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Concerning the Inequality of Hermite-Hadamard Generalized

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Abstract: In this paper, we present new extensions of the Hermite-Hadamard type generalized inequalities for convex functions, within the framework of a generalized operator integral. Results are general in nature.

Keywords: Fractional derivatives and integrals, Fractional integral inequalities, Generalized convex mapping

1 Introduction

A function $f:\mathscr{I}\subseteq\mathbb{R}\to\mathbb{R}$ is said to be convex on the interval $\mathscr{I}$, if the inequality

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y) \tag{1}$$

holds for all $x,y\in\mathscr{I}$ and $t\in[0,1]$. We say that f is concave if $-f$ is convex.

Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. The following inequalities

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{2}$$

holds. This double inequality is known in the literature as Hermite–Hadamard integral inequality for convex functions (see [1]). Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (4) we refer the reader to the recent papers [2,3,4,5,6,7,8].

Consider the classic biparameter Mittag–Leffler function $E_{a,b}(.)$, with $Re(a)>0$, $Re(b)>0$, defined by $E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)}$. For convenience, we will use the notation $E_{a,b}(t^{-\alpha})_k$ to designate the k-th term of $E_{a,b}(.)$ when necessary. Two classic examples are $E_{1,1}(z)=e^z$ and $E_{1,1}(z)_1=z$.

A non empty set $\mathscr{K}$ is called generalized e-convex set if

$$u+t\,E_{a,b}(v-u)\in\mathscr{K}, \tag{3}$$

holds for all $u,v\in\mathscr{K}$ and $t\in[0,1]$. A function $h:\mathscr{K}\subseteq\mathbb{R}\to\mathbb{R}$ is said to be generalized e-convex on a generalized e-convex set $\mathscr{K}$, if the inequality

$$h(u+t\,E_{a,b}(v-u))\leq th(v)+(1-t)h(u), \tag{4}$$

holds for all $u,v\in\mathscr{K}$ and $t\in[0,1]$ with $E_{a,b}(z)$ the Mittag–Leffler function. We say that h is e-concave if $-h$ is e-convex.

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*Remark.*Obviously, if in the (4) we have $E_{1,1}(z)_1$, then the classical definition of convexity is obtained.

In [9] a generalized fractional derivative was defined in the following way.

Definition 1.*Given a function $f:[0,+\infty)\to\mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$N_F^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon} \tag{5}$$

for all $t>0$, $\alpha\in(0,1)$ and $F(t,\alpha)$ is some function. Here, we will use some cases of F defined in terms of the Mittag–Leffler function $E_{a,b}(.)$ with $Re(a),Re(b)>0$.

If f is $\alpha-$differentiable on $(0,\alpha)$ and $\lim_{t\to 0^+}N_F^{(\alpha)}f(t)$ exists, then define $N_F^{(\alpha)}f(0)=\lim_{t\to 0^+}N_F^{(\alpha)}f(t)$. Note that, if f is differentiable, then $N_F^{(\alpha)}f(t)=F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative.

The function $E_{a,b}(z)$ was defined and studied by Mittag–Leffler in the year 1903. It is a direct generalization of the exponential function. This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others.

We consider the following examples:

I) $F(t,\alpha)\equiv 1$. In this case, we have the ordinary derivative.

II) $F(t,\alpha)=e^{(\alpha-1)t}$. This kernel satisfies the property that $F(t,\alpha)\to 1$ as $\alpha\to 1$ and yields a conformable derivative used in [10].

III)$F(t,\alpha)=t^\alpha$. With this kernel, we have $F(t,\alpha)\to t$ as $\alpha\to 1$. It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

IV) $F(t,\alpha)=t^{-\alpha}$. With this kernel, we have $F(t,\alpha)\to t^{-1}$ as $\alpha\to 1$ .This is the derivative N_3^α studied in [11]. As in the previous case, the results obtained have not been reported in the literature.

V)$F(t,\alpha)=E_{1,1}(t^{-\alpha})_1=t^\alpha$. With this kernel we have $F(t,\alpha)\to x$ as $\alpha\to 1$ (see [7]). It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

VI) Using the Robotov's Function given by

$$F(t,\alpha)=R_\alpha(\beta,t)=t^\alpha\sum_{k=0}^{\infty}\frac{\beta^k t^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)}= t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}).$$

Definition 2.*Let I be an interval $I\subseteq\mathbb{R}$, $a,t\in I$ and $\alpha\in\mathbb{R}$. The integral operator $J_{T,a}^\alpha$, right and left, is defined for every locally integrable function f on I as*

$$J_{T,a+}^\alpha(f)(t)=\int_a^t\frac{f(s)}{T(t-s,\alpha)}ds, t>a. \tag{6}$$

$$J_{T,b-}^\alpha(f)(t)=\int_t^b\frac{f(s)}{T(s-t,\alpha)}ds, b>t. \tag{7}$$

*Remark.*It is easy to see that the case of the J_T^α operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing some of the fractional integrals that already exist in the literature. For example, we have that

1) if $T(t,\alpha)=t^{1-\alpha}$, $T(t,\alpha)=\Gamma(\alpha)T(x-t,\alpha)$, then from (12) we have the right-sided Riemann–Liouville fractional integrals $(R_{a+}^\alpha f)(t)$. Similarly, from (13) we obtain the left-sided derivative of Riemann–Liouville fractional integral. Then its corresponding right differential operator is

$$({}^{RL}\ D_{a+}^\alpha f)(t)=\frac{d}{dt}(R_{a+}^{1-\alpha}f)(t).$$

Analogously, we obtain the left differential operator.
2) with $T(t,\alpha)=t^{1-\alpha}$, $T(t-x,\alpha)=\Gamma(\alpha)T(\ln t-\ln x,\alpha)t$, we obtain the right-sided Hadamard integral from (12). The left-sided Hadamard integral is obtained similarly from (13). The right derivative is

$$({}^{H}\ D_{a+}^\alpha f)(t)=t\frac{d}{dt}(H_{a+}^{1-\alpha}f)(t),$$

and in a similar way, we can obtain the left.
3) The right-sided Katugampola integral is obtained from (12) making

$$T(t,\alpha)=t^{1-\alpha},\quad e(t)=t^\rho,\quad T(t,\alpha)=\frac{\Gamma(\alpha)}{F(\rho,\alpha)}\frac{F(e(t)-e(x),\alpha)}{e'(t)},$$

analogously for the integral left fractional. In this case, the right derivative is

$$({}^{K}\ D_{a^+}^{\alpha,\rho}f)(t)=t^{1-\rho}\frac{d}{dt}\ K_{a^+}^{1-\alpha,\rho}f(t)= F(t,\rho)\frac{d}{dt}\ K_{a^+}^{1-\alpha,\rho}f(t),$$

and we can obtain the left derivative in the same way.
4) The solution of equation $(-\Delta)^{-\frac{\alpha}{2}}\phi(u)=-f(u)$ called Riesz potential, is given by the expression $\phi=C_n^\alpha\int_{R^n}\frac{f(v)}{|u-v|^{n-\alpha}}dv$, where C_n^α is a constant (see [12, 13, 14]). Obviously, this solution can be expressed in terms of the operator (12) very easily.
5) Obviously, we can define the lateral derivative operators (right and left) in the case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make

use of the fact that if f is differentiable, then $N_F^{\alpha} f(t) = F(t,\alpha) f'(t)$ where $f'(t)$ is the ordinary derivative. For the right derivative we have $\left(N_{F,a+}^{\alpha} f\right)(t) = N_F^{\alpha}\left[J_{T,a+}^{\alpha}(f)(t)\right] = \frac{d}{dx}\left[J_{T,a+}^{\alpha}(f)(t)\right] F(t,\alpha)$, similarly to the left.
6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined. For example, in [15] presented the definition of fractional integral of f with respecto to g of following way. Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$ having a continuous derivative $g'(x)$ on (a,b). The left-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by

$$I_{a+;g}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s) f(s)}{[g(t)-g(s)]^{1-\alpha}} ds, \quad x > a, \quad (8)$$

similarly the right lateral derivative is defined as well

$$I_{b-;g}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s) f(s)}{[g(s)-g(t)]^{1-\alpha}} ds, \quad x < b. \quad (9)$$

It will be very easy for the reader to build the kernel T in this case.

7) We can define the function space $L_{\alpha}^{p}[a,b]$ as the set of functions over $[a,b]$ such that $(J_{T,a+}^{\alpha}[f(t)]^p(b)) < +\infty$.

It is clear that it is necessary to define an integral operator associated with the N_F derivative, which will be used in this work, which allows us to obtain results associated with integration processes (see [16]). The following statement is analogous to the one known from the Ordinary Calculus (see [17,9]).

Theorem 1. *Let f be N-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

a) If f is differentiable ${}_{N_F}J_{t_0}^{\alpha}(N_F^{\alpha} f(t)) = f(t) - f(t_0)$.
b) $N_F^{\alpha}\left({}_{N_F}J_{t_0}^{\alpha} f(t)\right) = f(t)$.

An important property, and necessary, in our work is that established in the following result.

Theorem 2. *(Integration by parts) Let u and v be N-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

$${}_{N_F}J_{t_0}^{\alpha}((uN_F^{\alpha} v)(t)) = [uv(t) - uv(t_0)] - {}_{N_F}J_{t_0}^{\alpha}((vN_F^{\alpha} u)(t)) \quad (10)$$

In this work, to facilitate the calculations related to the definition of Generalized e-convex, we will use the following differential and integral operators as follows.

Definition 3. *Given a function $f : [0,+\infty) \to \mathbb{R}$. Then the N_4-derivative of f of order α is defined by*

$$N_4^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\, e^{t^{\alpha}}) - f(t)}{\varepsilon} \quad (11)$$

for all $t > 0$, $\alpha \in (0,1)$.

If f is N_4-differentiable in some $(0,\alpha)$, and $\lim_{t \to 0^+} N_4^{(\alpha)} f(t)$ exists, then define $N_4^{(\alpha)} f(0) = \lim_{t \to 0^+} N_4^{(\alpha)} f(t)$, note that if f is differentiable, then $N_4^{(\alpha)} f(t) = e^{t^{\alpha}} f'(t)$ where $f'(t)$ is the ordinary derivative.

In [17] the integral operator corresponding to the derivative N_F was defined as follows.

Definition 4. *Let I be an interval $I \subseteq \mathbb{R}$, $a,t \in I$, $\alpha \in \mathbb{R}$ and F an absolutly continuous function. The integral operator $J_{F,a+}^{a}$, right and left, is defined for every locally integrable function f on I as*

$$J_{F,a+}^{a}(f)(t) = \int_a^t \frac{f(s)}{F(t-s,\alpha)} ds, t > a. \quad (12)$$

$$J_{F,b-}^{a}(f)(t) = \int_t^b \frac{f(s)}{F(s-t,\alpha)} ds, b > t. \quad (13)$$

Then the corresponding integral operator of the Derivative N_4, is given by the following form.

Definition 5. *Let I be an interval $I \subseteq \mathbb{R}$, $a,t \in I$, $\alpha \in \mathbb{R}$. The integral operator $J_{F,a+}^{a}$, right and left, is defined for every locally integrable function f on I as*

$${}_{N_4}\ J_{a+}^{\alpha} f(b) = \int_a^b e^{-\left(\frac{b-x}{b-a}\right)^{\alpha}} (b-x)^{\alpha-1} f(x) dx, x > a. \quad (14)$$

$${}_{N_4}\ J_{b-}^{\alpha} f(a) = \int_a^b e^{-\left(\frac{x-a}{b-a}\right)^{\alpha}} (x-a)^{\alpha-1} f(x) dx, b > x. \quad (15)$$

Remark. As can be seen, the integral operator thus defined is a "weighted" Riemann-Liouville integral.

Also, from [7] we have the following.

Definition 6. *Let $\alpha \in \mathbb{R}$ and $a < b$. For each function $f \in L_{\alpha,0}[a,b]$, let us define the fractional integrals*

$${}_{N_3}J_{a+}^{\alpha} f(x) = \int_a^x (x-t)^{-\alpha} f(t) dt$$

$${}_{N_3}J_{b-}^{\alpha} f(x) = \int_x^b (t-x)^{-\alpha} f(t) dt$$

for every $x \in [a,b]$

The aim of our paper is to establish some generalized inequalities of Hermite–Hadamard type for e-convex functions using the generalized integral operator defined above.

2 Main Results

We will start with the following equality that will be useful in establishing.

Lemma 1. *Let $\alpha \in (0,1)$, $f : [a,b] \to [0,+\infty)$ be a differentiable function defined on $[a,b]$, with $0 < a < b$. If $f' \in L[a,b]$, then*

$$\frac{\alpha}{(b-a)^{\alpha+1}}\left[{}_{N_4} J^{\alpha}_{b^-} f(a) + {}_{N_4} J^{\alpha}_{a^+} f(b)\right] + \frac{(e^{-1}-1)(f(b)+f(a))}{b-a} = \int_0^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(at+(1-t)b)dt.$$

Proof. We can write I as follows:

$$I = \int_0^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(at+(1-t)b)dt = \int_0^1 e^{-(1-t)^{\alpha}} f'(at+(1-t)b)\,dt - \int_0^1 e^{-t^{\alpha}} f'(at+(1-t)b)\,dt.$$

Integrating by parts and using the change of variables $x = at+(1-t)b$, we have that

$$\begin{aligned}\int_0^1 e^{-(1-t)^{\alpha}} f'(at+(1-t)b)\,dt &= \frac{1}{b-a}\left(e^{-1}f(b)-f(a)\right) - \frac{\alpha}{b-a}\int_0^1 e^{-(1-t)^{\alpha}}(1-t)^{\alpha-1} f(at+(1-t)b)dt \\ &= \frac{1}{b-a}\left(e^{-1}f(b)-f(a)\right) + \frac{\alpha}{(b-a)^{\alpha+1}}\int_a^b (x-a)^{-\alpha} e^{-\left(\frac{x-a}{b-a}\right)^{\alpha}} f(x)dx \\ &= \frac{1}{b-a}\left(e^{-1}f(b)-f(a)\right) + \frac{\alpha}{(b-a)^{\alpha+1}}\ {}_{N_4} J^{\alpha}_{b^-} f(a).\end{aligned}$$

Using similar arguments as in the above, we deduce that

$$\int_0^1 e^{-t^{\alpha}} f'(at+(1-t)b)\,dt = \frac{1}{b-a}\left(f(b) - e^{-1}f(a)\right) - \frac{\alpha}{(b-a)^{\alpha}}\ {}_{N_F} J^{\alpha}_{a^+} f(b).$$

The desired equality follows by combining these two identities and rearranging the terms.

Corollary 1. *Under the conditions of the previous lemma, if f' is an increasing and generalized ϕ-convex function, then we have*

$$\frac{\alpha}{(b-a)^{\alpha+1}}\left[{}_{N_4} J^{\alpha}_{b^-} f(a) + {}_{N_4} J^{\alpha}_{a^+} f(b)\right] + \frac{(e-1)\{f(b)+f(a)\}}{(b-a)} \leq \int_0^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'\{b+t\,E_k(a-b)\}dt.$$

Theorem 3. *Let $\alpha \in (0,1)$, $f : [a,b] \to [0,+\infty)$ a differentiable function. If $f' \in L_1[a,b]$ and increasing, then*

$$\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[{}_{N_4} J^{\alpha}_{b^-} f(a) + {}_{N_4} J^{\alpha}_{a^+} f(b)\right] \leq \frac{(e-1)(f(b)+f(a))}{(b-a)}.$$

Proof. Consider the following.

$$\begin{aligned}&\int_0^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(b+t\,E_k(a-b))dt \\ &= \int_0^{1/2} \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(b+t\,E_k(a-b))dt + \int_{1/2}^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(b+t\,E_k(a-b))dt \\ &= \int_0^{1/2} \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] f'(b+t\,E_k(a-b))dt + \int_{1/2}^1 \left[e^{-(t)^{\alpha}} - e^{-(1-t)^{\alpha}}\right] f'(a+t\,E_k(b-a))dt \\ &= \int_0^{1/2} \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] \left[f'(b+t\,E_k(a-b)) - f'(a+t\,E_k(b-a))\right] dt.\end{aligned}$$

Since the integrand is non-negative, we obtain the desired inequality.

Theorem 4. *Let $\alpha \in (0,1)$, $f : [a,b] \to [0,+\infty)$ be a differentiable function, with $0 < a < b$. If $f' \in L[a,b]$ and increasing, and $|f'|$ is a generalized ϕ-convex function, then*

$$\frac{\alpha}{(b-a)^{\alpha+1}}\left[{}_{N_4} J^{\alpha}_{b^-} f(a) + {}_{N_4} J^{\alpha}_{a^+} f(b)\right] + \frac{(e-1)(f(b)+f(a))}{(b-a)} \leq \left(\Gamma\left(\frac{1}{\alpha},0\right) - \Gamma\left(\frac{1}{\alpha},1\right)\right) \times \left(\frac{|f'(a)|+|f'(b)|}{\alpha}\right),$$

with $\Gamma(a,x)$ the classical gamma function.

*Proof.*From the previous Corollary, we have

$$\frac{\alpha}{(b-a)^{\alpha+1}}\left[{}_{N_4}J^{\alpha}_{b^-}f(a)+{}_{N_4}J^{\alpha}_{a^+}f(b)\right] + \frac{(e-1)(f(b)+f(a))}{(b-a)} \le \int_0^1 \left[e^{-(1-t)^{\alpha}} - e^{-t^{\alpha}}\right] \left|f'(b+t\,E_k(a-b))\right|dt$$
$$\le \int_0^1 e^{-(1-t)^{\alpha}} \left|f'(b+t\,E_k(a-b))\right|dt + \int_0^1 e^{-t^{\alpha}} \left|f'(b+t\,E_k(a-b))\right|dt$$
$$\le \int_0^1 e^{-(1-t)^{\alpha}} \left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt + \int_0^1 e^{-t^{\alpha}} \left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt.$$

After calculating both integrals, we obtain the desired result.

Theorem 5.*Let $\alpha\in(0,1)$, $f:[a,b]\to[0,+\infty)$ be a generalized ϕ-convex and increasing function, with $1<a<b$, then*

$$\frac{\alpha}{b-a}\;{}_{N_3}J^{\alpha}_{a}f(b)\le \min \{\, f(b)\,(A-B)-f(a)C),\; f(a)\,(A-B)-f(b)C)\}, \tag{16}$$

where $A=\Gamma(\frac{2}{\alpha},1)-\Gamma(\frac{1}{\alpha},1)$, $B=\Gamma(\frac{2}{\alpha},0)-\Gamma(\frac{1}{\alpha},0)$ *and* $C=\Gamma(\frac{2}{\alpha},1)-\Gamma(\frac{2}{\alpha},0)$.

*Proof.*We observe that for $t\in[a,b]$, the following holds;

$$t^{\alpha}=\left(\frac{t}{t-a}\right)^{\alpha}(t-a)^{\alpha}\ge\left(\frac{a}{b-a}\right)^{\alpha}(t-a)^{\alpha} \tag{17}$$

and

$$t^{\alpha}=\left(\frac{t}{b-t}\right)^{\alpha}(b-t)^{\alpha}\ge\left(\frac{a}{b-a}\right)^{\alpha}(b-t)^{\alpha}. \tag{18}$$

From (17) and (18), we obtain, respectively

$$e^{\,-t^{\alpha}}\le e^{\,-\left(\frac{a}{b-a}\right)^{\alpha}\,(t-a)^{\alpha}} \tag{19}$$

and

$$e^{\,-t^{\alpha}}\le e^{\,-\left(\frac{a}{b-a}\right)^{\alpha}\,(b-t)^{\alpha}}. \tag{20}$$

From (19) we have

$$\begin{aligned}{}_{N_3}J^{\alpha}_{a}f(b) &\le (b-a)\int_a^b f(t)\,e^{-\,a^{\alpha}\,\left(\frac{t-a}{b-a}\right)^{\alpha}}dt\\ &\le (b-a)\int_0^1 f(a+(b-a)z)\,e^{-\,a^{\alpha}z^{\alpha}}dz\\ &\le (b-a)\int_0^1 f(a+z\,E_k(b-a))\,e^{-z^{\alpha}}dz\\ &\le (b-a)\int_0^1 [f(a)z+(1-z)f(b)]\,e^{-z^{\alpha}}dz\\ &\le \{\,f(b)\,(A-B)-f(a)C)\}\end{aligned}$$

Thus

$${}_{N_3}J^{\alpha}_{a}f(b)\le\{\,f(b)\,(A-B)-f(a)C)\}$$

Now using (20) and similar arguments as in the above, we obtain

$${}_{N_3}J^{\alpha}_{a}f(b)\le\{\,f(a)\,(A-B)-f(b)C)\}$$

We obtain the desired result from the last two inequalities.

Theorem 6.*Let $\alpha\in(0,1)$, $f:[a,b]\to[0,+\infty)$ be a generalized ϕ-convex and increasing function, with $0<a<b$, then*

$${}_{N_3}J^{\alpha}_{a}f(b)\le\frac{1}{(b-a)^3\alpha}\times\{A(f(b)-f(a))-B(bf(b)-af(a))\} \tag{21}$$

where $A=\Gamma(\frac{2}{\alpha},\,b^{\alpha})-\Gamma(\frac{2}{\alpha},\,a^{\alpha})$, *and* $B=\Gamma(\frac{1}{\alpha},\,b^{\alpha})-\Gamma(\frac{1}{\alpha},\,a^{\alpha})$.

*Proof.*Using change of variables with $t=bs+a(1-s)$ and the generalized ϕ-convexity of f yields

$$\begin{aligned}&\int_a^b f(t)\,e^{-\,t^{\alpha}}dt\\ &=\frac{1}{(b-a)}\times\int_0^1 f\,(bs+a(1-s))\;e^{\,-(bs+a(1-s))^{\alpha}}ds\\ &\le\frac{1}{(b-a)}\times\int_0^1 f\,(a+s\,E_k(b-a))\;e^{\,-(bs+a(1-s))^{\alpha}}ds\\ &\le\frac{1}{(b-a)}\times\int_0^1 [f(a)s+f(b)(1-s)]\;e^{\,-(bs+a(1-s))^{\alpha}}ds.\end{aligned}$$

Integrating the last expression, we obtain

$$\int_0^1 [f(a)s+f(b)(1-s)]\, e^{-(bs+a(1-s))^\alpha} ds$$
$$= \frac{1}{(b-a)^2\alpha}\left\{\left[\Gamma(\frac{2}{\alpha}, b^\alpha)-\Gamma(\frac{2}{\alpha}, a^\alpha)\right](f(b)-f(a)) - \left[\Gamma(\frac{1}{\alpha}, b^\alpha)-\Gamma(\frac{1}{\alpha}, a^\alpha)\right] \times (bf(b)-af(a))\right\}$$

Where we get the inequality sought.

The inequality (21) can be refined, if we use the notion of generalized ϕ-convexity directly, as the following result shows.

Theorem 7. *Let $\alpha \in (0,1)$, $f:[a,b]\to[0,+\infty)$ be a generalized ϕ-convex and increasing function, with $0<a<b$, then*

$$_{N_3}J_a^\alpha f(b) \leq \frac{f(a)}{E_k(b-a)(b-a)^2\alpha}$$
$$\left[\Gamma\left(\frac{2}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{2}{\alpha}, a^\alpha\right)\right]$$
$$-b\left[\Gamma\left(\frac{1}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{1}{\alpha}, a^\alpha\right)\right]$$
$$+f(b)\left[\Gamma\left(\frac{2}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{2}{\alpha}, a^\alpha\right)\right]$$
$$+a\left[\Gamma\left(\frac{1}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{1}{\alpha}, a^\alpha\right)\right]$$

Proof. Using change of variables with $t=bs+a(1-s)$, the ϕ-convexity of f and integrating, we have

$$\int_a^b f(t)e^{-t^\alpha} dt = \frac{1}{E_k(b-a)}$$
$$\int_0^{\frac{b-a}{E_k(b-a)}} f\{a+sE_k(b-a)\}e^{-\{a+sE_k(b-a)\}^\alpha} ds$$
$$\leq \frac{1}{E_k(b-a)}\int_0^{\frac{b-a}{E_k(b-a)}} [f(b)s+f(a)(1-s)] \times e^{-\{bs+a(1-s)\}^\alpha} ds$$
$$\frac{1}{E_k(b-a)(b-a)^2\alpha}$$
$$\times\{f(a)(A-bB)+f(b)(A+aB)\},$$

with

$$A=\left[\Gamma\left(\frac{2}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{2}{\alpha}, a^\alpha\right)\right]$$

and

$$B=\left[\Gamma\left(\frac{1}{\alpha}, \left((b-a)\frac{b-a}{E_k(b-a)}+a\right)^\alpha\right) - \Gamma\left(\frac{1}{\alpha}, a^\alpha\right)\right]$$

This completes the proof.

3 Conclusion

In the development of tgis article we have presented some new extensions of the Hermite-Hadamard type for convex functions, within the framework of a generalized integral operator integral with the kernel associated to the functions e^{t^s} which is a particular case of the generalized of the integral operator related to the N_F derivative extensively studied by the authors in [18,17]. At this point we point out that differents generalized convexity type, such as $s-$convex functions, $(s,m)-$convex functions and others, can be obtained. We hope that this work will stimulate progress in the line of research associated with the study of integral inequalities

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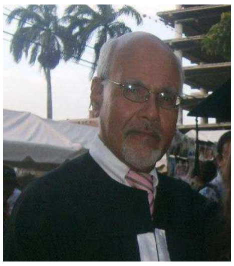

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