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Certain integral inequalities associated with the strongly harmonic $h-$convex functions

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Abstract: We study the concept of strongly harmonically h-convex functions and some examples and properties of them. Here, we develop few inequalities for this new class of functions, specifically these inequalities are: Hermite-Hadamard and Fejer. In addition, we establish some applications of our results to special media of non zero and non negative real numbers.

Keywords: h-convex functions, Hermite–Hadamard inequality, Fejer inequality, convex functions

1 Introduction

In the last few years, many extensions and generalizations have been studied for classical convexity, and the theory of inequalities has made necessary contributions in several fields of mathematics. An eminent subclass of convex functions is that of strongly convex functions established by B. T. Polyak [1] as follows: A function f is said to be strongly convex if there exists a $d > 0$ such that

$$f(\beta u_1 + (1-\beta)u_2) \leq \beta f(u_1) + (1-\beta)f(u_2) - d\beta(1-\beta)|u_1 - u_2|^2,$$

$\forall\, \beta \in [0,1]$.

Strongly convex functions are broadly used in applied economics, nonlinear optimization and many other branches of pure and applied mathematics. Since strong convexity is a nourishing of the concept of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. For more detailed information on strongly convex functions, see [2,3,4,5,6,7] and references therein.

In [8], Shi and Wang introduced the concept of harmonic set using the harmonic mean and which is used in some branches of science as electric circuits. It was defined as: A set $\hat{I} = [a_1, a_2] \subseteq \mathbb{R} \setminus \{0\}$ is said to be harmonic convex set if

$$\left(\frac{u_1 u_2}{tu_1 + (1-t)u_2}\right) \in \hat{I},$$

for all $u_1, u_2 \in \hat{I}$ and $t \in [0,1]$.

Also, with the use of weighted harmonic mean it s possible to define the harmonic convex functions: a function $f : \hat{I} \subset \mathbb{R} - \{0\} \to \mathbb{R}$ is said to be harmonic convex function on $\hat{I}$, if

$$f\left(\frac{u_1 u_2}{tu_1 + (1-t)u_2}\right) \leq tf(u_1) + (1-t)f(u_2), \quad (1)$$

for all $u_1, u_2 \in \hat{I}$ and $t \in [0,1]$.

In [9] and [10], Anderson *et.al.* and Iscan I. have studied many properties of this kind of generalized convex functions, also, Noor M.A. *et.al.* in [11,12,13,14] have established the geometric significance and characterization.

Our research are committed to the classical results based to convex functions defined by Jaques Hadamard [15] and Ch. Hermite [16].

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Preliminaries

Recently some articles have been published dealing with the following definitions [11].

Definition 1. *A set $\hat{I}=[a_1,a_2]\subseteq\mathbb{R}\setminus\{0\}$ is consider to be harmonic convex set if*

$$\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\in\hat{I},$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

Definition 2. *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic convex function on $\hat{I}$, if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le tf(u_1)+(1-t)f(u_2), \quad (2)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

A study of properties, geometric significance and characterizations regarding this type of functions is given in a publication of M.A. Noor et.al. [12].

Some others publications have been generalized the Definition 2 (See [11,13,14])

Definition 3. *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic $s-$convex function in the second sense on $\hat{I}$, for some $s\in(0,1]$ if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le t^sf(u_1)+(1-t)^sf(u_2), \quad (3)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

Definition 4. *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic $P-$convex function on $\hat{I}$, if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le f(u_1)+f(u_2), \quad (4)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

Definition 5. *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic $MT-$convex function on $\hat{I}$, if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le\frac{\sqrt{1-t}}{2\sqrt{t}}f(u_1)+\frac{\sqrt{t}}{2\sqrt{1-t}}f(u_2), \quad (5)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

Definition 6(See [11]). *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic $tgs-$convex function , if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le t(1-t)(f(u_1)+f(u_2)), \quad (6)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

Definition 7(See [13]). *Let $h:[0,1]\subset J\to\mathbb{R}$ a non-negative function. A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be harmonic $h-$convex function if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le g(t)f(u_1)+h(1-t)f(u_2), \quad (7)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

The inequality of Hermite-Hadamard and others has been established for all these types of generalized convex functions, as we can refer in [17,18,13].

About strongly harmonic convexity property of a function it will be necessary to cite some basics.

Definition 8(See [19]). *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be strongly harmonic convex function with modulus $d>0$ if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le tf(u_1)+(1-t)f(u_2) -dt(1-t)\left|\frac{u_1u_2}{u_1+u_2}\right|^2, \quad (8)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

M.A Noor *et.al.* in [13] presented the following definition.

Definition 9. *A function $f:\hat{I}\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is consider to be strongly generalized harmonic convex function with modulus $d>0$, for some $s\in[-1,1]$, if*

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)\le t^s(1-t)^s(f(u_1)+f(u_2)) -dt(1-t)\left|\frac{u_1u_2}{u_1+u_2}\right|^2, \quad (9)$$

$\forall\, u_1,u_2\in\hat{I}$ *and* $t\in[0,1]$.

For our main results, we will require the following Lemma, which is proved in [10].

Lemma 1. *Let $f:\hat{I}\to\mathbb{R}$ be a differentiable function on $int(\hat{I})$ and $a_1,a_2\in\hat{I}$ with $a_1<a_2$. If $f'\in L([a_1,a_2])$, then*

$$\frac{f(a_1)+f(a_2)}{2}-\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(x)}{x^2}dx =\frac{a_1a_2(a_2-a_1)}{2}\times \int_0^1\frac{1-2t}{(ta_2+(1-t)a_1)^2}f'\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)dt$$

2 On strongly harmonic $h-$convex functions

Now, we define the following definition.

Definition 10. *Let $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. The function $f:\mathbb{R}_+\to\mathbb{R}$ is consider to be strongly harmonic h−convex with modulus $d>0$ if*

$$f\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\le h(t)f(u_1)+h(1-t)f(u_2) -dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2, \tag{10}$$

$\forall\, u_1,u_2\in\mathbb{R}_+$ *and* $t\in[0,1]$.

Remark. (i)If $h(t)=t$, then a strongly harmonically $h-$convex functions is reduces to strongly harmonically convex function [20].

(ii)If $h(t)=t^s$ for some $s\in(0,1]$, then a strongly harmonic $h-$convex functions is reduces to strongly harmonic $s-$convex function in the second sense [19].

The following theorems are properties of the strongly harmonic $h-$convex functions.

Theorem 1. *Consider $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. Let $f_1,g_1:\mathbb{R}_+\to\mathbb{R}$ be two functions and $k\ge 0$,*

1. *If f_1,g_1 are strongly harmonic $h-$convex functions, then f_1+g_1 is a strongly harmonic $h-$convex function with modulus $2d$.*
2. *If f_1 is strongly harmonic $h-$convex function, then kf_1 is a strongly harmonic $h-$convex function..*

Proof. (1) Assume $u_1,u_2\in\mathbb{R}_+$ and $t\in[0,1]$,

$$\begin{aligned}(f_1+g_1)\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right) &= f_1\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right) +g_1\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\\ &\le h(t)f_1(u_1)+h(1-t)f_1(u_2)\\ &\quad +h(t)g_1(u_1)+h(1-t)g_1(u_2)-2dt(1-t)\left|\frac{u_1-u_2}{xy}\right|^2\\ &= h(t)(f_1+g_1)(u_1)\\ &\quad +h(1-t)(f_1+g_1)(u_2)-2dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2.\end{aligned}$$

Thus, f_1+g_1 is strongly harmonic $h-$convex function with mo-dulus $2d$.

(2) Assume $u_1,u_2\in\mathbb{R}_+$ and $t\in[0,1]$,

$$\begin{aligned}(kf_1)\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right) &= kf_1\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\\ &\le h(t)(kf_1)(u_1)\\ &\quad +h(1-t)(kf_1)(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2,\end{aligned}$$

so, kf_1 is a strongly harmonic $h-$convex functions with modulus d .

Hence the proof is complete.

Theorem 2. *Let $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. If $f_1,f_2:\mathbb{R}_+\to\mathbb{R}$ are strongly harmonic $h-$convex functions with modulus $d>0$, then $f:=\max\{f_1,f_2\}$ so is too.*

Proof. Assume $u_1,u_2\in$ and $t\in[0,1]$. Since f_1 and f_2 are harmonic $h-$convex function and $f:=\max\{f_1,f_2\}$, then we have

$$\begin{aligned}f_1\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right) &\le h(t)f_1(u_1)+h(1-t)f_1(u_1)\\ &\quad -dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2\\ &\le h(t)f(u_1)+h(1-t)f(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2,\end{aligned}$$

and

$$\begin{aligned}f_2\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right) &\le h(t)f_2(u_1)+h(1-t)f_2(u_2)\\ &\quad -dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2\\ &\le h(t)f(u_1)+h(1-t)f(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2.\end{aligned}$$

By the above we get

$$\begin{aligned}f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right) &= \max\left\{f_1\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right),f_2\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\right\}\\ &\le h(t)f(u_1)+h(1-t)f(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2.\end{aligned}$$

Thus, f is strongly harmonic $h-$convex functions. Hence the proof is complete.

Theorem 3.*Let $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. If $f_n:\mathbb{R}_+\to\mathbb{R}$ is a sequence of strongly harmonic $h-$convex functions with modulus $d>0$, converging pointwise to a function f on $\mathbb{R}_+$, then f is strongly harmonic $h-$convex function with modulus $d>0$.*

*Proof.*Consider $u_1,u_2\in\mathbb{R}_+$ and $t\in[0,1]$

$$\begin{aligned}&f\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\\&=\lim_{n\to\infty}f_n\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\\&\qquad\le\lim_{n\to\infty}\left[h(t)f_n(u_1)+h(1-t)f_n(u_2)-\right.\\&\qquad\qquad\left.dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2\right]\\&=h(t)f(u_1)+h(1-t)f(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2.\end{aligned}$$

Hence the proof is complete.

Theorem 4.*Let $f:\mathbb{R}_+\to\mathbb{R}$ be a harmonic convex function with $f(u_1)\ge u_1$, and $g:\mathbb{R}_+\to\mathbb{R}$ be a non-decreasing and strongly harmonic $h-$convex function with modulus $d>0$, such that $f(\mathbb{R}_+)\subseteq\mathbb{R}_+$, then $g\circ f$ is a strongly harmonic $h-$convex function.*

*Proof.*Since f is a harmonic convex function we have, for any $u_1,u_2\in\mathbb{R}_+$ and $t\in[0,1]$, we obtain,

$$f\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\le tf(u_1)+(1-t)f(u_2)$$

In addition, g is a non-decreasing function and is a strongly harmonic convex function, therefore

$$\begin{aligned}&g\left(f\left(\frac{u_1u_2}{tu_2+(1-t)u_1}\right)\right)\\&\qquad\le g\left(tf(u_1)+(1-t)f(u_2)\right)\\&\qquad\le h(t)g(f(u_1))+h(1-t)g(f(u_2))\\&\qquad\quad-dt(1-t)\left|\frac{f(u_1)-f(u_2)}{f(u_1)f(u_2)}\right|^2\\&\qquad\le h(t)(g\circ f)(u_1)+h(1-t)(g\circ f)(u_2)\\&\qquad\quad-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2.\end{aligned}$$

Thus, $g\circ f$ is a strongly harmonic $h-$convex function. Hence the proof is complete. □

3 Hermite-Hadamard type inequality

Theorem 5(Hermite-Hadamard type left-inequality). *Let $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. Let $f:\mathbb{R}_+\to\mathbb{R}$ be a strongly harmonic $h-$convex with modulus $d>0$. Then*

$$f\left(\frac{2a_1a_2}{a_1+a_2}\right)\le h(1/2)\frac{2a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1-\frac{d}{12}\left|\frac{a_2-a_1}{a_1a_2}\right|^2 \tag{11}$$

*Proof.*Since Let $f:\mathbb{R}_+\to\mathbb{R}$ is a strongly harmonic $h-$convex with modulus $d>0$, then

$$\begin{aligned}&f\left(\frac{2a_1+a_2}{a_1+a_2}\right)=\\&\qquad f\left(\frac{1}{\frac{1}{2}\frac{1}{\frac{a_1a_2}{ta_1+(1-t)a_2}}+\frac{1}{2}\frac{1}{\frac{a_1a_2}{ta_2+(1-t)a_1}}}\right)\\&\qquad\le h(1/2)f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)\\&\qquad\quad+h(1/2)f\left(\frac{a_1a_2}{ta_2+(1-t)a_1}\right)\\&\qquad\quad-dt(1-t)\left|\frac{a_2-a_1}{2a_1a_2}\right|^2\end{aligned}$$

By integration over the interval $[0,1]$, it is obtained that

$$\begin{aligned}&f\left(\frac{2a_1a_2}{a_1+a_2}\right)\le\\&h(1/2)\left\{\int_0^1 f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)dt\right.\\&\quad\left.+\int_0^1 f\left(\frac{a_1a_2}{ta_2+(1-t)a_1}\right)dt\right\}\\&\quad-d\left|\frac{a_2-a_1}{2a_1a_2}\right|^2\int_0^1 t(1-t)dt.\end{aligned}$$

Since

$$\int_0^1 f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)dt=\int_0^1 f\left(\frac{a_1a_2}{ta_2+(1-t)a_1}\right)dt$$
$$=\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1 \tag{12}$$

and

$$\int_0^1 t(1-t)dt=\frac{1}{6} \tag{13}$$

then we get our desired result.

Theorem 6(Hermite-Hadamard type right-inequality). *Let $h:[0,1]\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$.*

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a strongly harmonic $h-$convex with modulus $d > 0$. Then

$$\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1$$
$$\leq \max\left\{f(a_1)I_1(h)+f(a_2)I_2(h), f(a_2)I_1(h)+f(a_1)I_2(h)\right\}-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2$$

where

$$I_1(h)=\int_0^1 h(t)dt \text{ and } I_2(h)=\int_0^1 h(1-t)dt.$$

Proof. Since f is a strongly harmonic $h-$convex function for all $u_1,u_2 \in \mathbb{R}_+$ then we have

$$f\left(\frac{u_1u_2}{tu_1+(1-t)u_2}\right)$$
$$\leq h(t)f(u_1)+h(1-t)f(u_2)-dt(1-t)\left|\frac{u_1-u_2}{u_1u_2}\right|^2$$

In particular, if $u_1 = a_1$ and $u_2 = a_2$ then

$$f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)$$
$$\leq h(t)f(a_1)+h(1-t)f(a_2)-dt(1-t)\left|\frac{a_2-a_1}{a_1a_2}\right|^2.$$

Integrating over $t \in [0,1]$ it is obtained

$$\int_0^1 f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)dt$$
$$\leq f(a_1)\int_0^1 h(t)dt$$
$$+f(a_2)\int_0^1 h(1-t)dt-d\left|\frac{a_2-a_1}{a_1a_2}\right|^2\int_0^1 t(1-t)dt$$

Similarly

$$\int_0^1 f\left(\frac{a_1a_2}{ta_2+(1-t)a_1}\right)dt$$
$$\leq f(a_2)\int_0^1 h(t)dt$$
$$+f(a_1)\int_0^1 h(1-t)dt-d\left|\frac{a_2-a_1}{a_1a_2}\right|^2\int_0^1 t(1-t)dt$$

As it was mentioned in (12) and (13) we can write

$$\frac{a_1a_2}{a_1+a_2}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1 \leq f(a_1)I_1(h)$$
$$+f(a_2)I_2(h)-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2$$

and

$$\frac{a_1a_2}{a_1+a_2}\int_a^b\frac{f(u_1)}{u_1^2}du_1 \leq f(a_2)I_1(h)$$
$$+f(a_1)I_2(h)-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2,$$

where

$$I_1(h)=\int_0^1 h(t)dt \text{ and } I_2(h)=\int_0^1 h(1-t)dt.$$

Hence we easily deduced the desired result.

Corollary 1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a strongly harmonic convex function with modulus $d > 0$. Then*

$$f\left(\frac{2a_1a_2}{a_1+a_2}\right)$$
$$+\frac{d}{12}\left|\frac{a_2-a_1}{a_1a_2}\right|^2 \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1$$
$$\leq \frac{f(a_1)+f(a_2)}{2}-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2. \tag{14}$$

Proof. Using Theorem 5 and Theorem 6 with the function $h(t) = t$, for all $t \in [0,1]$, then it is attained the desired result. □

This last result coincides with Theorem 2.2 in [20].

Corollary 2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a strongly harmonic $s-$convex function in the second sense. Then*

$$2^{s-1}f\left(\frac{2a_1a_2}{a_1+a_2}\right)$$
$$+\frac{d}{12}\left|\frac{a_2-a_1}{a_1a_2}\right|^2 \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1$$
$$\leq \frac{f(a_1)+f(a_2)}{s+1}-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2.$$

Proof. Using Theorem 5 and Theorem 6 with the function $h(t) = t^s$, for all $t \in [0,1]$ and some $s \in (0,1]$ then it is attained the desired result.

Corollary 3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a strongly harmonic $P-$convex function. Then*

$$\frac{1}{2}f\left(\frac{2a_1a_2}{a_1+a_2}\right)$$
$$+\frac{d}{24}\left|\frac{a_2-a_1}{a_1a_2}\right|^2$$
$$\leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1 \leq f(a_1)$$
$$+f(a_2)-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2 \tag{15}$$

*Proof.*Using Theorem 5 and Theorem 6 with the function $h(t)=1$, for all $t\in[0,1]$, then it is attained the desired result.

Corollary 4.*Let $f:\mathbb{R}_+\to\mathbb{R}$ be a strongly harmonic $MT-$convex function. Then*

$$\frac{1}{2}f\left(\frac{2a_1a_2}{a_1+a_2}\right)+\frac{d}{24}\left|\frac{a_2-a_1}{a_1a_2}\right|^2\leq\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{\pi(f(a_1)+f(a_2))}{4}-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2.$$

*Proof.*Using Theorem 5 and Theorem 6 with the function $h(t)=1$, for all $t\in[0,1]$, then it is attained the desired result. □

Corollary 5.*Let $f:\mathbb{R}_+\to\mathbb{R}$ be a harmonic $tgs-$harmonic convex function. Then*

$$2f\left(\frac{2a_1a_2}{a_1+a_2}\right)+\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2\leq\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{f(a_1)+f(a_2)}{6}-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2.$$

*Proof.*Using Theorem 5 and Theorem 6 with the function $h(t)=t(1-t)$, for all $t\in[0,1]$, then it is attained the desired result. □

Corollary 6.*Let $f:\mathbb{R}_+\to\mathbb{R}$ be a generalized harmonic $tgs-$harmonic convex function. Then*

$$2^{1-2s}f\left(\frac{2a_1a_2}{a_1+a_2}\right)+\frac{2^{-2s}d}{3}\left|\frac{a_2-a_1}{a_1a_2}\right|^2\leq\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq(f(a_1)+f(a_2))B(s+1,s+1)-\frac{d}{6}\left|\frac{a_2-a_1}{a_1a_2}\right|^2,$$

where $B(x,y)$ is the Beta function.

*Proof.*Using Theorem 5 and Theorem 6 with the function $h(t)=t^s(1-t)^s$, for all $t\in[0,1]$, then it is attained the desired result.

*Remark.*If in Corollaries 1-5 we make $d=0$ it follows the following results:

1.for harmonic convex functions

$$f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{f(a_1)+f(a_2)}{2}\tag{16}$$

making coincidence with a result found in [10].

2.for harmonic $s-$convex function in the second sense

$$2^{s-1}f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{f(a_1)+f(a_2)}{s+1}\tag{17}$$

making coincidence with a result presented in [21].

3.for $P-$harmonic convex functions

$$\frac{1}{2}f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq f(a_1)+f(a_2)\tag{18}$$

making coincidence with Corollary 3.4 in [13].

4.for harmonic $MT-$ convex functions

$$\frac{1}{2}f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{\pi(f(a_1)+f(a_2))}{4}$$

5.for harmonic $tgs-$ convex functions

$$2f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq\frac{f(a_1)+f(a_2)}{6}\tag{19}$$

6.for harmonic generalized harmonic $tgs-$functions

$$2^{1-2s}f\left(\frac{2a_1a_2}{a_1+a_2}\right)\leq\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}du_1\leq(f(a_1)+f(a_2))B(s+1,s+1).$$

Now we discuss the right side of the Hermite-Hadamard inequality for the product of two strongly harmonic $h-$convex functions with modulus $d>0$.

Theorem 7. *Let $f,g:\hat{I}\to\mathbb{R}^+$ be two strongly harmonic $h-$convex functions with modulus $d>0$, $a_1,a_2\in\hat{I}$ with $a_1<a_2$. If $fg\in L([a_1,a_2])$ then*

$$\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)g(u_1)}{u_1^2}du_1 \leq M(a_1,a_2)\int_0^1(h(t))^2dt +N(a_1,a_2)\int_0^1 h(t)h(1-t)dt -d\left|\frac{a_1-a_2}{a_1a_2}\right|^2\left\{L(a_1,a_2)\int_0^1 t(1-t)h(t)dt -d\left|\frac{a_1-a_2}{a_1a_2}\right|^2B(3,3)\right\}$$

where

$$M(a_1,a_2)=f(a_1)g(a_1)+f(a_2)g(a_2),$$

$$N(a_1,a_2)=f(a_1)g(a_2)+f(a_2)g(a_1),$$

$$L(a_1,a_2)=f(a_1)+f(a_2)+g(a_1)+g(a_2)$$

and $B(\cdot,\cdot)$ is the Beta function.

Proof. Note that

$$\frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)g(u_1)}{u_1^2}du_1 =\int_0^1 f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)g\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)$$

and since f and g are strongly harmonic $h-$convex functions, we have

$$f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right) \leq h(t)f(a_1)+h(1-t)f(a_2)-dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2$$

and

$$g\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right) \leq h(t)g(a_1)+h(1-t)g(a_2)-dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2;$$

so

$$f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)g\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right) \leq h(t)h(t)f(a_1)g(a_1) +h(t)h(1-t)f(a_1)g(a_2)-h(t)f(a_1)dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2 +h(1-t)h(t)f(a_2)g(a_1)+h(1-t)h(1-t)f(a_2)g(a_2) -h(1-t)f(a_2)dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2 -h(t)g(a_1)dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2 -h(1-t)g(a_2)dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2 +d^2t^2(1-t)^2\left|\frac{a_1-a_2}{a_1a_2}\right|^4$$

Integrating over $t\in[0,1]$

$$\int_0^1 f\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)g\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right) \leq (f(a_1)g(a_1)+f(a_2)g(a_2))\int_0^1(h(t))^2dt +(f(a_1)g(a_2)+f(a_2)g(a_1))\int_0^1 h(t)h(1-t)dt -(f(a_1)+g(a_1)+f(a_2) +g(a_2))d\left|\frac{a_1-a_2}{a_1a_2}\right|^2\int_0^1 t(1-t)h(t)dt +d^2\left|\frac{a_1-a_2}{a_1a_2}\right|^4\int_0^1 t^2(1-t)^2dt.$$

Letting ,

$$M(a_1,a_2)=f(a_1)g(a_1)+f(a_2)g(a_2),$$

$$N(a_1,a_2)=f(a_1)g(a_2)+f(a_2)g(a_1)$$

and

$$L(a_1,a_2)=f(a_1)+f(a_2)+g(a_1)+g(a_2)$$

then we achieve with the desired result.

The following results are established with the use of Lemma 1 .

Theorem 8. *Consider $f:\hat{I}\to\mathbb{R}$ be a differentiable function on $int(\hat{I})$, $a_1,a_2\in\hat{I}$ with $a_1<a_2$, and $f'\in L([a_1,a_2])$. If*

$|f'|^v, v \geq 1$. *is a strongly harmonic $h-$convex function then*

$$\left|\frac{f(a_1)+f(a_2)}{2} - \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2}\frac{f(u_1)}{u_1^2}dx\right| \leq \frac{a_1a_2(a_2-a_1)}{2}\left(\frac{1}{u+1}\right)^{1/u} \times \left\{|f'(a_1)|^v I_1(h) + |f'(a_2)|^v I_2(h) - d\left|\frac{a_1-a_2}{a_1a_2}\right|^2 \mathscr{C}(a_1,a_2)\right\}^{1/v}$$

where

$$I_1(h) = \int_0^1 \frac{h(t)}{(ta_2+(1-t)a_1)^{2v}}dt,$$
$$I_2(h) = \int_0^1 \frac{h(1-t)}{(ta_2+(1-t)a_1)^{2v}}dt,$$

and

$$\mathscr{C}(a_1,a_2) = \frac{1}{(a_2-a_1)^3} \times \left\{\frac{1}{2(2v^2-5v+3)}\left(b^{3-2v}-a^{3-2v}\right) + \frac{(4v-5)}{4v^2-10v+6}\left(a_1a_2^{2-2v}-a^{2-2v}a_2\right)\right\}.$$

*Proof.*Using Lemma 1, Hölder's inequality and the strongly harmonic $h-$convexity of $|f'|^v$ then we have

$$\left|\frac{f(a_1)+f(a_2)}{2} - \frac{a_1a_2}{(a_2-a_1)}\int_{a_1}^{a_2}\frac{f(u_1)}{{u_1}^2}du_1\right|$$

$$\leq \frac{a_1a_2(a_2-a_1)}{2} \times \int_0^1 \left|\frac{1-2t}{(ta_2+(1-t)a_1)^2}\right|\left|f'\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)\right|dt$$

$$\leq \frac{a_1a_2(a_2-a_1)}{2}\left\{\int_0^1 |1-2t|^u dt\right\}^{1/u} \times \left\{\int_0^1 \frac{1}{(ta_2+(1-t)a_1)^{2v}}\left|f'\left(\frac{a_1a_2}{ta_1+(1-t)a_2}\right)\right|^q dt\right\}^{1/v}$$

$$\leq \frac{a_1a_2(a_2-a_1)}{2}\left(\frac{1}{u+1}\right)^{1/u} \times \left(\int_0^1 \frac{1}{(ta_2+(1-t)a_1)^{2v}}\right).$$

$$\left\{\left\{h(t)|f'(a_1)|^v + h(1-t)|f'(a_2)|^v - dt(1-t)\left|\frac{a_1-a_2}{a_1a_2}\right|^2\right\}dt\right\}^{1/v}$$

$$= \frac{a_1a_2(a_2-a_1)}{2}\left(\frac{1}{u+1}\right)^{1/v} \times \left\{|f'(a_1)|^v\int_0^1\frac{h(t)}{(ta_2+(1-t)a_1)^{2v}}dt + |f'(a_2)|^v\int_0^1\frac{h(1-t)}{(ta_2+(1-t)a_1)^{2v}}dt\right\}$$

$$-d\left|\frac{a_1-a_2}{a_1+a_2}\right|^2\int_0^1\frac{t(1-t)}{(ta_2+(1-t)a_1)^{2v}}dt\Bigg\}^{1/v}.$$

Letting

$$I_1(h) = \int_0^1 \frac{h(t)}{(ta_2+(1-t)a_1)^{2v}}dt, \quad , \quad I_2(h) = \int_0^1 \frac{h(1-t)}{(ta_2+(1-t)a_1)^{2v}}dt,$$

and

$$\mathscr{C}(a_1,a_2) = \int_0^1 \frac{t(1-t)}{(ta_2+(1-t)a_1)^{2v}}dt = \frac{1}{(a_2-a_1)^3}\left[\frac{1}{2(2v^2-5v+3)}\left(a_2^{3-2v}-a_1^{3-2v}\right) + \frac{(4v-5)}{4v^2-10v+6}\left(a_1+a_2^{2-2q}-a_1^{2-2v}a_2\right)\right]$$

Hence we get the desired result.

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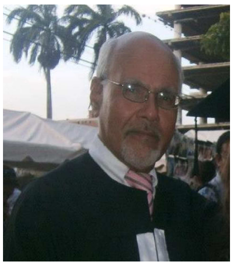

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