and non-negative. The joint prior distribution of μ and σ from (11) are as follows:

$$\pi(\mu,\sigma) \propto \frac{\sigma^{-(a_2+1)}e^{-\frac{1}{2\sigma}(b_1(\mu-a_1)^2+b_2)}}{\sqrt{\sigma}\Phi(\frac{a_1\sqrt{b_1}}{\sqrt{\sigma}})} \quad a_1,b_1>0,a_2,b_2,\mu,\sigma>0 \tag{12}$$

Therefore, the joint posterior distribution of the unknown parameters μ and σ, denoted by $\pi(\mu,\sigma|\mathbf{x})$ can be obtained by multiplying the likelihood function (6) by the joint prior distribution (12) . It can be written as:

$$\begin{aligned}\pi(\mu,\sigma|\mathbf{x}) = &\frac{\sigma^{-(a_2+m+1)}}{K\sqrt{\sigma}\Phi\left(\frac{a_1\sqrt{b_1}}{\sqrt{\sigma}}\right)}\frac{e^{-\sum_{i=1}^{m}\left(\psi^*(\xi_{x_{(i)}})-\xi_{x_{(i)}}\right)}}{(1-e^{-\psi(\xi)})^n}\\ &\times e^{-\frac{1}{2\sigma}(b_1(\mu-a_1)^2+b_2)}\prod_{i=1}^{m}\left(1-e^{-\psi^*(\xi_{x_{(i)}})}\right)^{R_i}, \quad a_1,b_1>0,a_2,b_2,\mu,\sigma>0,\end{aligned} \tag{13}$$

where K is a normalizing constant equal to

$$\begin{aligned}K = \int_0^\infty\int_0^\infty &\frac{\sigma^{-(a_2+m+1)}}{\sqrt{\sigma}\Phi\left(\frac{a_1\sqrt{b_1}}{\sqrt{\sigma}}\right)}\frac{e^{-\sum_{i=1}^{m}\left(\psi^*(\xi_{x_{(i)}})-\xi_{x_{(i)}}\right)}}{(1-e^{-\psi(\xi)})^n}\\ &\times e^{-\frac{1}{2\sigma}(b_1(\mu-a_1)^2+b_2)}\prod_{i=1}^{m}\left(1-e^{-\psi^*(\xi_{x_{(i)}})}\right)^{R_i} d\mu d\sigma.\end{aligned}$$

The loss function determines the effects of financial loss resulting from an inaccurate estimate of the unknown parameter. A commonly used loss function is SE loss function; symmetrical loss function that assigns equal losses to overestimation and underestimation. Let $\tilde{\varphi}(\nu)$ denote an estimate of $\varphi(\nu)$, a function of the parameter ν. The SE loss function is defined as:

$$\mathscr{L}_{SE}(\tilde{\varphi}(\nu),\varphi(\nu)) = (\tilde{\varphi}(\nu)-\varphi(\nu))^2$$

Where $\mathscr{L}_{SE}(.)$ is the SE loss function. Bayes estimate $\tilde{\varphi}_{SE}(\nu)$ of $\varphi(\nu)$ under this loss function is considered the mean of the posterior distribution. Hence, Bayes estimate of any function of μ and σ, say $\hat{\varphi}_{SE}(\mu,\sigma)$, under the SE loss function is

$$\tilde{\varphi}_{SE}(\mu,\sigma) = E(\varphi(\mu,\sigma)|\mathbf{x})$$

where

$$E(\varphi(\mu,\sigma)|\mathbf{x}) = \frac{\int_0^\infty\int_0^\infty \varphi(\mu,\sigma)\pi(\mu,\sigma|\mathbf{x})d\mu d\sigma}{\int_0^\infty\int_0^\infty \pi(\mu,\sigma|\mathbf{x})d\mu d\sigma}. \tag{14}$$

The GE loss function, which is an asymmetric loss function, was introduced by [14]. This function can be expressed as follows

$$\mathscr{L}_{GE}(\tilde{\varphi}(v),\varphi(v)) \propto (\frac{\tilde{\varphi}(v)}{\varphi(v)})^c - c\ln(\frac{\tilde{\varphi}(v)}{\varphi(v)}) - 1, \quad c \neq 0,$$

where c is a constant term. Bayes estimate $\tilde{\varphi}_{GE}(\mu,\sigma)$ of $\varphi(\mu,\sigma)$ under GE loss function is

$$\tilde{\varphi}_{GE}(\mu,\sigma) = ((E(\varphi(\mu,\sigma))^{-c}|\mathbf{x}))^{-\frac{1}{c}}$$

where

$$E(\varphi(\mu,\sigma)^{-c}|\mathbf{x})) = \frac{\int_0^\infty \int_0^\infty (\varphi(\mu,\sigma))^{-c}\pi(\mu,\sigma|\mathbf{x})d\sigma d\mu}{\int_0^\infty \int_0^\infty \pi(\mu,\sigma|\mathbf{x})d\sigma d\mu}. \tag{15}$$

Obviously, the calculation of integrals given by (14) and (15) cannot result in an explicit form. To obtain Bayes estimates of μ, σ , $S(t)$, and $h(t)$, MCMC algorithm is used to generate samples from joint posterior distribution. Further information can be found in references such as [15] and [16]. From (13), the conditional posterior distribution of μ given σ and $\mathbf{x}$, as follows

$$\pi(\mu|\sigma,\mathbf{x}) \propto \frac{e^{-\frac{b_1}{2\sigma}((\mu-a_1)^2)} e^{-\sum_{i=1}^m \psi^*(\xi_{x_{(i)}}) - \sum_{i=1}^m \xi_{x_{(i)}}}}{(1-e^{-\psi(\xi)})^n} \times \prod_{i=1}^m (1-e^{-\psi^*(\xi_{x_{(i)}})})^{R_i}, \tag{16}$$

and the conditional posterior distribution of σ given μ and $\mathbf{x}$ is

$$\pi(\sigma|\mu,\mathbf{x}) \propto \frac{\sigma^{-(a_2+m+1)} e^{-\frac{1}{2\sigma}(b_1(\mu-a_1)^2+b_2)}}{\sqrt{\sigma}\Phi(\frac{a_1\sqrt{b_1}}{\sqrt{\sigma}})} \frac{e^{-\sum_{i=1}^m \psi^*(\xi_{x_{(i)}}) - \sum_{i=1}^m \xi_{x_{(i)}}}}{(1-e^{-\psi(\xi)})^n} \prod_{i=1}^m (1-e^{-\psi^*(\xi_{x_{(i)}})})^{R_i}. \tag{17}$$

From (16) and (17), it is evident that the conditional posterior distributions of μ and σ do not belong to the standard form of statistical distributions. Handling such problem, M-H algorithm with normal proposal distribution is applied. The steps of M-H algorithm are as follows

Step 1. Start some initial guesses of μ and σ say $\mu^{(0)}$ and $\sigma^{(0)}$ respectively

Step 2. Put $j = 1$

Step 3. Generate $\mu^{(j)}$ and $\sigma^{(j)}$ from $\pi(\mu^{(j-1)}|\sigma^{(j-1)},\mathbf{x})$ and $\pi(\sigma^{(j-1)}|\mu^{(j)},\mathbf{x})$ with normal proposal distributions $N(\mu^{(j-1)},V(\mu))$ and $N(\sigma^{(j-1)},V(\sigma))$ respectively, where, $V(\mu)$ and $V(\sigma)$ are obtained from the main diagonal in approximate variance-covariance matrix.

Step 4. Generate a proposal μ^* from $N(\mu^{(j-1)},V(\mu))$ and σ^* from $N(\sigma^{(j-1)},V(\sigma))$.

Step 5. Compute the acceptance probabilities

$$\eta_\mu^* = \min\left(1, \frac{\pi(\mu^*|\sigma^{(j-1)},\mathbf{x})}{\pi(\mu^{(j-1)}|\sigma^{(j-1)},\mathbf{x})}\right),$$

and

$$\eta_\sigma^* = \min\left(1, \frac{\pi(\sigma^*|\mu^{(j)},\mathbf{x})}{\pi(\sigma^{(j-1)}|\mu^{(j)},\mathbf{x})}\right).$$

Step 6. Generate u_1 and u_2 from uniform distribution.

i.If $u_1 < \eta_\mu^*$ accept the proposal and set $\mu^{(j)} = \mu^*$, otherwise put $\mu^{(j)} = \mu^{(j-1)}$.

iiIf $u_2 < \eta_\sigma^*$ accept the proposal and set $\sigma^{(j)} = \sigma^*$, otherwise put $\sigma^{(j)} = \sigma^{(j-1)}$.

Step 7. Compute the survival function $S^{(j)}(t)$ and hazard function $h^{(j)}(t)$ as

$$S^{(j)}(t) = \frac{(1-e^{\psi^{*(j)}(\xi_t)})}{(1-e^{-\psi^{(j)}(\xi)})}, \qquad \mu,\sigma > 0, t > 0$$

and

$$h^{(j)}(t) = \frac{e^{-\psi^{*(j)}(\xi_t)-\xi_t}}{\sigma(1-e^{-\psi^{*(j)}(\xi_t)})}, \qquad \mu,\sigma > 0, t > 0.$$

Step 8. Set $j = j+1$.

Step 9. Repeat steps 3-8, M times and get $\mu^{(j)}$ and $\sigma^{(j)}$, $j = 1,2,...,M$.

In order to guarantee the convergence and to avoid the bias of the selection of initial guess the first simulated variates $M_\circ$ are deleted. Therefore, the selected samples $\mu^{(j)}, \sigma^{(j)}, S^{(j)}(t)$ and $h^{(j)}(t)$ for $j = M_\circ + 1, ..., M$.. For sufficiently large M Bayes estimates of $\phi = \mu, \sigma, S(t), h(t)$ under SE and GE loss function as

$$\tilde{\phi}_{MCMC} = \frac{1}{M - M_\circ} \sum_{j=M_\circ+1}^{M} \phi_\tau^{(j)}, \quad \tau = 1,2,3,4, \tag{18}$$

$$\tilde{\phi}_{MCMC} = \left(\frac{1}{M - M_\circ} \sum_{j=M_\circ+1}^{M} (\phi_\tau^{(j)})^{-c} \right)^{-\frac{1}{c}}, \quad \tau = 1,2,3,4, \tag{19}$$

respectively, where, $\phi_1 = \mu$, $\phi_2 = \sigma$, $\phi_3 = S(t)$ and $\phi_4 = h(t)$.

5 Monte Carlo simulation

In this section, extensive Monte Carlo simulations are conducted with the aim of comparing the performance of the previously proposed estimators of unknown parameters and examining the reliability characteristics $S(t)$ and $h(t)$ of the LTG distribution under progressive type-II censoring. Following the algorithm [13], 1,000 times progressive type-II censored samples are generated from the $LTG(\mu, \sigma)$ distribution by considering two different sets for the true values of μ and σ as $LTG(0.4, 0.8)$ and $LTG(2.0, 1.5)$. Thus, the corresponding actual values of the survival parameters $S(t)$ and $h(t)$ are 0.867 and 0.644 (at $t = 0.25$) when $LTG(0.4, 0.8)$ are 0.877 and 0.216 (at $t = 1$) when $LTG(2.0, 1.5)$. Considering $n = 50$ and 100, three different choices of the effective sample size m are used. As the progressive type-II censored life test is terminated when the number of failed subjects reaches (or exceeds) a certain value m, three percentages of failure information ($\frac{m}{n} \times 100$) such as 30%, 60%, and 90% are considered to determine the total number of failure units. Moreover, to assess the behavior of removal patterns $R_i, i = 1, 2, \ldots, m$ for both n and m, various censoring schemes (CSs) are considered namely:

$$\text{CS-1:} R_1 = n - m, \quad R_i = 0 \quad \text{for} \quad i \neq 1$$

$$\text{CS-2:} \begin{cases} R_{\frac{m}{2}} = n - m, R_i = 0 & \text{for } i \neq \frac{m}{2}, \text{ if } m \text{ even} \\ R_{\frac{m+1}{2}} = n - m, R_i = 0 & \text{for } i \neq \frac{m+1}{2}, \text{if } m \text{ odd} \end{cases}$$

$$\text{CS-3:} R_m = n - m, \quad R_i = 0 \quad \text{for} \quad i \neq m$$

To assign values for the hyper-parameters of the conjugate truncated normal and inverted gamma priors, two different informative sets $a_i, b_i, i = 1, 2$, for each set of μ and σ are used, namely:

– Prior-I: $(a_1, a_2, b_1, b_2) = (0.1, 3, 2, 3)$ and Prior-II: $(a_1, a_2, b_1, b_2) = (0.1, 4, 2, 5)$ when LTG(0.4,0.8).
– Prior-I: $(a_1, a_2, b_1, b_2) = (2, 3, 2, 6)$ and Prior-II: $(a_1, a_2, b_1, b_2) = (2, 4, 2, 9)$ when LTG(2.0,1.5).

It is worth mentioning that the values of $a_i, b_i, i = 1, 2$, are specified in such a way that the prior mean becomes the expected value of the target parameter, see [6]. Also, when $LTG(0.4, 0.8)$, the associated variances of μ ad σ using Prior-I are 0.160 and 0.563 and using Prior-II are 0.160 and 0.347 respectively. Similarly, when $LTG(2.0, 1.5)$ the associated variances of μ and σ using Prior-I are 0.701 and 2.250 and using Prior-II, they are 0.701 and 1.125, respectively. It is noteworthy that when the improper prior is available, the posterior distribution is similar to the proportional likelihood function. Hence, if prior information about the unknown parameters of interest is lacked, it is recommended to use the MLEs instead of Bayesian estimates. To calculate Bayesian estimators using M-H algorithm, 12,000 MCMC samples are generated, and then the first 2,000 simulated values of each unknown parameter are ignored. Therefore, utilizing 10,000 MCMC samples, the average Bayes estimates using SE and GE (for) $c = (-2, 0.02, +2)$ loss functions along with their associated 95% credible intervals are obtained. The MLEs of μ and σ are used as initial values in order to run the MCMC sampler. Therefore, it is observed that Markov chains reach the stationary condition quickly. For computational illustration, the average estimates, $\overline{\hat{\phi}}^*_\tau$, of $\phi = (\mu, \sigma, S(t), h(t))$ are given by

$$\overline{\hat{\phi}}^*_\tau = \frac{1}{G} \sum_{j=1}^{G} \hat{\phi}_\tau^{(j)}, \quad \tau = 1,2,3,4,$$

where, G is the number of replications, $\hat{\phi}$ is the desired estimate of ϕ, $\hat{\phi}_\tau^{(j)}$ denotes the calculated estimate of ϕ at j^{th} sample $\phi_1 = \mu$, $\phi_2 = \sigma$, $\phi_3 = S(t)$ and $\phi_4 = h(t)$. Using two criteria called RMSEs and MABs , point estimates values are compared using the following formula:

$$\mathrm{RMSE}(\hat{\phi}_\tau) = \sqrt{\frac{1}{G}\sum_{j=1}^{G}(\hat{\phi}_\tau^{(j)} - \phi_\tau)^2} \quad \text{and} \quad \mathrm{MAB}(\hat{\phi}_\tau) = \frac{1}{G}\sum_{j=1}^{G}\left|\hat{\phi}_\tau^{(j)} - \phi_\tau)^2\right| \quad , \tau = 1,2,3,4,$$

respectively. Moreover, the performance of 95% asymptotic/credible interval estimates is compared using their ACLs and CPs as

$$\mathrm{ACL}_{(1-p)}(\hat{\phi}_\tau) = \frac{1}{G}\sum_{j=1}^{G}\left(U^*(\hat{\phi}_\tau^{(j)}) - L^*(\hat{\phi}_\tau^{(j)})\right), \quad \text{and} \quad \mathrm{CP}_{(1-p)}(\phi_\tau) = \frac{1}{G}\sum_{j=1}^{G} I^*\left(L^*(\hat{\phi}_\tau^{(j)}), U^*(\hat{\phi}_\tau^{(j)})\right), \quad \tau = 1,2,3,4,$$

and respectively, where, $I^*(.)$ is the indicator function. $L^*(.)$ and $U^*(.)$ denote the lower and upper bounds of $(1-p)100\%$ the asymptotic (or credible) interval, respectively. All computational algorithms are coded in R statistical programming language software version 4.1.2 via two packages, namely the maxLik package by [5] and the coda package proposed by [16] . These packages have been recently recommended by [2] and [4]. The average estimates , RMSEs and, MABs of $\mu, \sigma, S(t)$ and $h(t)$ are reported in Tables 1-8. Furthermore, the ACLs and CPs for 95% interval estimates $\mu, \sigma, S(t)$ and $h(t)$ are listed in Tables 9-12. From Tables 1-12, it is observed that:

– Generally, both point and interval estimates of the unknown parameters and reliability characteristics of the LTG distribution have the advantage the smallest RMSEs, MABs,and ACLs and the highest CPs.

– As n (or m) increases, the performances of the proposed estimates in terms of minimum RMSEs and MABs become even better as expected. Similarly, the total number of progressive censoring **R** decreases.

– Bayes MCMC estimates perform better than those based on the likelihood function. Similarly, credible intervals are better than asymptotic intervals for all unknown parameters.

– Because the variance of Prior-II is less than Prior-I, Bayes estimates under Prior-II have performed more efficiently in terms of the smallest RMSEs, MABs, and ACLs as well as the highest CPs.

– In most cases, it is obvious that the estimates of all unknown parameters relative to GE loss function performed better if compared to the SE loss function in respect to the lowest RMSEs and MABs values. This indicates that the use of SE loss function gives an equal weight to underestimation and overestimation because of its symmetrical nature.

– Comparing the censoring scheme 1, 2, and 3 of the point estimates with respect to the RMSEs and MABs of the classical and Bayes estimates of μ and $S(t)$, CS-1 is smaller than CS-3. While for σ and $h(t)$, the errors are smaller using CS-3 than CS-1. Similarly, the behavior of the best CS is observed for the interval estimates of μ, σ, $S(t)$, and $h(t)$ in respect of the lowest ACLs and the highest CPs.

– As μ and σ increase, the RMSEs, MABs and ACLs for the MLEs of μ decrease while the CPs increase. It is also noticed that the RMSEs, MABs and ACLs for Bayes MCMC estimates of μ increase while the CPs decrease.

– As μ and σ increase, the RMSEs, MABs and ACLs for all estimates of σ and $S(t)$ increase while those associated with $h(t)$ decrease. As μ and σ increase, the CPs for all estimates of σ and $S(t)$ decrease while those associated with $h(t)$ tend to increase.

– Extending the results of [12] in the case of LTG distribution under complete sampling to incomplete (progressive type-II censoring) sampling and may be obtained as a special case by putting $R_i = 0, i = 1,2,\ldots,m$ and $m = n$.

– According to the stated of observations, it is recommended to apply a Bayesian estimation that applies M-H algorithm to estimate the unknown parameters or the reliability characteristics of the LTG distribution under progressively type-II censoring.

6 Numerical examples

This section analyzes both simulated and real data sets to indicate how the proposed methodologies can be applied in real phenomenon.

6.1 Example 1: (Simulated data)

In this example, using Monte Carlo simulation, some different progressive type-II censored samples for various choices of n, m and R are generated from LTG distribution for $(\mu,\sigma)=(1,2)$. All simulated samples ($S_i^{(m:n)}, i=1,2,3$ for each) are reported in Table 13. Briefly, the censoring scheme $R=(1,0,0,0,1)$ has been referred to by $R=(1,0^{*}3,1)$. Using generated samples, the maximum likelihood estimates along with their two-sided asymptotic confidence intervals of the unknown parameters μ, σ, $S(t)$ and $h(t)$ are calculated. The corresponding actual value of $S(t)$ and $h(t)$ at distinct time $t=1.5$ becomes 0.670 and 0.330 respectively. Furthermore, Bayes estimates are obtained by both SE and GE (for $c=(-5,+5)$) loss functions using three different sets of the hyper-parameters a_i, b_i, $i=1,2$ namely; Prior-I: $a_i,b_i=0,i=1,2$; Prior-II: $(a_1,a_2)=(2,4)$ and $b_i=2,i=1,2$ and Prior-III: $(a_1,a_2)=(2,4)$ and $b_i=5,i=1,2$.

Furthermore, Bayes estimates are obtained by both SE and GE (for $c=(-5,+5)$) loss functions using three different sets of the hyper-parameters a_i, b_i, $i=1,2$ namely; Prior-I: $a_i,b_i=0,i=1,2$; Prior-II: $(a_1,a_2)=(2,4)$ and $b_i=2,i=1,2$ and Prior-III: $(a_1,a_2)=(2,4)$ and $b_i=5,i=1,2$. To develop Bayes estimates and their BCIs, 30,000 MCMC samples are simulated with ignoring the first 5,000 iterations. The calculated maximum likelihood estimates of μ and σ are used as initial guess values to run the MCMC algorithm. The point estimates (with their standard errors) and the interval estimates (with their standard lengths) of μ, σ, $S(t)$, and $h(t)$ are computed and listed in Tables 14 and 15 respectively.

6.2 Example 2: (Real data)

To indicate how the proposed estimation methods can be utilized in a practical situation, a real-life data set is analyzed. It consists of the time between failures for thirty repairable mechanical equipment items. The times of these items in order are: 0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23, 1.24, 1.43, 1.46, 1.49, 1.74, 1.82, 1.86, 1.97, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73. This data set was originally provided by [9] and has been recently discussed by [4].

A resulting question emerges about whether the equipment data set fits the LTG distribution or not. In order to check the validity of the proposed distribution, The MLEs are used to obtain Kolmogorov-Smirnov distance and its P-value based on the complete equipment data set. The MLEs $\hat{\mu}$ and $\hat{\sigma}$ along with standard errors of the LTG parameters μ and σ are 0.9545(0.2289) and 0.8530(0.1668) respectively. Hence, Kolmogorov-Smirnov distance is 0.079 with P-value 0.992. This implies that the LTG distribution highly fits the equipment data. Furthermore, analytical steps of likelihood iterations cannot frequently verify the existence and uniqueness of the computed MLEs. Using the complete equipment data set, a contour plot of the log-likelihood function with respect to the unknown parameters μ and σ is graphed as displayed in Figure 1 . It indicates that the most suitable starting values of μ and σ are quite close to 0.9545 and 0.8530 respectively. Thereby, the existence and uniqueness of the MLEs $\hat{\mu}$ and $\hat{\sigma}$. According to the methodologies discussed in this study, the complete equipment data set based on different choices of m and R, is used to generate six different progressive type-II censored samples as indicated in Table 17. Because prior information about the LTG parameters is lacked, Bayes estimates are obtained via MCMC sampler using improper priors, i.e., $a_i,b_i=0,i=1,2$ relative to SE and GE (for $c=(-3,+3)$). Therefore, the hyper-parameter values are taken to be 0.0001. Using the M-H algorithm, $30,000$ MCMC iterations are performed with discarding the first $5,000$ draws as a burn-in sample.

Table 1: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of μ when $(\mu,\sigma) = (0.4, 0.8)$ using MLEs and Bayesian estimates under choices of m and n
.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
$c\rightarrow$						-2	-0.02	2	-2	-0.02	2
50	15	1	1.6863	0.7377	0.3424	0.7537	0.7300	0.6913	0.3437	0.3412	0.3385
			4.6900	0.3805	0.0646	0.3634	0.3314	0.2945	0.0615	0.0588	0.0563
			0.8957	0.3479	0.0578	0.3628	0.3300	0.2913	0.0615	0.0588	0.0563
		2	1.2001	0.7605	0.3422	0.7739	0.7506	0.7205	0.3435	0.3410	0.3383
			5.3515	0.3891	0.0648	0.3763	0.3515	0.3286	0.0617	0.0590	0.0565
			1.3734	0.3644	0.0580	0.3759	0.3506	0.3286	0.0617	0.0590	0.0565
		3	1.7780	0.7656	0.3365	0.7784	0.7555	0.7272	0.3378	0.3351	0.3322
			5.9257	0.3922	0.0704	0.3790	0.3564	0.3333	0.0678	0.0649	0.0622
			2.3929	0.3682	0.0560	0.3787	0.3555	0.3333	0.0678	0.0649	0.0622
	30	1	0.8874	0.5451	0.3630	0.5582	0.5319	0.5050	0.4374	0.4366	0.4357
			2.2061	0.1840	0.0433	0.2080	0.1235	0.0975	0.0379	0.0374	0.0362
			0.5924	0.1481	0.0390	0.2079	0.1234	0.0970	0.0379	0.0374	0.0362
		2	1.0066	0.5390	0.3674	0.5523	0.5256	0.4982	0.4392	0.4383	0.4375
			2.3277	0.1886	0.0444	0.2158	0.1258	0.0988	0.0392	0.0383	0.0375
			0.7245	0.1526	0.0373	0.2158	0.1256	0.0982	0.0392	0.0383	0.0375
		3	1.5799	0.5363	0.3691	0.5492	0.5234	0.4970	0.4407	0.4398	0.4390
			4.4276	0.1805	0.0446	0.2185	0.1321	0.1055	0.0407	0.0399	0.0390
			1.2676	0.1452	0.0404	0.2185	0.1319	0.1050	0.0407	0.0398	0.0390
	45	1	0.7435	0.4968	0.4370	0.3480	0.2723	0.4704	0.3638	0.3621	0.3603
			1.9093	0.1370	0.0408	0.1492	0.1277	0.0105	0.0339	0.0318	0.0299
			0.4711	0.1067	0.0321	0.1492	0.1277	0.0086	0.0339	0.0318	0.0299
		2	0.8147	0.4473	0.4387	0.3548	0.2840	0.4237	0.3683	0.3664	0.3644
			2.0064	0.1519	0.0420	0.1523	0.1160	0.0258	0.0356	0.0336	0.0317
			0.5446	0.1308	0.0335	0.1523	0.1160	0.0237	0.0356	0.0336	0.0317
		3	0.8359	0.4026	0.4402	0.3569	0.2835	0.3779	0.3701	0.3682	0.3661
			2.6797	0.1520	0.0416	0.1583	0.1165	0.0721	0.0397	0.0379	0.0362
			0.5627	0.1314	0.0374	0.1582	0.1165	0.0704	0.0397	0.0379	0.0362
100	30	1	0.9662	0.7469	0.3630	0.7628	0.7209	0.6823	0.3462	0.3444	0.3426
			2.3990	0.3720	0.0588	0.3544	0.3223	0.2854	0.0561	0.0544	0.0527
			0.6715	0.3388	0.0538	0.3537	0.3209	0.2823	0.0559	0.0542	0.0525
		2	1.0473	0.7636	0.3718	0.7759	0.7465	0.7141	0.3475	0.3458	0.3441
			4.0184	0.3884	0.0604	0.3744	0.3475	0.3167	0.0576	0.0558	0.0540
			0.7653	0.3614	0.0551	0.3739	0.3465	0.3141	0.0574	0.0556	0.0538
		3	1.7780	0.7675	0.3783	0.7787	0.7521	0.7206	0.3453	0.3435	0.3416
			5.6225	0.3906	0.0612	0.3789	0.3531	0.3232	0.0585	0.0567	0.0549
			1.7058	0.3664	0.0637	0.3784	0.3521	0.3206	0.0584	0.0565	0.0547
	60	1	0.7067	0.3148	0.3453	0.3404	0.2858	0.0914	0.3638	0.3621	0.3603
			2.1399	0.1628	0.0366	0.3086	0.1143	0.0596	0.0254	0.0229	0.0206
			0.4002	0.1368	0.0266	0.3086	0.1142	0.0596	0.0254	0.0229	0.0206
		2	0.6737	0.3242	0.3466	0.3371	0.2839	0.0714	0.3728	0.3707	0.3685
			2.2694	0.1711	0.0395	0.3229	0.1162	0.0629	0.0315	0.0293	0.0272
			0.4331	0.1445	0.0301	0.3205	0.1161	0.0629	0.0315	0.0293	0.0272
		3	1.2060	0.3256	0.3444	0.3360	0.2836	0.0667	0.3794	0.3771	0.3746
			3.8608	0.1640	0.0444	0.3294	0.1165	0.0640	0.0377	0.0366	0.0357
			0.9201	0.1378	0.0373	0.3272	0.1164	0.0640	0.0377	0.0366	0.0357
	90	1	0.4742	0.3159	0.3825	0.5059	0.4880	0.1815	0.3832	0.3817	0.3801
			0.6645	0.0830	0.0302	0.0233	0.0150	0.0432	0.0200	0.0184	0.0168
			0.2189	0.0673	0.0248	0.0221	0.0119	0.0431	0.0199	0.0183	0.0168
		2	0.4977	0.3132	0.3823	0.4555	0.4394	0.1842	0.3831	0.3815	0.3800
			0.9959	0.0991	0.0302	0.0570	0.0410	0.0453	0.0201	0.0185	0.0170
			0.2461	0.0753	0.0248	0.0555	0.0394	0.0452	0.0200	0.0185	0.0169
		3	0.5127	0.3122	0.3823	0.4110	0.3944	0.1921	0.3831	0.3815	0.3799
			1.3940	0.1222	0.0302	0.1072	0.0894	0.0521	0.0201	0.0186	0.0170
			0.2618	0.1010	0.0248	0.1059	0.0880	0.0520	0.0201	0.0185	0.0169

Table 2: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of μ when $(\mu,\sigma) = (2.0, 1.5)$ using MLEs and Bayesian estimates under choices of m and n
.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
c →						-2	-0.02	2	-2	-0.02	2
50	15	1	1.7698	2.5135	1.8526	2.5285	2.5058	2.4816	1.8537	1.8515	1.8492
			0.8375	0.5713	0.1609	0.5300	0.5077	0.4842	0.1509	0.1486	0.1464
			0.3268	0.5184	0.1474	0.5285	0.5058	0.4816	0.1508	0.1485	0.1463
		2	1.8870	2.5652	1.8419	2.6378	2.6177	2.5961	1.8431	1.8407	1.8383
			2.0389	0.6695	0.1716	0.6388	0.6191	0.5981	0.1618	0.1594	0.1570
			0.5089	0.6284	0.1581	0.6378	0.6177	0.5961	0.1617	0.1593	0.1569
		3	1.3687	2.5932	1.8322	2.6466	2.6304	2.6128	1.8334	1.8310	1.8285
			6.4963	0.6718	0.1810	0.6474	0.6315	0.6144	0.1717	0.1692	0.1667
			0.8182	0.6389	0.1678	0.6466	0.6304	0.6128	0.1715	0.1690	0.1666
	30	1	1.9741	2.1439	2.0804	1.7156	1.6744	1.6316	2.0807	2.0801	2.0794
			0.3139	0.4049	0.0822	0.3020	0.2740	0.2462	0.0738	0.0732	0.0725
			0.2361	0.3369	0.0740	0.3017	0.2735	0.2455	0.0738	0.0731	0.0725
		2	1.9757	2.1363	2.0734	1.7545	1.7265	1.6983	2.0738	2.0731	2.0725
			0.3414	0.3422	0.0886	0.3031	0.2757	0.2488	0.0768	0.0761	0.0755
			0.2368	0.2912	0.0806	0.3028	0.2752	0.2481	0.0767	0.0761	0.0754
		3	1.9692	2.1351	2.0764	1.7519	1.7248	1.6972	2.0767	2.0761	2.0754
			0.3773	0.3414	0.0973	0.3689	0.3264	0.2855	0.0808	0.0801	0.0794
			0.2797	0.2886	0.0920	0.3684	0.3256	0.2844	0.0807	0.0801	0.0794
	45	1	1.9811	2.0370	1.9516	1.9053	1.8977	1.8896	1.9524	1.9508	1.9492
			0.2746	0.1553	0.0681	0.1105	0.1012	0.0936	0.0305	0.0285	0.0266
			0.2159	0.1252	0.0512	0.1104	0.1012	0.0935	0.0303	0.0282	0.0263
		2	1.9818	2.0220	1.9698	1.9053	1.8976	1.8896	1.9707	1.9688	1.9668
			0.2748	0.1560	0.0684	0.1105	0.1024	0.0948	0.0334	0.0314	0.0295
			0.2156	0.1264	0.0506	0.1104	0.1024	0.0947	0.0332	0.0312	0.0293
		3	1.9821	2.0099	1.9728	1.9065	1.8988	1.8908	1.9737	1.9718	1.9697
			0.2859	0.1560	0.0740	0.1093	0.1024	0.0947	0.0509	0.0492	0.0476
			0.2246	0.1260	0.0550	0.1092	0.1023	0.0947	0.0508	0.0492	0.0476
100	30	1	1.9645	2.5171	1.8683	2.5247	2.5023	2.4786	1.8692	1.8674	1.8656
			0.2570	0.5671	0.1385	0.5262	0.5042	0.4811	0.1292	0.1276	0.1260
			0.2014	0.5148	0.1270	0.5247	0.5023	0.4786	0.1288	0.1271	0.1255
		2	1.9646	2.6278	1.8737	2.5766	2.5537	2.5290	1.8745	1.8729	1.8712
			0.3639	0.6162	0.1443	0.5780	0.5555	0.5315	0.1348	0.1331	0.1313
			0.2695	0.5662	0.1323	0.5766	0.5537	0.5290	0.1344	0.1326	0.1308
		3	1.9133	2.6385	1.8597	2.6044	2.5819	2.5576	1.8607	1.8588	1.8569
			0.6700	0.6416	0.1529	0.6057	0.5837	0.5601	0.1436	0.1417	0.1398
			0.2293	0.5942	0.1409	0.6044	0.5819	0.5576	0.1431	0.1412	0.1393
	60	1	1.9911	1.6950	1.9530	2.1480	2.1399	2.1318	1.9538	1.9521	1.9504
			0.2002	0.1890	0.0738	0.1396	0.1317	0.1239	0.0107	0.0086	0.0072
			0.1907	0.1488	0.0546	0.1392	0.1311	0.1231	0.0093	0.0075	0.0063
		2	1.9896	1.7404	1.9958	2.1404	2.1323	2.1241	1.9971	1.9944	1.9915
			0.2022	0.1908	0.0744	0.1408	0.1328	0.1249	0.0397	0.0359	0.0319
			0.1552	0.1496	0.0649	0.1404	0.1323	0.1241	0.0384	0.0344	0.0303
		3	1.9876	1.7382	2.0364	2.1392	2.1311	2.1231	2.0384	2.0344	2.0303
			0.2471	0.1958	0.0849	0.1483	0.1404	0.1325	0.0496	0.0479	0.0463
			0.1567	0.1562	0.0768	0.1480	0.1399	0.1318	0.0496	0.0479	0.0462
	90	1	1.9950	1.9015	1.9976	2.0391	2.0350	2.0310	1.9984	1.9967	1.9949
			0.1967	0.0883	0.0590	0.0180	0.0152	0.0130	0.0064	0.0070	0.0065
			0.1545	0.0698	0.0502	0.0144	0.0125	0.0110	0.0053	0.0056	0.0053
		2	1.9950	1.9014	1.9969	2.0240	2.0201	2.0163	1.9978	1.9961	1.9943
			0.2057	0.0932	0.0590	0.0283	0.0247	0.0211	0.0065	0.0072	0.0082
			0.1615	0.0707	0.0502	0.0240	0.0201	0.0170	0.0053	0.0057	0.0066
		3	1.9952	1.9026	1.9969	2.0117	2.0080	2.0044	1.9978	1.9961	1.9943
			0.1967	0.1000	0.0595	0.0418	0.0377	0.0338	0.0082	0.0072	0.0079
			0.1545	0.0752	0.0508	0.0391	0.0350	0.0310	0.0066	0.0057	0.0063

Table 3: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of σ when $(\mu,\sigma) = (0.4,0.8)$ using MLEs and Bayesian estimates under choices of m and n
.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
$c\rightarrow$						-2	-0.02	2	-2	-0.02	2
50	15	1	0.8027	0.9009	0.7210	0.9176	0.8839	0.8474	0.7214	0.7205	0.7196
			0.2539	0.5869	0.0911	0.5536	0.5122	0.4673	0.0790	0.0781	0.0772
			0.1991	0.5382	0.0733	0.5536	0.5122	0.4672	0.0789	0.0780	0.0771
		2	0.7680	0.9119	0.7224	0.9297	0.8940	0.8556	0.7229	0.7220	0.7211
			0.3008	0.5925	0.0913	0.5637	0.5215	0.4759	0.0804	0.0795	0.0786
			0.2320	0.5428	0.0791	0.5637	0.5215	0.4758	0.0804	0.0795	0.0786
		3	0.7918	1.3401	0.8455	1.3602	1.3195	1.2758	0.8462	0.8448	0.8434
			0.4940	0.5403	0.0579	0.5111	0.4724	0.4312	0.0744	0.0709	0.0674
			0.3645	0.4919	0.0470	0.5111	0.4724	0.4311	0.0742	0.0707	0.0672
	30	1	0.7993	1.3332	0.8459	1.3536	1.3122	1.2672	0.8466	0.8452	0.8437
			0.1888	0.2128	0.0615	0.1298	0.0940	0.0878	0.0472	0.0458	0.0443
			0.1493	0.1693	0.0505	0.1297	0.0940	0.0878	0.0466	0.0452	0.0437
		2	0.8089	1.2919	0.8492	1.3111	1.2724	1.2311	0.8499	0.8484	0.8469
			0.2112	0.5824	0.0903	0.5586	0.5176	0.4736	0.0742	0.0708	0.0673
			0.1651	0.5332	0.0734	0.5585	0.5176	0.4735	0.0741	0.0706	0.0671
		3	0.7932	0.8110	0.7884	0.8240	0.7986	0.7751	0.7896	0.7872	0.7849
			0.2521	0.2013	0.0559	0.1176	0.0840	0.0475	0.0468	0.0454	0.0439
			0.1895	0.1598	0.0450	0.1176	0.0839	0.0474	0.0462	0.0448	0.0434
	45	1	0.8009	0.8104	0.7850	0.7293	0.7206	0.7123	0.8203	0.8190	0.8178
			0.1518	0.1475	0.0457	0.0787	0.0691	0.0556	0.0203	0.0191	0.0178
			0.1197	0.1221	0.0403	0.0787	0.0691	0.0556	0.0203	0.0190	0.0178
		2	0.8039	0.8099	0.7864	0.7309	0.7213	0.7122	0.8185	0.8173	0.8161
			0.1604	0.1473	0.0832	0.0878	0.0793	0.0708	0.0505	0.0490	0.0475
			0.1256	0.1224	0.0614	0.0877	0.0793	0.0707	0.0499	0.0484	0.0469
		3	0.7913	1.3382	0.8455	0.8463	0.8405	0.8345	0.8741	0.8706	0.8671
			0.1622	0.1466	0.0454	0.0471	0.0412	0.0353	0.0186	0.0174	0.0162
			0.1282	0.1212	0.0399	0.0463	0.0405	0.0345	0.0185	0.0173	0.0161
100	30	1	0.7975	1.3428	0.8444	1.3637	1.3215	1.2755	0.8742	0.8707	0.8673
			0.1800	0.5567	0.0817	0.5265	0.4852	0.4406	0.0748	0.0719	0.0684
			0.1402	0.5061	0.0777	0.5265	0.4852	0.4406	0.0748	0.0717	0.0683
		2	0.8123	1.3061	0.8421	1.3265	1.2852	1.2406	0.8742	0.8707	0.8672
			0.2372	0.5883	0.0832	0.5602	0.5196	0.4756	0.0804	0.0795	0.0746
			0.1816	0.5401	0.0791	0.5602	0.5195	0.4755	0.0804	0.0795	0.0746
		3	0.8656	0.7246	0.8186	0.7286	0.7207	0.7131	0.7214	0.7205	0.7196
			0.4214	0.2013	0.0567	0.1176	0.0869	0.0840	0.0468	0.0454	0.0439
			0.3066	0.1598	0.0460	0.1176	0.0869	0.0839	0.0462	0.0448	0.0434
	60	1	0.7976	0.7249	0.8196	0.8486	0.8428	0.8368	0.8451	0.8437	0.8423
			0.1419	0.1102	0.0549	0.0501	0.0444	0.0406	0.0435	0.0420	0.0386
			0.1122	0.0954	0.0440	0.0494	0.0437	0.0400	0.0428	0.0414	0.0378
		2	0.8042	0.7260	0.8179	0.8494	0.8437	0.8378	0.8428	0.8414	0.8400
			0.1562	0.1125	0.0583	0.0794	0.0715	0.0475	0.0457	0.0443	0.0429
			0.1222	0.0975	0.0474	0.0794	0.0714	0.0474	0.0451	0.0437	0.0423
		3	0.8146	0.8434	0.8723	0.9176	0.8839	0.8474	0.8192	0.8181	0.8169
			0.1952	0.1075	0.0364	0.0494	0.0435	0.0376	0.0193	0.0181	0.0169
			0.1450	0.0931	0.0303	0.0486	0.0428	0.0368	0.0192	0.0181	0.0169
	90	1	0.8010	0.8457	0.8725	0.7738	0.8236	0.7816	0.7839	0.7861	0.7978
			0.1112	0.0845	0.0375	0.0266	0.0247	0.0179	0.0159	0.0139	0.0067
			0.0879	0.0670	0.0311	0.0262	0.0236	0.0170	0.0147	0.0124	0.0059
		2	0.8032	0.8465	0.8724	0.7737	0.8231	0.7830	0.7853	0.7876	0.7974
			0.1148	0.0845	0.0452	0.0267	0.0251	0.0193	0.0172	0.0152	0.0068
			0.0903	0.0672	0.0399	0.0263	0.0240	0.0184	0.0161	0.0139	0.0059
		3	0.8067	0.9009	0.7210	1.2735	1.3585	0.8434	0.8448	0.8462	1.3176
			0.1221	0.0834	0.0361	0.0254	0.0242	0.0162	0.0142	0.0122	0.0064
			0.0942	0.0663	0.0301	0.0249	0.0231	0.0151	0.0128	0.0104	0.0059

Table 4: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of σ when $(\mu,\sigma) = (2.0, 1.5)$ using MLEs and Bayesian estimates under choices of m and n

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n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
c →						-2	-0.02	2	-2	-0.02	2
50	15	1	1.5205	2.0761	1.3973	2.0962	2.0554	2.0106	1.3980	1.3967	1.3954
			0.4409	0.6135	0.1054	0.5637	0.5242	0.4809	0.0986	0.0973	0.0961
			0.3387	0.5459	0.0970	0.5636	0.5240	0.4805	0.0986	0.0973	0.0960
		2	1.4828	2.0440	1.4033	2.0636	2.0240	1.9805	1.4040	1.4027	1.4014
			0.6267	0.6448	0.1110	0.5963	0.5557	0.5110	0.1046	0.1033	0.1021
			0.4189	0.5770	0.1030	0.5962	0.5554	0.5106	0.1046	0.1033	0.1020
		3	1.5176	2.0162	1.4079	2.0352	1.9968	1.9551	1.4085	1.4073	1.4061
			0.3854	0.5861	0.1008	0.5353	0.4970	0.4554	0.0939	0.0927	0.0915
			0.2954	0.5186	0.0925	0.5352	0.4968	0.4551	0.0939	0.0927	0.0915
	30	1	1.4881	1.7265	1.5745	1.7406	1.7128	1.6854	1.5759	1.5732	1.5705
			0.2811	0.3613	0.0908	0.2855	0.2534	0.2221	0.0673	0.0648	0.0623
			0.2260	0.2863	0.0718	0.2850	0.2529	0.2217	0.0661	0.0636	0.0612
		2	1.4777	1.7687	1.5647	1.7850	1.7529	1.7217	1.5659	1.5635	1.5611
			0.2925	0.4029	0.1005	0.3254	0.2885	0.2531	0.0771	0.0744	0.0716
			0.2329	0.3196	0.0797	0.3248	0.2880	0.2526	0.0759	0.0732	0.0705
		3	1.4783	1.8060	1.5648	1.8248	1.7880	1.7526	1.5661	1.5636	1.5612
			0.2636	0.3167	0.0905	0.2409	0.2131	0.1857	0.0672	0.0647	0.0622
			0.2107	0.2510	0.0716	0.2406	0.2128	0.1854	0.0659	0.0635	0.0611
	45	1	1.4907	1.5572	1.4754	1.5709	1.5437	1.5163	1.4779	1.4731	1.4684
			0.2131	0.2174	0.0892	0.0718	0.0449	0.0337	0.0294	0.0287	0.0207
			0.1701	0.1764	0.0785	0.0709	0.0437	0.0316	0.0269	0.0260	0.0182
		2	1.4896	1.5552	1.4716	1.5696	1.5412	1.5125	1.4740	1.4693	1.4646
			0.2144	0.2193	0.0895	0.0742	0.0471	0.0337	0.0372	0.0329	0.0253
			0.1714	0.1789	0.0787	0.0734	0.0459	0.0317	0.0354	0.0307	0.0221
		3	1.4897	1.5595	1.4755	1.5734	1.5459	1.5182	1.4779	1.4731	1.4683
			0.2117	0.2157	0.0890	0.0706	0.0427	0.0190	0.0294	0.0253	0.0163
			0.1693	0.1748	0.0784	0.0696	0.0412	0.0163	0.0269	0.0221	0.0132
100	30	1	1.4970	2.0683	1.6186	2.0877	2.0484	2.0050	1.6220	1.6152	1.6083
			0.2926	0.6354	0.1592	0.5378	0.4972	0.4520	0.1239	0.1168	0.1085
			0.2359	0.5693	0.1277	0.5376	0.4969	0.4513	0.1236	0.1165	0.1083
		2	1.4770	2.0176	1.6200	2.0376	1.9969	1.9513	1.6236	1.6165	1.6094
			0.4024	0.5910	0.1614	0.5486	0.5055	0.1107	0.1259	0.1188	0.1097
			0.2894	0.5233	0.1291	0.5484	0.5050	0.1104	0.1246	0.1185	0.1094
		3	1.4921	1.9031	1.6200	1.9215	1.8843	1.8439	1.6236	1.6165	1.6094
			0.2571	0.4826	0.1614	0.4216	0.3845	0.3442	0.1223	0.1155	0.0624
			0.2044	0.4171	0.1291	0.4215	0.3843	0.3439	0.1220	0.1152	0.0619
	60	1	1.4903	1.4345	1.5828	1.4381	1.4309	1.4233	1.5843	1.5813	1.5782
			0.1980	0.1208	0.1140	0.0985	0.0953	0.0920	0.0519	0.0442	0.0369
			0.1548	0.1009	0.0906	0.0983	0.0951	0.0918	0.0514	0.0437	0.0363
		2	1.4852	1.4600	1.5967	1.4637	1.4563	1.4486	1.5983	1.5951	1.5918
			0.2046	0.1258	0.1213	0.5878	0.0997	0.0963	0.1031	0.0770	0.0695
			0.1595	0.1054	0.0968	0.5877	0.0995	0.0960	0.1029	0.0767	0.0691
		3	1.4855	1.4667	1.6012	1.4707	1.4627	1.4544	1.6029	1.5995	1.5960
			0.1845	0.1121	0.1083	0.0845	0.0815	0.0784	0.0459	0.0376	0.0325
			0.1446	0.0900	0.0888	0.0843	0.0813	0.0782	0.0456	0.0373	0.0299
	90	1	1.4983	1.4601	1.5731	1.4632	1.4571	1.4507	1.5745	1.5718	1.5691
			0.1489	0.1024	0.0572	0.0439	0.0380	0.0321	0.0267	0.0250	0.0233
			0.1164	0.0811	0.0445	0.0425	0.0361	0.0294	0.0249	0.0232	0.0215
		2	1.4978	1.4675	1.5241	1.4706	1.4644	1.4579	1.5249	1.5232	1.5215
			0.1497	0.1039	0.0991	0.0504	0.0445	0.0389	0.0757	0.0730	0.0703
			0.1169	0.0824	0.0786	0.0493	0.0429	0.0368	0.0745	0.0718	0.0691
		3	1.4978	1.4670	1.5211	1.4701	1.4639	1.4575	1.5219	1.5203	1.5187
			0.1477	0.1020	0.0550	0.0435	0.0376	0.0297	0.0239	0.0222	0.0206
			0.1154	0.0808	0.0423	0.0421	0.0356	0.0293	0.0219	0.0203	0.0187

Table 5: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of $S(t)$ when $(\mu,\sigma) = (0.4, 0.8)$ using MLEs and Bayesian estimates under choices of m and n
.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
c →						-2	-0.02	2	-2	-0.02	2
50	15	1	0.8885	0.9253	0.8491	0.9259	0.9257	0.9254	0.8491	0.8490	0.8490
			0.0532	0.0405	0.0193	0.0584	0.0582	0.0580	0.0183	0.0183	0.0184
			0.0404	0.0389	0.0183	0.0584	0.0582	0.0579	0.0183	0.0183	0.0184
		2	0.9028	0.9264	0.8492	0.9268	0.9266	0.9264	0.8492	0.8492	0.8491
			0.0648	0.0399	0.0194	0.0588	0.0586	0.0584	0.0184	0.0185	0.0185
			0.0509	0.0354	0.0184	0.0588	0.0586	0.0584	0.0184	0.0185	0.0185
		3	0.9036	0.9258	0.8485	0.9263	0.9261	0.9259	0.8485	0.8485	0.8484
			0.0680	0.0547	0.0200	0.0593	0.0591	0.0589	0.0190	0.0190	0.0191
			0.0522	0.0548	0.0190	0.0593	0.0591	0.0589	0.0190	0.0190	0.0191
	30	1	0.8907	0.8915	0.8678	0.8917	0.8914	0.8911	0.8627	0.8627	0.8627
			0.0476	0.0275	0.0055	0.0228	0.0225	0.0223	0.0045	0.0046	0.0046
			0.0350	0.0239	0.0047	0.0228	0.0225	0.0222	0.0045	0.0045	0.0046
		2	0.8953	0.8907	0.8683	0.8909	0.8906	0.8903	0.8630	0.8630	0.8629
			0.0523	0.0280	0.0055	0.0234	0.0231	0.0228	0.0048	0.0048	0.0048
			0.0385	0.0244	0.0049	0.0234	0.0231	0.0228	0.0048	0.0048	0.0048
		3	0.8926	0.8901	0.8689	0.8903	0.8900	0.8897	0.8625	0.8625	0.8625
			0.0524	0.0287	0.0058	0.0242	0.0239	0.0236	0.0050	0.0050	0.0050
			0.0398	0.0251	0.0052	0.0242	0.0239	0.0236	0.0050	0.0050	0.0050
	45	1	0.8870	0.8863	0.8714	0.8865	0.8862	0.8858	0.8678	0.8678	0.8677
			0.0440	0.0214	0.0052	0.0108	0.0104	0.0100	0.0009	0.0008	0.0008
			0.0314	0.0179	0.0046	0.0108	0.0104	0.0099	0.0007	0.0007	0.0007
		2	0.8852	0.8823	0.8713	0.8825	0.8821	0.8818	0.8687	0.8687	0.8686
			0.0441	0.0229	0.0055	0.0150	0.0147	0.0143	0.0015	0.0014	0.0014
			0.0324	0.0199	0.0046	0.0150	0.0146	0.0143	0.0012	0.0012	0.0011
		3	0.8860	0.8780	0.8717	0.8782	0.8779	0.8774	0.8692	0.8692	0.8691
			0.0427	0.0258	0.0057	0.0191	0.0187	0.0184	0.0019	0.0019	0.0019
			0.0328	0.0228	0.0048	0.0190	0.0187	0.0183	0.0017	0.0017	0.0016
100	30	1	0.8847	0.9258	0.8726	0.9254	0.9252	0.9249	0.8726	0.8725	0.8725
			0.0443	0.0301	0.0092	0.0420	0.0418	0.0406	0.0052	0.0051	0.0050
			0.0308	0.0323	0.0075	0.0360	0.0351	0.0325	0.0051	0.0051	0.0050
		2	0.8913	0.9267	0.8726	0.9265	0.9262	0.9260	0.8726	0.8725	0.8725
			0.0498	0.0308	0.0092	0.0480	0.0475	0.0468	0.0052	0.0051	0.0050
			0.0372	0.0332	0.0075	0.0325	0.0323	0.0321	0.0051	0.0051	0.0050
		3	0.9014	0.9262	0.8726	0.9259	0.9257	0.9255	0.8726	0.8725	0.8725
			0.0628	0.0311	0.0092	0.0550	0.0547	0.0545	0.0052	0.0051	0.0050
			0.0444	0.0387	0.0075	0.0451	0.0540	0.0538	0.0051	0.0050	0.0050
	60	1	0.8797	0.8474	0.8627	0.8477	0.8471	0.8465	0.8714	0.8714	0.8714
			0.0343	0.0275	0.0051	0.0173	0.0178	0.0184	0.0039	0.0039	0.0039
			0.0239	0.0225	0.0043	0.0173	0.0178	0.0184	0.0038	0.0038	0.0038
		2	0.8824	0.8491	0.8630	0.8494	0.8488	0.8483	0.8713	0.8713	0.8713
			0.0381	0.0277	0.0055	0.0181	0.0187	0.0192	0.0040	0.0040	0.0040
			0.0280	0.0242	0.0048	0.0181	0.0186	0.0192	0.0039	0.0039	0.0039
		3	0.8865	0.8499	0.8625	0.8502	0.8497	0.8491	0.8717	0.8717	0.8716
			0.0426	0.0281	0.0056	0.0198	0.0204	0.0210	0.0043	0.0042	0.0042
			0.0286	0.0233	0.0051	0.0198	0.0204	0.0210	0.0042	0.0042	0.0041
	90	1	0.8754	0.8631	0.8678	0.8633	0.8630	0.8628	0.8678	0.8678	0.8677
			0.0283	0.0146	0.0052	0.0043	0.0046	0.0048	0.0009	0.0008	0.0008
			0.0208	0.0117	0.0047	0.0042	0.0045	0.0047	0.0007	0.0007	0.0007
		2	0.8754	0.8631	0.8687	0.8632	0.8630	0.8627	0.8683	0.8683	0.8683
			0.0291	0.0148	0.0053	0.0044	0.0046	0.0048	0.0012	0.0012	0.0011
			0.0213	0.0118	0.0045	0.0043	0.0045	0.0048	0.0010	0.0009	0.0009
		3	0.8756	0.8631	0.8692	0.8632	0.8630	0.8628	0.8689	0.8689	0.8688
			0.0294	0.0150	0.0061	0.0044	0.0046	0.0048	0.0017	0.0016	0.0016
			0.0215	0.0120	0.0051	0.0043	0.0045	0.0047	0.0014	0.0014	0.0013

Table 6: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of $S(t)$ when $(\mu,\sigma) = (2.0, 1.5)$ using MLEs and Bayesian estimates under choices of m and n

.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
c →						-2	-0.02	2	-2	-0.02	2
50	15	1	0.8685	0.9062	0.8614	0.8374	0.8356	0.8337	0.8566	0.8565	0.8564
			0.0435	0.0329	0.0190	0.0338	0.0350	0.0363	0.0203	0.0204	0.0205
			0.0345	0.0296	0.0162	0.0337	0.0350	0.0362	0.0202	0.0203	0.0204
		2	0.8740	0.9112	0.8594	0.9176	0.9175	0.9173	0.8569	0.8568	0.8567
			0.0463	0.0374	0.0208	0.0406	0.0404	0.0402	0.0206	0.0207	0.0208
			0.0356	0.0345	0.0180	0.0405	0.0403	0.0401	0.0206	0.0207	0.0208
		3	0.8776	0.9146	0.8576	0.9243	0.9241	0.9239	0.8551	0.8550	0.8549
			0.0599	0.0407	0.0225	0.0473	0.0471	0.0469	0.0221	0.0222	0.0223
			0.0462	0.0377	0.0197	0.0472	0.0470	0.0468	0.0221	0.0222	0.0223
	30	1	0.8788	0.8959	0.8672	0.9068	0.9065	0.9063	0.8907	0.8906	0.8905
			0.0404	0.0274	0.0158	0.0228	0.0296	0.0293	0.0128	0.0128	0.0127
			0.0322	0.0225	0.0133	0.0256	0.0294	0.0291	0.0128	0.0127	0.0126
		2	0.8801	0.8952	0.8699	0.8434	0.8422	0.8409	0.8900	0.8899	0.8898
			0.0405	0.0277	0.0159	0.0244	0.0291	0.0293	0.0130	0.0129	0.0128
			0.0325	0.0227	0.0134	0.0242	0.0290	0.0292	0.0130	0.0129	0.0128
		3	0.8812	0.8947	0.8703	0.8433	0.8421	0.8409	0.8901	0.8900	0.8899
			0.0465	0.0283	0.0163	0.0377	0.0375	0.0372	0.0135	0.0134	0.0133
			0.0371	0.0231	0.0139	0.0376	0.0373	0.0370	0.0135	0.0134	0.0133
	45	1	0.8781	0.8663	0.8906	0.8665	0.8661	0.8656	0.8748	0.8746	0.8745
			0.0377	0.0224	0.0104	0.0107	0.0111	0.0115	0.0027	0.0028	0.0029
			0.0308	0.0185	0.0079	0.0107	0.0111	0.0115	0.0024	0.0025	0.0026
		2	0.8784	0.8659	0.8899	0.8661	0.8657	0.8652	0.8721	0.8720	0.8719
			0.0378	0.0225	0.0106	0.0109	0.0113	0.0117	0.0052	0.0053	0.0054
			0.0309	0.0187	0.0082	0.0109	0.0113	0.0117	0.0050	0.0052	0.0053
		3	0.8786	0.8661	0.8901	0.8663	0.8659	0.8654	0.8703	0.8702	0.8701
			0.0390	0.0224	0.0123	0.0110	0.0115	0.0119	0.0069	0.0070	0.0071
			0.0318	0.0186	0.0102	0.0110	0.0115	0.0119	0.0068	0.0069	0.0070
100	30	1	0.8762	0.9066	0.8565	0.9063	0.9061	0.9058	0.8615	0.8613	0.8612
			0.0301	0.0311	0.0182	0.0294	0.0291	0.0289	0.0157	0.0158	0.0160
			0.0241	0.0306	0.0154	0.0292	0.0289	0.0287	0.0157	0.0158	0.0159
		2	0.8769	0.9175	0.8569	0.9113	0.9111	0.9108	0.8594	0.8593	0.8592
			0.0318	0.0314	0.0206	0.0339	0.0341	0.0339	0.0177	0.0179	0.0180
			0.0251	0.0310	0.0175	0.0338	0.0340	0.0337	0.0177	0.0178	0.0180
		3	0.8788	0.9242	0.8550	0.9147	0.9145	0.9142	0.8577	0.8575	0.8574
			0.0440	0.0315	0.0220	0.0389	0.0417	0.0436	0.0195	0.0197	0.0198
			0.0349	0.0312	0.0193	0.0387	0.0416	0.0435	0.0195	0.0196	0.0198
	60	1	0.8790	0.8365	0.8674	0.8962	0.8957	0.8952	0.8714	0.8714	0.8714
			0.0281	0.0234	0.0101	0.0179	0.0174	0.0169	0.0057	0.0058	0.0058
			0.0226	0.0301	0.0082	0.0178	0.0173	0.0168	0.0057	0.0057	0.0058
		2	0.8793	0.8428	0.8753	0.8954	0.8949	0.8944	0.8713	0.8713	0.8713
			0.0284	0.0253	0.0122	0.0183	0.0178	0.0173	0.0058	0.0059	0.0059
			0.0229	0.0300	0.0100	0.0183	0.0178	0.0173	0.0058	0.0059	0.0059
		3	0.8798	0.8427	0.8808	0.8950	0.8945	0.8940	0.8713	0.8713	0.8713
			0.0337	0.0258	0.0124	0.0190	0.0186	0.0181	0.0097	0.0097	0.0098
			0.0273	0.0308	0.0115	0.0190	0.0185	0.0180	0.0097	0.0097	0.0098
	90	1	0.8781	0.8747	0.8714	0.8672	0.8672	0.8671	0.8675	0.8674	0.8673
			0.0277	0.0109	0.0080	0.0068	0.0069	0.0070	0.0019	0.0020	0.0021
			0.0223	0.0090	0.0067	0.0068	0.0069	0.0070	0.0018	0.0019	0.0021
		2	0.8782	0.8720	0.8713	0.8700	0.8699	0.8698	0.8753	0.8752	0.8751
			0.0278	0.0111	0.0080	0.0072	0.0073	0.0073	0.0038	0.0037	0.0035
			0.0223	0.0093	0.0068	0.0072	0.0073	0.0073	0.0037	0.0036	0.0034
		3	0.8783	0.8703	0.8713	0.8703	0.8703	0.8702	0.8809	0.8807	0.8806
			0.0288	0.0112	0.0080	0.0099	0.0100	0.0100	0.0058	0.0059	0.0059
			0.0232	0.0093	0.0068	0.0099	0.0100	0.0100	0.0058	0.0059	0.0059

Table 7: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of $h(t)$ when $(\mu,\sigma) = (0.4, 0.8)$ using MLEs and Bayesian estimates under choices of m and n
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n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
c →						-2	-0.02	2	-2	-0.02	2
50	15	1	0.5971	0.3418	0.7465	0.3495	0.3352	0.3244	0.7472	0.7458	0.7444
			0.2948	0.2813	0.1092	0.2528	0.1626	0.1412	0.1050	0.1036	0.1021
			0.2092	0.2023	0.1042	0.1727	0.1425	0.1190	0.1049	0.1035	0.1020
		2	0.5085	0.3378	0.7457	0.3451	0.3316	0.3214	0.7463	0.7450	0.7436
			0.3134	0.2446	0.1073	0.2407	0.1591	0.1357	0.1031	0.1018	0.1004
			0.2418	0.1763	0.1024	0.1697	0.1289	0.1155	0.1031	0.1017	0.1003
		3	0.5699	0.3419	0.7483	0.3486	0.3361	0.3265	0.7490	0.7476	0.7462
			0.3714	0.2300	0.1065	0.2276	0.1481	0.1248	0.1023	0.1010	0.0996
			0.2754	0.1722	0.1016	0.1576	0.1180	0.1046	0.1022	0.1009	0.0995
	30	1	0.5545	0.5560	0.6338	0.5490	0.5379	0.5272	0.6344	0.6332	0.6319
			0.2222	0.1382	0.0352	0.1170	0.1063	0.0953	0.0187	0.0181	0.0174
			0.1594	0.1209	0.0297	0.1169	0.1062	0.0951	0.0186	0.0180	0.0174
		2	0.5412	0.5700	0.6302	0.5527	0.5418	0.5312	0.6320	0.6307	0.6294
			0.2409	0.1302	0.0331	0.1130	0.1025	0.0916	0.0176	0.0170	0.0164
			0.1712	0.1109	0.0280	0.1129	0.1024	0.0914	0.0175	0.0169	0.0163
		3	0.5438	0.5871	0.6284	0.5554	0.5444	0.5338	0.6291	0.6277	0.6262
			0.2429	0.1271	0.0303	0.1104	0.0999	0.0889	0.0162	0.0156	0.0150
			0.1791	0.1041	0.0259	0.1103	0.0998	0.0887	0.0162	0.0155	0.0149
	45	1	0.5693	0.5433	0.6305	0.5661	0.5473	0.5324	0.6344	0.6300	0.6290
			0.1960	0.1281	0.0293	0.1119	0.0970	0.0782	0.0184	0.0173	0.0158
			0.1399	0.1128	0.0230	0.1117	0.0968	0.0780	0.0180	0.0164	0.0149
		2	0.5646	0.5471	0.6316	0.5800	0.5614	0.5464	0.6309	0.6311	0.6301
			0.1998	0.1249	0.0293	0.0978	0.0829	0.0643	0.0157	0.0155	0.0142
			0.1426	0.1097	0.0230	0.0977	0.0828	0.0642	0.0148	0.0146	0.0132
		3	0.5757	0.5498	0.6298	0.5979	0.5776	0.5611	0.6303	0.6293	0.6283
			0.2019	0.1234	0.0285	0.0831	0.0667	0.0463	0.0132	0.0120	0.0108
			0.1446	0.1006	0.0223	0.0830	0.0665	0.0462	0.0122	0.0109	0.0097
100	30	1	0.5850	0.3399	0.6086	0.3477	0.3333	0.3226	0.6101	0.6071	0.6041
			0.2157	0.1261	0.0556	0.1613	0.1144	0.1044	0.0342	0.0372	0.0402
			0.1476	0.1182	0.0441	0.1211	0.1143	0.1043	0.0340	0.0370	0.0400
		2	0.5494	0.3359	0.6086	0.3430	0.3298	0.3198	0.6101	0.6071	0.6041
			0.2323	0.1231	0.0556	0.1475	0.1109	0.1015	0.0342	0.0372	0.0402
			0.1683	0.1142	0.0441	0.1074	0.1108	0.1013	0.0340	0.0370	0.0400
		3	0.5267	0.3401	0.6086	0.3467	0.3343	0.3248	0.6101	0.6071	0.6041
			0.3094	0.1216	0.0556	0.1467	0.1099	0.0994	0.0342	0.0372	0.0402
			0.2164	0.1041	0.0441	0.1064	0.1098	0.1015	0.0340	0.0370	0.0400
	60	1	0.5974	0.7552	0.6338	0.7630	0.7476	0.7321	0.6616	0.6611	0.6605
			0.1583	0.1253	0.0342	0.1149	0.1036	0.0881	0.0187	0.0173	0.0159
			0.1070	0.1143	0.0288	0.1138	0.1035	0.0880	0.0179	0.0165	0.0151
		2	0.5852	0.7479	0.6314	0.7549	0.7411	0.7274	0.6603	0.6597	0.6590
			0.1737	0.1108	0.0322	0.1108	0.0971	0.0834	0.0169	0.0149	0.0139
			0.1242	0.1128	0.0273	0.1108	0.0970	0.0833	0.0160	0.0141	0.0131
		3	0.5678	0.7443	0.6284	0.7509	0.7380	0.7252	0.6627	0.6621	0.6615
			0.1973	0.1100	0.0303	0.1068	0.0939	0.0812	0.0147	0.0138	0.0129
			0.1259	0.1105	0.0259	0.1067	0.0939	0.0811	0.0140	0.0130	0.0120
	90	1	0.6130	0.6557	0.6613	0.6310	0.6332	0.6319	0.6591	0.6523	0.6455
			0.1261	0.0685	0.0271	0.0165	0.0156	0.0146	0.0160	0.0097	0.0050
			0.0895	0.0528	0.0218	0.0158	0.0148	0.0138	0.0150	0.0082	0.0040
		2	0.6126	0.6555	0.6600	0.6321	0.6295	0.6281	0.6588	0.6522	0.6457
			0.1296	0.0674	0.0264	0.0159	0.0143	0.0130	0.0156	0.0096	0.0050
			0.0923	0.0518	0.0212	0.0152	0.0134	0.0121	0.0147	0.0081	0.0039
		3	0.6115	0.6552	0.6624	0.6584	0.6277	0.6261	0.6292	0.6521	0.6457
			0.1321	0.0661	0.0258	0.0153	0.0120	0.0108	0.0132	0.0094	0.0050
			0.0922	0.0511	0.0208	0.0143	0.0109	0.0097	0.0122	0.0080	0.0039

Table 8: Average estimates (first line), root mean squared error (second line), and mean absolute biases (third line) of $h(t)$ when $(\mu,\sigma) = (2.0, 1.5)$ using MLEs and Bayesian estimates under choices of m and n
.

n				SE		GE					
Prior→	m	CS	MLE	I	II	I			II		
$c\rightarrow$						-2	-0.02	2	-2	-0.02	2
50	15	1	0.2327	0.2483	0.2024	0.1476	0.1433	0.1393	0.2487	0.2479	0.2472
			0.0897	0.0532	0.0210	0.0853	0.0813	0.0769	0.0365	0.0357	0.0350
			0.0676	0.0391	0.0173	0.0852	0.0812	0.0767	0.0364	0.0357	0.0349
		2	0.2356	0.2502	0.2032	0.1426	0.1381	0.1342	0.2506	0.2499	0.2491
			0.0896	0.0501	0.0208	0.0819	0.0780	0.0736	0.0347	0.0339	0.0332
			0.0614	0.0358	0.0172	0.0818	0.0778	0.0733	0.0347	0.0339	0.0332
		3	0.2333	0.2520	0.2033	0.1393	0.1348	0.1307	0.2524	0.2517	0.2509
			0.0869	0.0453	0.0208	0.0767	0.0728	0.0686	0.0327	0.0320	0.0313
			0.0584	0.0307	0.0171	0.0766	0.0727	0.0683	0.0327	0.0320	0.0313
	30	1	0.2166	0.2221	0.1454	0.1880	0.1854	0.1826	0.2223	0.2218	0.2213
			0.0628	0.0471	0.0176	0.0341	0.0306	0.0280	0.0154	0.0157	0.0159
			0.0497	0.0361	0.0167	0.0339	0.0306	0.0279	0.0154	0.0157	0.0159
		2	0.2172	0.1403	0.2195	0.1894	0.1867	0.1839	0.2198	0.2192	0.2186
			0.0562	0.0369	0.0122	0.0321	0.0293	0.0267	0.0147	0.0149	0.0152
			0.0439	0.0343	0.0098	0.0320	0.0292	0.0266	0.0147	0.0149	0.0152
		3	0.2154	0.1369	0.2190	0.1895	0.1869	0.1842	0.2193	0.2187	0.2181
			0.0537	0.0350	0.0119	0.0318	0.0291	0.0266	0.0141	0.0144	0.0146
			0.0421	0.0323	0.0095	0.0317	0.0290	0.0265	0.0141	0.0144	0.0146
	45	1	0.2162	0.1867	0.2004	0.2030	0.2018	0.2006	0.2005	0.2003	0.2000
			0.0503	0.0368	0.0171	0.0155	0.0143	0.0131	0.0064	0.0059	0.0055
			0.0407	0.0315	0.0156	0.0153	0.0141	0.0129	0.0064	0.0059	0.0054
		2	0.2161	0.1881	0.2017	0.2039	0.2026	0.2013	0.2018	0.2016	0.2013
			0.0487	0.0359	0.0165	0.0149	0.0136	0.0123	0.0039	0.0034	0.0029
			0.0392	0.0304	0.0149	0.0147	0.0134	0.0121	0.0038	0.0032	0.0027
		3	0.2157	0.1882	0.2011	0.2040	0.2026	0.2013	0.2013	0.2010	0.2008
			0.0483	0.0346	0.0160	0.0149	0.0135	0.0122	0.0035	0.0030	0.0024
			0.0390	0.0302	0.0143	0.0147	0.0133	0.0120	0.0033	0.0027	0.0022
100	30	1	0.2199	0.1451	0.2347	0.1474	0.1429	0.1389	0.2348	0.2346	0.2343
			0.0609	0.0497	0.0205	0.0557	0.0488	0.0216	0.0192	0.0189	0.0187
			0.0477	0.0472	0.0191	0.0446	0.0387	0.0215	0.0191	0.0188	0.0186
		2	0.2242	0.1333	0.2330	0.1351	0.1316	0.1284	0.2331	0.2329	0.2327
			0.0531	0.0458	0.0204	0.0510	0.0445	0.0176	0.0190	0.0187	0.0184
			0.0404	0.0427	0.0189	0.0400	0.0344	0.0176	0.0189	0.0186	0.0184
		3	0.2231	0.1288	0.2349	0.1304	0.1273	0.1244	0.2350	0.2348	0.2345
			0.0496	0.0357	0.0186	0.0488	0.0332	0.0172	0.0173	0.0170	0.0168
			0.0374	0.0310	0.0172	0.0286	0.0230	0.0171	0.0172	0.0170	0.0167
	60	1	0.2149	0.2756	0.2219	0.2799	0.2712	0.2615	0.2221	0.2216	0.2212
			0.0428	0.0386	0.0135	0.0334	0.0155	0.0100	0.0065	0.0077	0.0088
			0.0342	0.0356	0.0121	0.0334	0.0152	0.0096	0.0062	0.0075	0.0086
		2	0.2160	0.2648	0.2162	0.2677	0.2617	0.2549	0.2166	0.2158	0.2150
			0.0364	0.0328	0.0121	0.0218	0.0159	0.0094	0.0063	0.0058	0.0053
			0.0292	0.0247	0.0089	0.0217	0.0157	0.0089	0.0062	0.0057	0.0052
		3	0.2153	0.2641	0.2091	0.2668	0.2613	0.2552	0.2097	0.2085	0.2073
			0.0350	0.0312	0.0120	0.0209	0.0155	0.0092	0.0015	0.0014	0.0017
			0.0282	0.0251	0.0089	0.0208	0.0154	0.0092	0.0013	0.0012	0.0013
	90	1	0.2150	0.2346	0.2134	0.2357	0.2335	0.2312	0.2137	0.2130	0.2123
			0.0359	0.0297	0.0125	0.0128	0.0105	0.0093	0.0026	0.0032	0.0038
			0.0287	0.0240	0.0111	0.0125	0.0115	0.0092	0.0022	0.0029	0.0036
		2	0.2150	0.2343	0.2134	0.2354	0.2332	0.2309	0.2137	0.2131	0.2124
			0.0345	0.0294	0.0125	0.0125	0.0103	0.0090	0.0025	0.0031	0.0038
			0.0275	0.0238	0.0111	0.0125	0.0112	0.0091	0.0022	0.0029	0.0036
		3	0.2149	0.2341	0.2134	0.2352	0.2330	0.2308	0.2137	0.2131	0.2124
			0.0343	0.0293	0.0125	0.0123	0.0101	0.0084	0.0025	0.0031	0.0038
			0.0275	0.0236	0.0111	0.0122	0.0110	0.0078	0.0022	0.0029	0.0036

Table 9: The Average confidence lengths and coverage probabilities for 95% asymptotic and credible intervals of μ under choices of m and n.

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(μ,σ) Prior→	n	m	CS	ACI		BCI I		BCI II	
				ACL	CP	ACL	CP	ACL	CP
(0.4,0.8)	50	15	1	1.3497	0.865	0.5720	0.955	0.1096	0.978
			2	1.4166	0.845	0.5854	0.953	0.1096	0.973
			3	1.7865	0.763	0.6237	0.951	0.1133	0.972
		30	1	1.2966	0.871	0.5239	0.962	0.0935	0.987
			2	1.3609	0.861	0.5612	0.960	0.0938	0.986
			3	1.4868	0.801	0.6194	0.960	0.0943	0.984
		45	1	1.2646	0.881	0.4535	0.975	0.0870	0.990
			2	1.2963	0.880	0.4561	0.974	0.0870	0.989
			3	1.3047	0.872	0.4618	0.972	0.0874	0.986
	100	30	1	1.2042	0.878	0.5107	0.958	0.0940	0.979
			2	1.3457	0.872	0.5111	0.955	0.0969	0.975
			3	1.4092	0.851	0.5283	0.959	0.1001	0.974
		60	1	1.0495	0.885	0.4327	0.967	0.0884	0.989
			2	1.1315	0.877	0.4395	0.965	0.0912	0.984
			3	1.1523	0.859	0.4364	0.964	0.0984	0.984
		90	1	1.0026	0.916	0.3070	0.977	0.0733	0.988
			2	1.0293	0.913	0.3192	0.979	0.0740	0.986
			3	1.0229	0.900	0.3516	0.980	0.0741	0.985
(2.0,1.5)	50	15	1	1.2890	0.951	0.9252	0.950	0.2387	0.963
			2	1.5875	0.913	0.9380	0.947	0.2473	0.960
			3	1.9600	0.898	0.9442	0.944	0.2499	0.958
		30	1	1.1780	0.955	0.8897	0.955	0.2082	0.967
			2	1.2441	0.944	0.9267	0.952	0.2135	0.966
			3	1.4352	0.941	0.9286	0.950	0.2214	0.966
		45	1	1.1052	0.965	0.5148	0.965	0.2043	0.970
			2	1.1066	0.960	0.5188	0.964	0.2055	0.968
			3	1.1543	0.954	0.5202	0.961	0.2065	0.968
	100	30	1	0.9602	0.959	0.8310	0.958	0.1974	0.967
			2	1.0583	0.932	0.8959	0.951	0.2374	0.964
			3	1.3911	0.922	0.8981	0.945	0.2444	0.961
		60	1	0.7782	0.960	0.7421	0.960	0.1954	0.971
			2	0.7813	0.958	0.7592	0.956	0.2051	0.965
			3	0.9600	0.952	0.8760	0.952	0.2054	0.966
		90	1	0.7559	0.969	0.4424	0.967	0.1407	0.974
			2	0.7561	0.966	0.4428	0.965	0.1410	0.972
			3	0.7917	0.957	0.4427	0.965	0.1416	0.972

Table 10: The Average confidence lengths and coverage probabilities for 95% asymptotic and credible intervals of σ under choices of m and n.

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(μ,σ)				ACI		BCI			
Prior→	n	m	CS			I		II	
				ACL	CP	ACL	CP	ACL	CP
(0.4,0.8)	50	15	1	0.9136	0.841	0.8212	0.901	0.1689	0.914
			2	1.0393	0.877	0.8381	0.897	0.1692	0.905
			3	1.2530	0.905	0.8036	0.911	0.1672	0.925
		30	1	0.6897	0.907	0.6600	0.921	0.1290	0.928
			2	0.7746	0.903	0.6839	0.914	0.1317	0.923
			3	0.8080	0.917	0.6297	0.923	0.1287	0.937
		45	1	0.5762	0.913	0.5104	0.930	0.1040	0.939
			2	0.5796	0.919	0.5154	0.925	0.1060	0.930
			3	0.5957	0.929	0.5098	0.936	0.1040	0.944
	100	30	1	0.6769	0.911	0.8204	0.919	0.1478	0.924
			2	0.8165	0.889	0.8308	0.909	0.1499	0.918
			3	1.1818	0.924	0.7855	0.932	0.1458	0.945
		60	1	0.4971	0.926	0.3001	0.931	0.1271	0.940
			2	0.5591	0.919	0.3150	0.923	0.1287	0.936
			3	0.6468	0.936	0.2831	0.941	0.1266	0.956
		90	1	0.4253	0.941	0.2697	0.949	0.1016	0.957
			2	0.4337	0.939	0.2708	0.943	0.1015	0.948
			3	0.4445	0.944	0.2679	0.955	0.1011	0.968
(2.0,1.5)	50	15	1	1.6449	0.827	0.9949	0.895	0.3337	0.904
			2	2.2377	0.810	1.0114	0.876	0.3367	0.900
			3	1.4665	0.873	0.9705	0.901	0.3322	0.911
		30	1	1.0695	0.880	0.8620	0.910	0.2329	0.913
			2	1.1151	0.851	0.9141	0.907	0.2440	0.909
			3	1.0109	0.901	0.8100	0.916	0.2323	0.919
		45	1	0.8432	0.908	0.7318	0.922	0.1972	0.925
			2	0.8480	0.905	0.7424	0.920	0.2427	0.921
			3	0.8374	0.921	0.7283	0.929	0.1949	0.935
	100	30	1	1.1167	0.874	0.9893	0.905	0.2933	0.916
			2	1.4773	0.862	0.9971	0.901	0.2938	0.914
			3	0.9981	0.903	0.9394	0.923	0.2915	0.933
		60	1	0.7535	0.904	0.4367	0.916	0.2282	0.935
			2	0.7746	0.877	0.4588	0.910	0.2341	0.930
			3	0.7124	0.917	0.4223	0.935	0.2224	0.948
		90	1	0.5951	0.933	0.3575	0.940	0.1655	0.941
			2	0.5989	0.931	0.3609	0.935	0.1671	0.939
			3	0.5920	0.940	0.3568	0.946	0.1579	0.944

Table 11: The Average confidence lengths and coverage probabilities for 95% asymptotic and credible intervals of $S(t)$ under choices of m and n.

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(μ,σ) Prior→	n	m	CS	ACI		BCI I		BCI II	
				ACL	CP	ACL	CP	ACL	CP
(0.4,0.8)	50	15	1	0.1375	0.917	0.0666	0.928	0.0235	0.934
			2	0.1561	0.913	0.0676	0.925	0.0239	0.931
			3	0.1864	0.904	0.0757	0.916	0.0245	0.922
		30	1	0.1327	0.923	0.0661	0.931	0.0171	0.939
			2	0.1348	0.914	0.0670	0.927	0.0177	0.932
			3	0.1540	0.911	0.0672	0.923	0.0181	0.929
		45	1	0.1313	0.930	0.0495	0.939	0.0148	0.947
			2	0.1315	0.931	0.0528	0.937	0.0149	0.946
			3	0.1357	0.929	0.0565	0.935	0.0150	0.944
	100	30	1	0.1024	0.929	0.0540	0.940	0.0230	0.947
			2	0.1135	0.921	0.0631	0.932	0.0232	0.939
			3	0.1508	0.917	0.0724	0.929	0.0234	0.935
		60	1	0.0997	0.932	0.0511	0.942	0.0161	0.949
			2	0.1012	0.931	0.0565	0.941	0.0171	0.948
			3	0.1191	0.929	0.0587	0.936	0.0179	0.945
		90	1	0.0989	0.937	0.0353	0.949	0.0125	0.955
			2	0.0991	0.935	0.0379	0.945	0.0126	0.952
			3	0.1034	0.933	0.0421	0.944	0.0127	0.951
(2.0,1.5)	50	15	1	0.1564	0.912	0.1041	0.923	0.0402	0.927
			2	0.1678	0.904	0.1065	0.918	0.0411	0.921
			3	0.2335	0.884	0.1269	0.901	0.0413	0.907
		30	1	0.1504	0.918	0.0741	0.927	0.0332	0.932
			2	0.1514	0.911	0.0747	0.921	0.0336	0.925
			3	0.1788	0.899	0.0752	0.910	0.0338	0.914
		45	1	0.1500	0.926	0.0519	0.936	0.0273	0.940
			2	0.1504	0.923	0.0547	0.933	0.0290	0.937
			3	0.1557	0.920	0.0567	0.928	0.0295	0.931
	100	30	1	0.1145	0.920	0.0704	0.930	0.0321	0.934
			2	0.1207	0.911	0.0707	0.925	0.0353	0.927
			3	0.1720	0.902	0.0709	0.916	0.0391	0.919
		60	1	0.1080	0.930	0.0560	0.938	0.0277	0.943
			2	0.1092	0.927	0.0569	0.936	0.0329	0.939
			3	0.1312	0.915	0.0623	0.922	0.0336	0.928
		90	1	0.1072	0.932	0.0487	0.941	0.0217	0.946
			2	0.1075	0.928	0.0534	0.939	0.0216	0.942
			3	0.1117	0.926	0.0558	0.935	0.0219	0.940

Using Table 17, the classical and Bayes estimates with their standard errors of the unknown parameters μ and σ as well as the reliability characteristics $S(t)$ and $h(t)$ (at distinct time $t = 1.5$) are calculated and listed in Table 18. Moreover, two-sided 95% asymptotic/credible interval estimates with their lengths are also calculated and listed in Table 18. It shows that the point estimates of μ, σ, $S(t)$ and $h(t)$ obtained by maximum likelihood and Bayesian estimation methods are quite close to each other as expected. Similar behavior is also observed in the case of interval estimates. To evaluate the convergence of the MCMC draws at each iteration (x-axis) and the sampled values (y-axis), trace plots with their sample mean (solid (—) lines) and two bounds of 95% credible intervals (dashed (– – –) lines) of the simulated 25,000 samples of μ, σ, $S(t)$ and $h(t)$ (using samples $S_1^{(15:30)}$ and $S_1^{(25:30)}$ as an example) are plotted in Figure 2. It is evident that the Metropolis-Hastings algorithm sampler converges well. It also shows that discarding the first 5,000 iterations as burn-in is an appropriate size to remove the effect of the initial guesses. Also, using the Gaussian kernel based on 25,000 chain values, the marginal posterior density estimates of the unknown parameters μ, σ, $S(t)$ and $h(t)$ with their histograms are displayed in Figure 3. In each histogram plot, the sample mean is depicted as vertical dash-dotted line (:). It shows that all the generated posterior estimates have been well approximated to the theoretical posterior density functions. Lastly, the numerical results of the proposed point/interval estimates using complete equipment data demonstrate the LTG lifetime model well.

Table 12: The Average confidence lengths and coverage probabilities for 95% asymptotic and credible intervals of $h(t)$ under choices of m and n.
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(μ,σ)	n	m	CS	ACI		BCI			
Prior→						I		II	
				ACL	CP	ACL	CP	ACL	CP
(0.4,0.8)	50	15	1	0.7050	0.905	0.4427	0.931	0.1268	0.934
			2	0.7551	0.919	0.4202	0.933	0.1257	0.941
			3	0.8322	0.928	0.4154	0.934	0.1253	0.945
		30	1	0.5462	0.934	0.3192	0.940	0.1031	0.953
			2	0.5730	0.937	0.3164	0.944	0.0994	0.954
			3	0.6548	0.940	0.3148	0.945	0.0955	0.957
		45	1	0.5385	0.939	0.2738	0.942	0.0941	0.957
			2	0.5507	0.940	0.2626	0.945	0.0928	0.956
			3	0.5684	0.943	0.2534	0.947	0.0923	0.960
	100	30	1	0.4647	0.929	0.3839	0.936	0.1253	0.946
			2	0.5149	0.930	0.3643	0.939	0.1250	0.952
			3	0.6443	0.937	0.3567	0.940	0.1245	0.954
		60	1	0.4060	0.935	0.2734	0.940	0.0998	0.954
			2	0.4204	0.937	0.2611	0.948	0.0982	0.957
			3	0.4972	0.945	0.2541	0.949	0.0952	0.963
		90	1	0.3952	0.943	0.2654	0.955	0.0735	0.962
			2	0.4070	0.944	0.2531	0.958	0.0718	0.964
			3	0.4267	0.944	0.2428	0.960	0.0716	0.966
(2.0,1.5)	50	15	1	0.3275	0.927	0.1657	0.940	0.0509	0.948
			2	0.2781	0.930	0.1323	0.944	0.0503	0.950
			3	0.2588	0.934	0.1268	0.945	0.0487	0.954
		30	1	0.2347	0.935	0.0983	0.950	0.0379	0.956
			2	0.2018	0.938	0.0981	0.956	0.0378	0.960
			3	0.1934	0.944	0.0967	0.955	0.0375	0.964
		45	1	0.1973	0.944	0.0894	0.953	0.0307	0.963
			2	0.1897	0.945	0.0883	0.957	0.0301	0.964
			3	0.1888	0.946	0.0877	0.957	0.0282	0.965
	100	30	1	0.2295	0.940	0.0974	0.945	0.0471	0.954
			2	0.1860	0.939	0.0885	0.947	0.0412	0.956
			3	0.1708	0.940	0.0832	0.948	0.0343	0.960
		60	1	0.1672	0.941	0.0803	0.951	0.0362	0.961
			2	0.1407	0.941	0.0797	0.955	0.0359	0.962
			3	0.1352	0.946	0.0797	0.956	0.0341	0.963
		90	1	0.1391	0.947	0.0601	0.966	0.0275	0.967
			2	0.1332	0.949	0.0595	0.966	0.0272	0.969
			3	0.1326	0.954	0.0583	0.968	0.0271	0.972

7 Conclusion

In this paper, we developed maximum likelihood and Bayesian estimates for the unknown parameters of the LTG distribution and reliability characteristics under progressive type II censoring. The proposed estimation methods were compared with existing ones in terms of various statistical measures such as RMSEs, MABs, and CPs. We also provided a comprehensive analysis of the estimation methods, including Bayesian MCMC estimation, which outperformed maximum likelihood estimation in reducing RMSEs and MABs.The results of the study revealed that both point and interval estimates of the unknown parameters and reliability characteristics of the LTG distribution have the advantage of the smallest RMSEs, MABs, and ACLs, and the highest CPs. Moreover, Bayes MCMC estimates performed better than those based on the likelihood function, and credible intervals were better than asymptotic intervals for all unknown parameters. It was also observed that the variance of Prior-II was less than Prior-I, and Bayes estimates under Prior-II performed more efficiently in terms of the smallest RMSEs, MABs, and ACLs, as well as the highest CPs. The use of SE loss function gave equal weight to underestimation and overestimation, leading to symmetrical results, while the GE loss function performed better in terms of the lowest RMSEs and MABs values. Additionally, it was noticed that the best censoring scheme varied depending on the estimates, and as the sample size increased, the root mean square errors, mean absolute biases, and average confidence length for the MLEs of some parameters decreased while the coverage

Table 13: Different progressive type-II censored samples from LTG distribution.

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(n,m)	$S_i^{m:n}$	Scheme	Progressive type II censored samples
(40,20)	$S_1^{20:40}$	(20,0^{*}19)	0.1029, 2.5646, 2.6061, 2.7184, 2.8407, 3.0269, 3.1155, 3.2346, 3.2923, 3.3537, 3.6245, 4.1259, 4.3659, 5.8869, 6.3897, 6.4626, 7.1265, 7.4080, 9.1758, 9.1781.
	$S_2^{20:40}$	(0^{*}9,20,0^{*}10)	0.1029, 0.2988, 0.3216, 0.4796, 0.6374, 0.9526, 1.0861, 1.1015, 1.1059, 1.2218, 3.6245, 4.1259, 4.3659, 5.8869, 6.3897, 6.4626, 7.1265, 7.4080, 9.1758, 9.1781
	$S_3^{20:40}$	(0^{*}19,20)	0.1029, 0.2988, 0.3216, 0.4796, 0.6374, 0.9526, 1.0861, 1.1015, 1.1059, 1.2218, 1.2924, 1.2974, 1.4785, 1.5162, 1.6075, 1.7217, 1.8347, 1.8776, 2.1712, 2.2297.
(40,30)	$S_1^{30:40}$	(10,0^{*}29)	0.1029, 1.2974, 1.4785, 1.5162, 1.6075, 1.7217, 1.8347, 1.8776, 2.1712, 2.2297, 2.2461, 2.5646, 2.6061, 2.7184, 2.8407, 3.0269, 3.1155, 3.2346, 3.2923, 3.3537, 3.6245, 4.1259, 4.3659, 5.8869, 6.3897, 6.4626, 7.1265, 7.4080, 9.1758, 9.1781.
	$S_2^{30:40}$	(0^{*}19,10,0^{*}10)	0.1029, 0.2988, 0.3216, 0.4796, 0.6374, 0.9526, 1.0861, 1.1015, 1.1059, 1.2218, 1.2924, 1.2974, 1.4785, 1.5162, 1.6075, 1.7217, 1.8347, 1.8776, 2.1712, 2.2297, 3.6245, 4.1259, 4.3659, 5.8869, 6.3897, 6.4626, 7.1265, 7.4080, 9.1758, 9.1781.
	$S_3^{30:40}$	(0^{*}29,10)	0.1029, 0.2988, 0.3216, 0.4796, 0.6374, 0.9526, 1.0861, 1.1015, 1.1059, 1.2218, 1.2924, 1.2974, 1.4785, 1.5162, 1.6075, 1.7217, 1.8347, 1.8776, 2.1712, 2.2297, 2.2461, 2.5646, 2.6061, 2.7184, 2.8407, 3.0269, 3.1155, 3.2346, 3.2923, 3.3537.

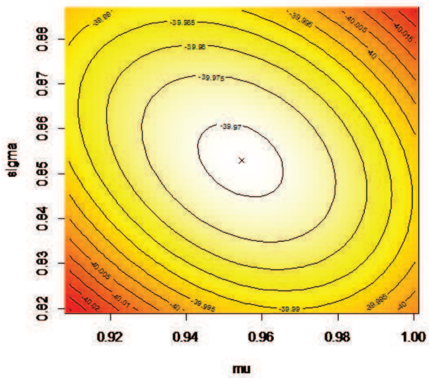

Fig. 1: Contour plot of μ and σ under the equipment data.

probabilities increased. On the other hand, the root mean square errors, mean absolute biases, and average confidence length for the Bayes MCMC estimates of some parameters increased while the coverage probabilities decreased. In conclusion, this study extends the results for the LTG distribution under progressive type-II censoring. A Bayesian estimation method is proposed that applies the M-H algorithm to estimate the unknown parameters or the reliability characteristics of the LTG distribution under progressively type-II censoring. This method is recommended due to its efficiency in reducing RMSEs and MABs while maintaining high CPs. These findings are beneficial for researchers and practitioners working in the field of reliability analysis, as they provide more accurate estimates of the LTG distribution parameters and reliability characteristics.

Table 14: Point estimates and standard errors of μ, σ, $S(t)$, and $h(t)$ using MLEs and Bayesian estimates under choices of m, n and simulated data sets.

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Sample	Parameter	MLE	SE			GE					
$c\rightarrow$						-5			+5		
Prior$\rightarrow$			I	II	III	I	II	III	I	II	III
$S_1^{(20:40)}$	μ	3.3942	3.3143	3.3911	3.3943	3.3240	3.3925	3.3943	3.2844	3.3869	3.3942
		0.4938	0.0016	0.0006	0.0001	0.0153	0.0151	0.0147	0.0150	0.0145	0.0144
	σ	2.0336	2.0194	2.0439	2.0341	2.0320	2.0461	2.0341	1.9818	2.0371	2.0340
		0.3678	0.0014	0.0006	0.0001	0.0003	0.0002	0.0002	0.0002	0.0002	0.0001
	$S(1.5)$	0.9256	0.9198	0.9247	0.9255	0.9201	0.9247	0.9255	0.9186	0.9245	0.9255
		0.0457	0.0002	0.0001	0.0001	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	$h(1.5)$	0.1071	0.1128	0.1074	0.1071	0.1155	0.1078	0.1071	0.1036	0.1061	0.1071
		0.0448	0.0002	0.0001	0.0001	0.0014	0.0014	0.0014	0.0014	0.0014	0.0014
$S_2^{(20:40)}$	μ	1.0473	0.9693	1.0512	1.0477	1.0126	1.0558	1.0477	0.6333	1.0368	1.0475
		2.2984	0.0019	0.0006	0.0001	0.0023	0.0003	0.0002	0.0004	0.0003	0.0001
	σ	3.2841	3.2195	3.2884	3.2844	3.2311	3.2899	3.2844	3.1843	3.2840	3.2843
		0.9760	0.0017	0.0006	0.0001	0.0082	0.0081	0.0080	0.0081	0.0078	0.0075
	$S(1.5)$	0.7782	0.7708	0.7785	0.7782	0.7711	0.7785	0.7782	0.7701	0.7784	0.7782
		0.0620	0.0001	0.0001	0.0001	0.0007	0.0007	0.0006	0.0006	0.0006	0.0005
	$h(1.5)$	0.1909	0.1986	0.1907	0.1908	0.1996	0.1908	0.1908	0.1957	0.1903	0.1908
		0.0481	0.0001	0.0001	0.0001	0.0009	0.0009	0.0009	0.0009	0.0008	0.0008
$S_3^{(20:40)}$	μ	1.5575	1.4830	1.5571	1.5577	1.5011	1.5601	1.5577	1.4205	1.5482	1.5576
		0.3787	0.0015	0.0006	0.0001	0.0036	0.0035	0.0034	0.0035	0.0034	0.0027
	σ	1.4855	1.4990	1.5017	1.4862	1.5161	1.5048	1.4862	1.4469	1.4924	1.4861
		0.4296	0.0014	0.0006	0.0001	0.0035	0.0032	0.0033	0.0032	0.0031	0.0030
	$S(1.5)$	0.6859	0.6766	0.6874	0.6860	0.6780	0.6877	0.6860	0.6723	0.6867	0.6860
		0.0635	0.0003	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0001	0.0001
	$h(1.5)$	0.3829	0.3973	0.3801	0.3827	0.4031	0.3811	0.3827	0.3810	0.3770	0.3826
		0.1007	0.0004	0.0002	0.0002	0.0005	0.0003	0.0003	0.0003	0.0002	0.0002
$S_1^{(30:40)}$	μ	2.5238	2.4497	2.5218	2.5239	2.4608	2.5236	2.5240	2.4148	2.5163	2.5239
		0.3658	0.0015	0.0006	0.0001	0.0096	0.0092	0.0091	0.0096	0.0091	0.0089
	σ	1.7372	1.7426	1.7501	1.7377	1.7550	1.7527	1.7378	1.7055	1.7424	1.7376
		0.2999	0.0013	0.0006	0.0001	0.0019	0.0016	0.0017	0.0017	0.0016	0.0015
	$S(1.5)$	0.8469	0.8382	0.8464	0.8469	0.8389	0.8465	0.8469	0.8360	0.8460	0.8469
		0.0556	0.0002	0.0001	0.0001	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	$h(1.5)$	0.2048	0.2138	0.2044	0.2048	0.2165	0.2048	0.2048	0.2057	0.2030	0.2048
		0.0512	0.0002	0.0001	0.0001	0.0008	0.0008	0.0008	0.0007	0.0007	0.0006
$S_2^{(30:40)}$	μ	0.1819	0.2786	0.2304	0.1840	0.3407	0.2459	0.1843	0.0031	0.1449	0.1832
		2.5662	0.0012	0.0005	0.0001	0.0063	0.0054	0.0052	0.0050	0.0048	0.0042
	σ	2.7736	2.7159	2.7791	2.7740	2.7270	2.7808	2.7740	2.6821	2.7740	2.7739
		0.7793	0.0016	0.0006	0.0001	0.0051	0.0049	0.0046	0.0049	0.0049	0.0043
	$S(1.5)$	0.7055	0.7024	0.7078	0.7057	0.7029	0.7079	0.7057	0.7010	0.7076	0.7057
		0.0620	0.0002	0.0001	0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
	$h(1.5)$	0.2600	0.2658	0.2582	0.2599	0.2673	0.2584	0.2599	0.2613	0.2575	0.2599
		0.0508	0.0002	0.0001	0.0001	0.0006	0.0005	0.0004	0.0005	0.0004	0.0003
$S_3^{(30:40)}$	μ	1.5829	1.5073	1.5822	1.5831	1.5244	1.5851	1.5832	1.4483	1.5735	1.5830
		0.3435	0.0014	0.0006	0.0001	0.0037	0.0036	0.0033	0.0037	0.0033	0.0028
	σ	1.4470	1.4711	1.4634	1.4477	1.4841	1.4664	1.4477	1.4322	1.4544	1.4476
		0.2858	0.0012	0.0006	0.0001	0.0036	0.0035	0.0035	0.0035	0.0034	0.0033
	$S(1.5)$	0.6879	0.6788	0.6894	0.6880	0.6801	0.6896	0.6880	0.6747	0.6886	0.6880
		0.0622	0.0003	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001	0.0001
	$h(1.5)$	0.3886	0.3984	0.3856	0.3884	0.4029	0.3866	0.3884	0.3854	0.3826	0.3883
		0.0770	0.0004	0.0002	0.0002	0.0005	0.0004	0.0004	0.0004	0.0003	0.0003

Table 15: Interval Estimates with their Lengths of μ, σ, $S(t)$, and $h(t)$ using MLEs under Choices of m, n

.

Sample Prior→	Parameter	ACI Lower	Upper	Length
$S_1^{(20:40)}$	μ	2.4264	4.3620	1.9356
	σ	1.3127	2.7545	1.4418
	$S(1.5)$	0.8359	1.0152	0.1792
	$h(1.5)$	0.0193	0.1949	0.1756
$S_2^{(20:40)}$	μ	0.0000	5.5520	5.5520
	σ	1.3711	5.1970	3.8258
	$S(1.5)$	0.6568	0.8996	0.2429
	$h(1.5)$	0.0967	0.2851	0.1884
$S_3^{(20:40)}$	μ	0.8152	2.2997	1.4845
	σ	0.6435	2.3275	1.6840
	$S(1.5)$	0.5614	0.8104	0.2490
	$h(1.5)$	0.1854	0.5803	0.3949
$S_1^{(20:40)}$	μ	1.8069	3.2407	1.4338
	σ	1.1494	2.3249	1.1755
	$S(1.5)$	0.7380	0.9559	0.2179
	$h(1.5)$	0.1044	0.3052	0.2008
$S_2^{(20:40)}$	μ	0.0000	5.2115	5.2115
	σ	1.2463	4.3010	3.0547
	$S(1.5)$	0.5840	0.8271	0.2430
	$h(1.5)$	0.1604	0.3596	0.1992
$S_3^{(20:40)}$	μ	0.9096	2.2562	1.3465
	σ	0.8868	2.0072	1.1204
	$S(1.5)$	0.5661	0.8097	0.2437
	$h(1.5)$	0.2377	0.5394	0.3017

Table 16: Different progressive type-II censored samples from equipment data set

.

(n,m)	$S_i^{m:n}$	Scheme	Progressive type II censored samples
(30,15)	$S_1^{15:30}$	(15,0^{*}14)	0.11, 1.43, 1.46, 1.49, 1.74, 1.82, 1.86, 1.97, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73
	$S_2^{15:30}$	(0^{*}7,15,0^{*}7)	0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73
	$S_3^{15:30}$	(0^{*}14,15)	0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23
(30,25)	$S_1^{25:30}$	(5,0^{*}24)	0.11, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23, 1.24, 1.43, 1.46, 1.49, 1.74, 1.82, 1.86, 1.97, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73
	$S_2^{25:30}$	(0^{*}14,5,0^{*}14)	0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23, 1.82, 1.86, 1.97, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73
	$S_3^{25:30}$	(0^{*}24,5)	0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23, 1.24, 1.43, 1.46, 1.49, 1.74, 1.82, 1.86, 1.97, 2.23, 2.37

Table 17: Interval Estimates with their Lengths of μ, σ, $S(t)$, and $h(t)$ using Bayesian Estimation under Choices of m, n, and Simulated Data Sets

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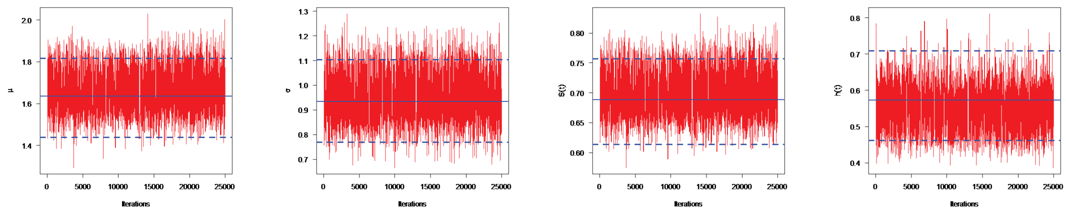
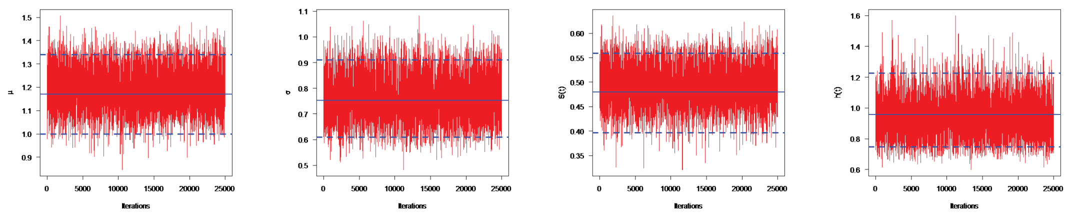
Sample Prior→	Parameter	BCI								
		I			II			III		
		Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
$S_1^{(20:40)}$	μ	2.8141	3.8063	0.9922	3.2007	3.5820	0.3813	3.3746	3.4141	0.0395
	σ	1.6097	2.4873	0.8776	1.8586	2.2335	0.3749	2.0145	2.0541	0.0395
	$S(1.5)$	0.8656	0.9671	0.1015	0.9036	0.9453	0.0418	0.9233	0.9278	0.0044
	$h(1.5)$	0.0651	0.1622	0.0971	0.0887	0.1255	0.0368	0.1051	0.1090	0.0039
$S_2^{(20:40)}$	μ	0.4109	1.5499	1.1390	0.8573	1.2445	0.3872	1.0281	1.0676	0.0395
	σ	2.6959	3.7644	1.0685	3.0968	3.4827	0.3859	3.2648	3.3043	0.0395
	$S(1.5)$	0.7287	0.8055	0.0768	0.7650	0.7910	0.0260	0.7769	0.7796	0.0026
	$h(1.5)$	0.1639	0.2421	0.0782	0.1781	0.2044	0.0264	0.1895	0.1922	0.0027
$S_3^{(20:40)}$	μ	0.9993	1.9137	0.9143	1.3694	1.7436	0.3742	1.5381	1.5776	0.0395
	σ	1.0830	1.9741	0.8910	1.3158	1.6923	0.3764	1.4666	1.5061	0.0395
	$S(1.5)$	0.5875	0.7571	0.1696	0.6499	0.7225	0.0726	0.6821	0.6898	0.0077
	$h(1.5)$	0.2859	0.5508	0.2650	0.3302	0.4401	0.1099	0.3769	0.3885	0.0116
$S_1^{(20:40)}$	μ	1.9755	2.8957	0.9202	2.3338	2.7096	0.3758	2.5043	2.5438	0.0394
	σ	1.3698	2.1795	0.8097	1.5682	1.9370	0.3688	1.7182	1.7577	0.0395
	$S(1.5)$	0.7683	0.9055	0.1372	0.8183	0.8756	0.0573	0.8439	0.8500	0.0061
	$h(1.5)$	0.1529	0.2844	0.1314	0.1787	0.2317	0.0529	0.2020	0.2076	0.0056
$S_2^{(20:40)}$	μ	0.0124	0.7427	0.7302	0.0720	0.4062	0.3342	0.1645	0.2038	0.0392
	σ	2.2488	3.2115	0.9628	2.5929	2.9700	0.3770	2.7544	2.7937	0.0393
	$S(1.5)$	0.6488	0.7484	0.0996	0.6878	0.7263	0.0385	0.7036	0.7077	0.0041
	$h(1.5)$	0.2153	0.3280	0.1127	0.2383	0.2805	0.0422	0.2577	0.2621	0.0044
$S_3^{(20:40)}$	μ	1.0273	1.9232	0.8959	1.3950	1.7680	0.3729	1.5635	1.6029	0.0394
	σ	1.1204	1.8842	0.7638	1.2839	1.6484	0.3645	1.4281	1.4676	0.0394
	$S(1.5)$	0.5933	0.7579	0.1646	0.6515	0.7254	0.0740	0.6840	0.6920	0.0080
	$h(1.5)$	0.2959	0.5296	0.2337	0.3351	0.4460	0.1109	0.3824	0.3944	0.0121



$S_1^{(15:30)}$



$S_1^{(25:30)}$

Fig. 2: MCMC trace plots of *mu*, σ, $S(t)$ and $h(t)$ from equipment data set.

Table 18: The point estimates with their (standard errors) and interval estimates with their [lengths]

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Sample c→	Parameter	MLE	SE	GE -3	GE +3	ACI	BCI
$S_1^{(15:30)}$	μ	1.7346	1.6347	1.6402	1.6236	(1.1870,2.2822)	(1.4378,1.8176)
		(0.2794)	(0.0006)	(0.0006)	(0.0007)	[1.0952]	[0.3797]
	σ	1.0108	0.9337	0.9411	0.9187	(0.6189,1.4027)	(0.7697,1.1032)
		(0.1999)	(0.0005)	(0.0004)	(0.0006)	[0.7838]	[0.3334]
	$S(1.5)$	0.7195	0.6879	0.6872	0.6841	(0.5378,0.9011)	(0.6137,0.7566)
		(0.0927)	(0.0002)	(0.0002)	(0.0002)	[0.3633]	[0.1429]
	$h(1.5)$	0.4932	0.5720	0.5789	0.5586	(0.2064,0.7801)	(0.4615,0.7088)
		(0.1463)	(0.0004)	(0.0005)	(0.0004)	[0.5737]	[0.2473]
$S_2^{(15:30)}$	μ	0.9824	0.8799	0.8911	0.8557	(0.0000,2.3135)	(0.6743,1.0762)
		(0.6792)	(0.0006)	(0.0006)	(0.0008)	[2.3135]	[0.4019]
	σ	1.4310	1.3378	1.3447	1.3235	(0.5290,2.3331)	(1.1502,1.5242)
		(0.4602)	(0.0006)	(0.0005)	(0.0007)	[1.8041]	[0.3740]
	$S(1.5)$	0.5814	0.5458	0.5470	0.5433	(0.4042,0.7586)	(0.4913,0.5941)
		(0.0904)	(0.0002)	(0.0002)	(0.0002)	[0.3543]	[0.1028]
	$h(1.5)$	0.4835	0.5400	0.5442	0.5322	(0.1957,0.7712)	(0.4584,0.6418)
		(0.1468)	(0.0003)	(0.0004)	(0.0003)	[0.5756]	[0.1834]
$S_3^{(15:30)}$	μ	0.9611	0.8797	0.8869	0.8647	(0.6575,1.2647)	(0.7270,1.0360)
		(0.1549)	(0.0005)	(0.0005)	(0.0006)	[0.6072]	[0.3090]
	σ	0.6943	0.6278	0.6379	0.6076	(0.3315,1.0572)	(0.4796,0.7923)
		(0.1851)	(0.0005)	(0.0004)	(0.0005)	[0.7257]	[0.3127]
	$S(1.5)$	0.3758	0.3163	0.3230	0.3002	(0.1846,0.5669)	(0.2191,0.4055)
		(0.0975)	(0.0003)	(0.0003)	(0.0005)	[0.3823]	[0.1864]
	$h(1.5)$	1.1342	1.3409	1.3739	1.2798	(0.4056,1.8627)	(0.9882,1.8181)
		(0.3717)	(0.0013)	(0.0015)	(0.0009)	[1.4571]	[0.8299]
		(0.1463)	(0.0004)	(0.0005)	(0.0004)	[0.5737]	[0.2473]
$S_1^{(25:30)}$	μ	1.2566	1.1700	1.1770	1.1580	(0.9001,1.6131)	(0.9997,1.3393)
		(0.1819)	(0.0005)	(0.0005)	(0.0006)	[0.7130]	[0.3397]
	σ	0.8148	0.7521	0.7600	0.7364	(0.5282,1.1014)	(0.6105,0.9102)
		(0.1462)	(0.0004)	(0.0003)	(0.0004)	[0.5733]	[0.2997]
	$S(1.5)$	0.5287	0.4799	0.4834	0.4723	(0.3705,0.6868)	(0.3966,0.5593)
		(0.0807)	(0.0002)	(0.0002)	(0.0003)	[0.3163]	[0.1626]
	$h(1.5)$	0.8279	0.9579	0.9736	0.9281	(0.4561,1.1997)	(0.7481,1.2259)
		(0.1897)	(0.0007)	(0.0009)	(0.0006)	[0.7436]	[0.4778]
$S_2^{(25:30)}$	μ	0.9443	0.8509	0.8612	0.8288	(0.2661,1.6225)	(0.6678,1.0361)
		(0.3460)	(0.0006)	(0.0005)	(0.0006)	[1.3564]	[0.3683]
	σ	1.0280	0.9493	0.9572	0.9331	(0.5505,1.5051)	(0.7850,1.1224)
		(0.2441)	(0.0006)	(0.0004)	(0.0005)	[0.9547]	[0.3374]
	$S(1.5)$	0.4806	0.4337	0.4366	0.4276	(0.3277,0.6335)	(0.3612,0.5010)
		(0.0780)	(0.0002)	(0.0002)	(0.0003)	[0.3058]	[0.1398]
	$h(1.5)$	0.7170	0.8178	0.8281	0.7983	(0.3736,1.0604)	(0.6607,1.0171)
		(0.1752)	(0.0005)	(0.0007)	(0.0005)	[0.6868]	[0.3564]
$S_3^{(25:30)}$	μ	0.9732	0.8904	0.8985	0.8731	(0.5987,1.3473)	(0.7195,1.0596)
		(0.1911)	(0.0005)	(0.0004)	(0.0006)	[0.7486]	[0.3401]
	σ	0.7892	0.7274	0.7359	0.7103	(0.4724,1.1048)	(0.5822,0.8884)
		(0.1613)	(0.0004)	(0.0003)	(0.0004)	[0.6324]	[0.3062]
	$S(1.5)$	0.4144	0.3635	0.3682	0.3532	(0.2646,0.5642)	(0.2801,0.4441)
		(0.0764)	(0.0002)	(0.0002)	(0.0003)	[0.2996]	[0.1640]
	$h(1.5)$	0.9707	1.1140	1.1340	1.0771	(0.5178,1.4236)	(0.8623,1.4357)
		(0.2311)	(0.0009)	(0.0010)	(0.0007)	[0.9057]	[0.5734]

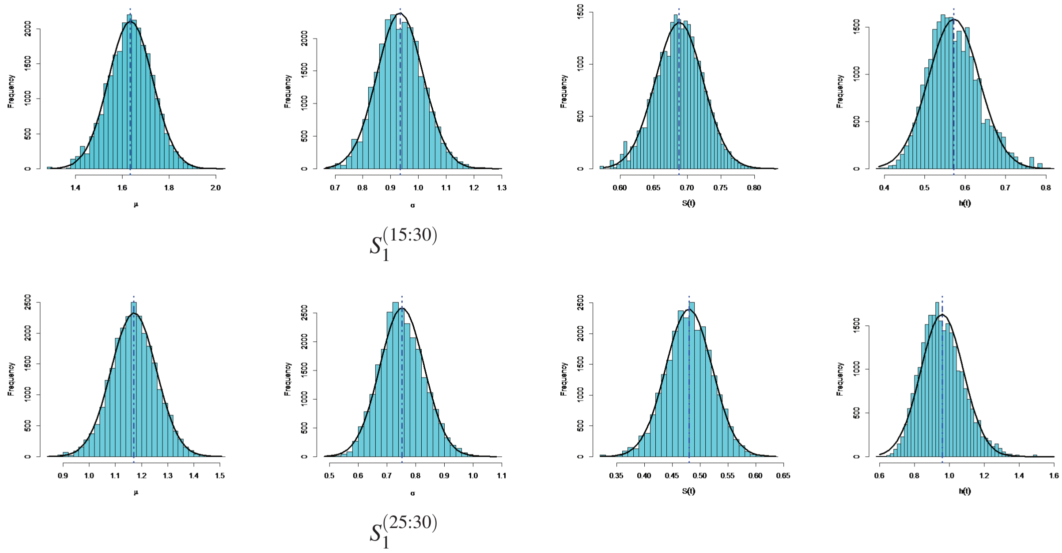

Fig. 3: Histogram and kernel estimates of *mu*, σ, $S(t)$ and $h(t)$ from equipment data set.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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