R-Norm Information Measure with Applications in Multi Criteria Decision Making Technique under Intuitionistic Fuzzy Set Environment

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Abstract: The main aim of this research article is to define a new information measure for quantifying fuzziness in the intuitionistic fuzzy set environment. For this purpose, we present R-norm intuitionistic fuzzy measure that quantifies the amount of vagueness or fuzziness of a particular fuzzy set. We prove that this measure is a valid measure of intuitionistic fuzzy entropy by making it satisfy essential properties. Also, some mathematical properties are used to check the validation of the measure. In the end, a practical example of decision-making is illustrated in terms of Multi Criteria Decision Making problem that presents the application of the proposed measure.

1 Introduction

The field of fuzzy sets has offered significant applications in various areas of study. This concept, introduced by Zadeh[1], characterised by membership function, gives the amount of vagueness/ambiguity associated with an element. In fuzzy set theory, membership function gives the degree of belongingness of an element to the fuzzy set and it usually lies between 0 and 1. In certain situations, the degree of non-belongingness of an element cannot be calculated by subtracting the degree of belongingness from unity. There might be some degree of hesitation to assign the degree of belongingness and non-belongingness of the element to the fuzzy set under study. A generalization of fuzzy sets characterized by two functions expressing belongingness and non-belongingness respectively was introduced by Atanassov [2], incorporating the degree of hesitation associated with the element called Intuitionistic Fuzzy sets (IFs). Atanassov [3] also discussed the application part of IFs. Further, IFs have found applications in many areas since these are flexible in handling uncertainty and vagueness and are also a tool for defining imperfect facts that have inexact information. IFs have found applications in medical diagnosis [4] [5], pattern recognition [6] [7] [8], market prediction [9], electoral system [10], MCDM [11], [12], etc.

1.1 Preliminaries

Following are some basic concepts related to IFs:

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1.1.1 Definition:

Suppose Y is a non-empty fuzzy set drawn from universal set Z, then we have an ordered triplet given by:

$$Z = \{(y, \eta_z(y), \phi_z(y)) \mid y \in Y\}$$

Where $\eta_z(y), \phi_z(y) : Y \to [0,1]$ denotes the degree of belongingness and non-belongingness respectively such that $0 \leq \eta_z(y) + \phi_z(y) \leq 1 \forall y$. Also, $1 - \eta_z(y) - \phi_z(y)$ denotes the degree of hesitancy of the element y .

1.1.2 Definition

Let P & Q ε FS(y) be such that some basic relations and operations on IFs are defined as:

1. $P \subseteq Q$ if $\eta_P(y_i) \leq \eta_Q(y_i), \phi_P(y_i) \geq \phi_Q(y_i), \forall y \varepsilon Y$

2. $P \supseteq Q$ if $\eta_P(y_i) \geq \eta_Q(y_i), \phi_P(y_i) \leq \phi_Q(y_i), \forall y \varepsilon Y$

3. $P = Q$ if $\eta_P(y_i) = \eta_Q(y_i), \phi_P(y_i) = \phi_Q(y_i), \forall y \varepsilon Y$

4. $P \cup Q = \{(y, max(\eta_P(y_i), \eta_Q(y_i)), min(\phi_P(y_i), \phi_Q(y_i))) \forall y \varepsilon Y\}$

5. $P \cap Q = \{(y, min(\eta_P(y_i), \eta_Q(y_i)), max(\phi_P(y_i), \phi_Q(y_i))) \forall y \varepsilon Y\}$

6. $P^c = \{(y(\phi_P(y, \eta_P(y_i)))) \forall y \varepsilon Y\}$

2 Generalized R-Norm Intuitionistic Fuzzy Measure

WWe state the following R-norm intuitionistic fuzzy measure (RIFM):

$$I_R^{\alpha,\beta}(Z) = \frac{R+\alpha-\beta}{R-\beta} \left(\sum_{i=1}^{n} \left\{ 1 - \left(\eta_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) + \phi_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) + w_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) \right) \right)^{\frac{\alpha}{R+\alpha-\beta}} \right\} \tag{1}$$

$$\text{where}$$

$$w_z(y_i) = 1 - \eta_z(y_i) - \phi_z(y_i), (\eta_z(y_i) + \phi_z(y_i)) \leq 1, R \neq \beta, 0 < (\alpha, \beta) \leq 1$$

For convenience, we put

$$T = \frac{R+\alpha-\beta}{\alpha} \quad i.e$$

$$I_R^{\alpha,\beta}(Z) = \frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^{n} \left\{ 1 - \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \right\} \tag{2}$$

$$\text{where}$$

$$w_z(y_i) = 1 - w_z(y_i) - w_z(y_i), (w_z(y_i) + w_z(y_i)) \leq 1, R \neq \beta, 0 < (\alpha, \beta) \leq 1$$

We verify that (1) is suitable intuitionistic fuzzy measure (IFM) by making use of four important properties given by Luca and Termini [13] viz. Sharpness, Maximality, Resolution and Symmetry.

2.1 Properties of the R-Norm Intuitionistic Fuzzy Measure

2.1.1 **Sharpness**: $I_R^{\alpha,\beta}(Z) = 0$ iff Z is a crisp set i.e either $\eta_z(y_i) = 1, \phi_z(y_i) = 0, \eta_z(y_i) = 0, \phi_z(y_i) = 1 \ \forall \ y \varepsilon Y$

Proof: we assume that $I_R^{\alpha,\beta}(Z) = 0; R, S > 0$ and $R \neq S$ Therefore, we can write

$$\frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^{n} \left\{ 1 - \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \right\} = 0$$

$$\implies \sum_{i=1}^{n} (1) - \sum_{i=1}^{n} \left\{ \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \right\} = 0$$

$$\implies n = \sum_{i=1}^{n} \left\{ \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \right\}$$

which is only possible when $\eta_z(y_i) = 1, \phi_z(y_i) = 0, \eta_z(y_i) = 0, \phi_z(y_i) = 1$
Conversely, let us suppose that Z is a crisp set i.e., $\eta_z(y_i) = 1, \phi_z(y_i) = 0, \eta_z(y_i) = 0, \phi_z(y_i) = 1$

$$\frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^{n} \left\{ 1 - (1+0+0)^{\frac{1}{T}} \right\} = 0$$

$$\implies I_R^{\alpha,\beta}(Z) = 0$$

which proves property 2.1.1.

This property is also verified mathematically as:

Table 1: Verification of Property 2.1.1

R	α	β	$\eta_z(y_i)$	$\phi_z(y_i)$	$I_R^{\alpha,\beta}(Z)$	$\eta_z(y_i)$	$\phi_z(y_i)$	$I_R^{\alpha,\beta}(Z)$
0.67	0.73	0.31	1	0	0	0	1	0
0.67	0.23	0.87	1	0	0	0	1	0
25	0.44	0.65	1	0	0	0	1	0
25	0.92	0.13	1	0	0	0	1	0
3	0.62	0.55	1	0	0	0	1	0
3	0.34	0.42	1	0	0	0	1	0

2.1.2 **Maximality**: $I_R^{\alpha,\beta}(Z)$ gives the maximum value i.e, 1 iff, $\eta_z(y_i) = \phi_z(y_i) \ \forall \ y_i \varepsilon Y$

Proof: To find the maximum value of $I_R^{\alpha,\beta}(Z)$ we differentiate (1) with respect to $\eta_z(y_i)$ and $\phi_z(y_i)$. The following result is obtained:

$$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)} = - \left(\frac{R+\alpha-\beta}{R-\beta} \right) \sum_{i=1}^{n} \left\{ \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \left(\eta_z^{T-1}(y_i) + w_z^{T-1}(y_i) \right) \right\}$$

&

$$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)} = - \left(\frac{R+\alpha-\beta}{R-\beta} \right) \sum_{i=1}^{n} \left\{ \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}} \left(\phi_z^{T-1}(y_i) + w_z^{T-1}(y_i) \right) \right\}$$

For verifying the convexity, we calculate the second differentiation of above functions, that is:

$$\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2} = T \sum_{i=1}^{n} \left[\left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-2} \left(\phi_z^{T-1}(y_i) - w_z^{T-1}(y_i) \right)^2 - \right.$$

$$\left. \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-1} \left(\phi_z^{T-2}(y_i) + w_z^{T-2}(y_i) \right) \right] \qquad \&$$

$$\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2} = T \sum_{i=1}^{n} \left[\left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-2} \left(\phi_z^{T-1}(y_i) - w_z^{T-1}(y_i) \right)^2 - \right.$$

$$\left. \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-1} \left(\phi_z^{T-2}(y_i) + w_z^{T-2}(y_i) \right) \right]$$

Also

$$\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)\phi_z(y_i)} = \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)\eta_z(y_i)}$$

$$= T \sum_{i=1}^{n} \left[\left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-2} \left(\eta_z^{T-1}(y_i) - w_z^{T-1}(y_i) \right) \left(\phi_z^T(y_i) - w_z^{T-1}(y_i) \right) - \right.$$

$$\left. \left(\eta_z^T(y_i) + \phi_z^T(y_i) + w_z^T(y_i) \right)^{\frac{1}{T}-1} \left(w_z^{T-2}(y_i) \right) \right]$$

$I_R^{\alpha,\beta}(Z)$ has maximum value when $\eta_z^T(y_i) = \phi_z^T(y_i) = w_z^T(y_i) = \frac{1}{3} \ \forall i = 1,2,...,n$ called the critical point of the given IFM.

To check the concavity of any function we make use of Hessian matrix as:

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} \end{bmatrix}$$

and show that it is negative semi-definite i.e.,

$$\frac{\partial^2}{\partial x^2} < 0, \frac{\partial^2}{\partial y^2} < 0 \ and \ \left[\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - \left(\frac{\partial^2}{\partial x \partial y} \right)^2 \right] > 0$$

$$Also, \ \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$$

Now applying above Hessian Matrix to prove that $I_R^{\alpha,\beta}(Z)$ is concave function of Z at the stationary point $\frac{1}{3}$ as

$$\begin{bmatrix} \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2} & \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)\partial \phi_z(y_i)} \\ \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)\partial \phi_z(y_i)} & \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \phi_z(y_i)]^2} \end{bmatrix}$$

The Hessian Matrix is calculated as

$$\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2}\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \phi_z(y_i)]^2} - \left(\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)\partial \phi_z(y_i)} \right)^2$$

$$where \ \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2} = \frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \phi_z(y_i)]^2}$$

Here two cases arises $R < 1$ and $R > 1$.

Since $\forall i = 1,2,...n, I_R^{\alpha,\beta}(Z)$ satisfies the conditions of negative semi-definite Hessian matrix showing that (1) is concave function with global maximum at $\eta_z(y_i) = \phi_z(y_i) = w_z(y_i) = \frac{1}{3}$. Thus $I_R^{\alpha,\beta}(Z)$ gives maximum value iff Z is most fuzzy set

Table 2: Computation of Hessian Matrix for different values of parameters

R	α	β	$\eta_z(y_i)$	$\phi_z(y_i)$	$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)}$	$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)}$	$\frac{\partial^2 I_R^{\alpha,\beta}(Z)}{[\partial \eta_z(y_i)]^2}$	H.M
0.67	0.73	0.31	1/3	1/3	0	0	-6.232662	29.13455
0.67	0.23	0.87	1/3	1/3	0	0	-1186.733	1056251
25	0.44	0.65	1/3	1/3	0	0	-114.9006	9901.612
25	0.92	0.13	1/3	1/3	0	0	-58.30607	2549.698
3	0.62	0.55	1/3	1/3	0	0	-173.4049	22551.95
3	0.34	0.42	1/3	1/3	0	0	-313.4991	73434.92

2.1.3 Resolution: $I_R^{\alpha,\beta}(Z) \geq I_R^{\alpha,\beta}(Z*)$ where Z* is sharpened or crisper form of Z i.e., we have for $max\, Z[\eta_z(y_i), \phi_z(y_i)] \leq \frac{1}{3}, \eta_{z*}(y_i) \geq \eta_z(y_i), \phi_{z*}(y_i) \geq \phi_z(y_i)$, also $min\, Z[\eta_z(y_i), \phi_z(y_i)] \geq \frac{1}{3}$, if we have $\eta_{z*}(y_i) \leq \eta_z(y_i), \phi_{z*}(y_i) \geq \phi_z(y_i)$,

Proof: To show that the given measure in (1) is increasing function for $\eta_z(y_i)$ and decreasing for $\phi_2(y_i)$, we partially differentiate $I_R^{\alpha,\beta}(Z)$ with respect to $\eta_z(y_i)$ & $\phi_z(y_i)$ respectively.

Also we have $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)} \geq 0$ if $\eta_z(y_i) \leq \phi_z(y_i)$ which means that $I_R^{\alpha,\beta}(Z)$ is an increasing function. And, for $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)} \leq 0$ with $\eta_z(y_i) \geq \phi_z(y_i)$, we have $I_R^{\alpha,\beta}(Z)$ is decreasing function ith respect to $\eta_z(y_i)$. Additionally, the same holds for $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)} \geq 0$ & $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)} \leq 0$ with $\eta_z(y_i) \geq \phi_z(y_i)$ & $\eta_z(y_i) \leq \phi_z(y_i)$ with respect to $\phi_z(y_i)$. Moreover from property 2.1.2, $I_R^{\alpha,\beta}(Z)$ is a concave function of Z.

Thus, if $max\, Z[\eta_z(y_i), \phi_z(y_i)] \leq \frac{1}{3}$, then $\eta_{z*}(y_i) \geq \eta_z(y_i), \phi_{z*}(y_i) \geq \phi_z(y_i)$, which shows that $\frac{1}{3} \geq \eta_z(y_i) \geq \eta_{z*}(y_i), \frac{1}{3} \geq \phi_z(y_i) \geq \phi_{z*}(y_i)$ and the same holds for $w_{z(y_i)}$ i.e. $\frac{1}{3} \geq w_z(y_i) \geq w_{z*}(y_i)$.
We show the above results with the help of numerical data in table given below as:

Case 1: For increasing function $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)} \geq 0$ and with $max\, Z[\eta_z(y_i), \phi_z(y_i)] \leq \frac{1}{3}$, we have $\eta_z(y_i) \leq \phi_z(y_i)$ under the conditions $\eta_{z*}(y_i) \leq \eta_z(y_i)$ and $\phi_{z*}(y_i) \leq \phi_z(y_i)$

Table 3: Verification of Resolution numerically for Case I

R	α	β	$\eta_z(y_i)$	$\phi_z(y_i)$	$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)}$	$I_R^{\alpha,\beta}(Z)$	$\phi_{z*}(y_i)$	$\eta_{z*}(y_i)$	$I_R^{\alpha,\beta}(Z*)$
0.67	0.73	0.31	0.03	0.13	7.91	3.19	0.01	0.12	2.42
0.67	0.43	0.82	0.24	0.29	12.99	5.74	0.23	0.29	4.73
43	0.65	0.22	0.06	0.13	5.07	1.75	0.05	0.04	1.22
43	0.99	0.50	0.20	0.30	5.11	1.77	0.10	0.10	1.23
			0.08	0.27			0.07	0.20	

Case 2: For decreasing function $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)} \leq 0$ having $min\, Z[\eta_z(y_i), \phi_z(y_i)] \geq \frac{1}{3}$, we have $\eta_z(y_i) \geq \phi_z(y_i)$ under the conditions $\eta_{z*}(y_i) \geq \eta_z(y_i)$ and $\phi_{z*}(y_i) \geq \phi_z(y_i)$

Similarly $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)} \geq 0$ when $x \geq y$ and $\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \phi_z(y_i)} \leq 0$ when $x \leq y$, the function is increasing and decreasing respectivily. Therefore, $I_R^{\alpha,\beta}(Z) \geq I_R^{\alpha,\beta}(Z*)$

2.1.4 Symmetry: $I_R^{\alpha,\beta}(Z) = I_R^{\alpha,\beta}(Z^c) \forall Z \varepsilon IFs(Z)$

. **Proof:** We have from the definition of IFs $Z^c = [(y, \phi_z(y_i), \eta_z(y_i)) | y \varepsilon Y]$. Thus, it can be easily proven that $I_R^{\alpha,\beta}(Z) = I_R^{\alpha,\beta}(Z^c)$.

As the measure given in (1) contains all the four properties of valid IFM. Thus, $I_R^{\alpha,\beta}(Z)$ is a valid measure of IFs.

Table 4: Verification of Resolution numerically for Case II

R	α	β	$\eta_z(y_i)$	$\phi_z(y_i)$	$\frac{\partial I_R^{\alpha,\beta}(Z)}{\partial \eta_z(y_i)}$	$I_R^{\alpha,\beta}(Z)$	$\eta_{z*}(y_i)$	$\phi_{z*}(y_i)$	$I_R^{\alpha,\beta}(Z*)$
0.52	0.48	0.51	0.52	0.41	-6.70	4.80	0.53	0.42	4.22
	0.99	0.10	0.50	0.40	-5.80	4.09	0.52	0.46	3.62
67	0.22	0.43	0.45	0.33	-5.01	2.58	0.50	0.37	2.39
	0.66	0.74	0.39	0.34	-5.04	2.60	0.46	0.49	2.41
			0.56	0.37			0.57	0.39	

2.2 Particular Cases of the R-Norm Intuitionistic Fuzzy Measure

1. For $w_z(y_i) = 0 \forall y_i \varepsilon Y$ the measure given in (1) reduces to the R-norm fuzzy information measure given by Peerzada et al. [14] and considering utilities, tends to Sofi et al. [15]
2. If $\alpha = 1, \beta = 1$ given RIFM (1) becomes the basic R-norm Information measure given by Boekee and Lubbe [16].
3. $R \to 1, \alpha = 1 and \beta = 1$, then the RIFM(1) tends to Shannon's [17] measure of entropy.

2.3 Joint Entropy

If we have two IFs P and Q defined over $Y = \{y_1, y_2, ..., y_n\}$. We define the partition of Y as:

$$Y_1 = \{y_i \varepsilon Y : P \subseteq Q\} = \{y_i \varepsilon Y : \eta_p(y_i) \le \eta_q(y_i), \phi_p(y_i) \ge \phi_q(y_i)\}$$

And

$$Y_2 = \{y_i \varepsilon Y : P \supseteq Q\} = \{y_i \varepsilon Y : \eta_p(y_i) \ge \eta_q(y_i), \phi_p(y_i) \le \phi_q(y_i)\}$$

The joint entropy between P and Q is defined as:

$$I_R^{\alpha,\beta}(P \cup Q) = \frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^n \left[1 - \left\{ \eta_{P \cup Q}^T(y_i) + \phi_{P \cap Q}^T(y_i) + (1 - \eta_{P \cup Q}^T(y_i) + \phi_{P \cap Q}^T(y_i)) \right\}^{\frac{1}{T}} \right]$$

$$= \frac{R+\alpha-\beta}{R-\beta} \sum_{y_i \varepsilon Y_1}^n \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i) \right)^{\frac{1}{T}} \right\} +$$

$$\frac{R+\alpha-\beta}{R-\beta} \sum_{y_i \varepsilon Y_2}^n \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i) \right)^{\frac{1}{T}} \right\}.$$

$$I_R^{\alpha,\beta}(P \cap Q) = \frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^n \left[1 - \left\{ \eta_{P \cap Q}^T(y_i) + \phi_{P \cup Q}^T(y_i) + (1 - \eta_{P \cap Q}^T(y_i) + \phi_{P \cup Q}^T(y_i)) \right\}^{\frac{1}{T}} \right]$$

$$= \frac{R+\alpha-\beta}{R-\beta} \sum_{y_i \varepsilon Y_1}^n \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i) \right)^{\frac{1}{T}} \right\} +$$

$$\frac{R+\alpha-\beta}{R-\beta} \sum_{y_i \varepsilon Y_2}^n \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i) \right)^{\frac{1}{T}} \right\}.$$

Theorem 1: For a universal set Y=$\{y_1, y_2, ..., y_n\}$, if P and Q are two IFs where $P = \{y_i, \eta_p(y_i), \phi_p(y_i) | y_i \varepsilon Y\}$ and $Q = \{y_i, \eta_q(y_i), \phi_q(y_i) | y_i \varepsilon Y\}$ such that whichever $P \subseteq Q$ or $P \supseteq Q$ then $I_R^{\alpha,\beta}(P \cup Q) + I_R^{\alpha,\beta}(P \cap Q) = I_R^{\alpha,\beta}(P) + I_R^{\alpha,\beta}(Q)$

Proof: If we have two IFs P and Q defined over Y=$\{y_1, y_2, ..., y_n\}$. We define the partition of Y as:

$$Y_1 = \{y_i \varepsilon Y : P \subseteq Q\} = \{y_i \varepsilon Y : \eta_p(y_i) \le \eta_q(y_i), \phi_p(y_i) \ge \phi_q(y_i)\}$$

And

$$Y_2 = \{y_i \varepsilon Y : P \supseteq Q\} = \{y_i \varepsilon Y : \eta_p(y_i) \ge \eta_q(y_i), \phi_p(y_i) \le \phi_q(y_i)\}$$

Therefore,

$$I_R^{\alpha,\beta}(P \cup Q) + I_R^{\alpha,\beta}(P \cap Q) = \left[\frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^{n} \left[1 - \left\{ \eta_{P\cup Q}^T(y_i) + \phi_{P\cap Q}^T(y_i) + \left(1 - \eta_{P\cup Q}^T(y_i) + \phi_{P\cap Q}^T(y_i)\right) \right\}^{\frac{1}{T}} \right] + \right.$$

$$\left. \sum_{i=1}^{n} \left[1 - \left\{ \eta_{P\cap Q}^T(y_i) + \phi_{P\cup Q}^T(y_i) + \left(1 - \eta_{P\cap Q}^T(y_i) + \phi_{P\cup Q}^T(y_i)\right) \right\}^{\frac{1}{T}} \right] \right]$$

$$= \frac{R+\alpha-\beta}{R-\beta} \left[\sum_{y_i \varepsilon Y_1}^{n} \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i)\right)^{\frac{1}{T}} \right\} + \sum_{y_i \varepsilon Y_2}^{n} \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i)\right)^{\frac{1}{T}} \right\} + \right.$$

$$\left. \sum_{y_i \varepsilon Y_1}^{n} \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i)\right)^{\frac{1}{T}} \right\} + \sum_{y_i \varepsilon Y_2}^{n} \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i)\right)^{\frac{1}{T}} \right\} \right].$$

$$= \frac{R+\alpha-\beta}{R-\beta} \left[\sum_{y_i \varepsilon Y_1}^{n} \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i)\right)^{\frac{1}{T}} \right\} + \sum_{y_i \varepsilon Y_2}^{n} \left\{ 1 - \left(\eta_Q^T(y_i) + \phi_Q^T(y_i) + w_Q^T(y_i)\right)^{\frac{1}{T}} \right\} + \right.$$

$$\left. \sum_{y_i \varepsilon Y_1}^{n} \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i)\right)^{\frac{1}{T}} \right\} + \sum_{y_i \varepsilon Y_2}^{n} \left\{ 1 - \left(\eta_P^T(y_i) + \phi_P^T(y_i) + w_P^T(y_i)\right)^{\frac{1}{T}} \right\} \right].$$

$$= I_R^{\alpha,\beta}(P) + I_R^{\alpha,\beta}(Q)$$

Hence the result.

3 Behaviour Of The Proposed R-Norm Intuitionistic Fuzzy Measure

We study the behaviour of RIFM proposed in (1) at different values of α, β & R by considering IFs
$Z = \{(0.2, 0.6), (0.6, 0.4), (0.5, 0.2), (0.3, 0.4), (0.1, 0.7), (0.3, 0.3)\}$
In figure 1,the relation between parameters is studied by fixing β and R at $\{0.97, 13\}, \{0.72, 3\}$ & $\{0.5, 0.85\}$ respectively and varying α over its range. It is observed from the above figure that with an increase in the value of α , RIFM (1) also increases. This relation remains same for other values of β and R.

We have analyzed the influence of β on RIFM (1) in figure 2 by fixing α and R at $\{0.97, 13\}, \{0.72, 3\}$ & $\{0.5, 0.85\}$ respectively. It is evident from the figure that there is a positive relation between β and RIFM (1). We also infer that for smaller values of α and R ,the slope remains steep.

We study the relation between R and RIFM (1) by taking different values of R over its range and fix α and β at $\{0.27, 0.32\}, \{0.5, 0.5\}$ & $\{0.92, 0.21\}$ respectively. The values of RIFM (1) obtained for these parameters are plotted in figure 3. We conclude that RIFM (1)decreases as we increase the value of R and the same trend follows for other values ofα and β.

4 Application Of The Proposed R-Norm Intuitionistic Fuzzy Measure

This section provides the use of decision-making method for explaining the MCDM approach in terms of IFs theory. For this, we consider u substitutes as $E_1, E_2, ..., E_u$ that are evaluated by decision maker for v diverse features as $F_1, F_2, ..., F_v$. Also, we assume that $\tau_{ij} = \langle \eta_{ij}, \phi_{ij} \rangle$ be the IF number where η_{ij} denotes that E_i satisfies the feature F_j while as ϕ_{ij} denotes that substitute E_i does not satisfy the feature F_j . Here, from the definition of IFs we have the conditions that $\langle \eta_{ij} + \phi_{ij} \rangle \leq 1$ and $\eta_{ij}, \phi_{ij} \varepsilon [0, 1]$ Therefore, on the basis of result maker inclinations τ_{ij}, the values are summarized in the form of decision matrix.

$$M = \begin{bmatrix} & F_1 & F_2 & F_3 & --- & F_v \\ E_1 & \langle \eta_{11}, \phi_{11} \rangle & \langle \eta_{12}, \phi_{12} \rangle & \langle \eta_{13}, \phi_{13} \rangle & --- & \langle \eta_{1v}, \phi_{1v} \rangle \\ E_2 & \langle \eta_{21}, \phi_{21} \rangle & \langle \eta_{22}, \phi_{22} \rangle & \langle \eta_{23}, \phi_{23} \rangle & --- & \langle \eta_{2v}, \phi_{2v} \rangle \\ & . & . & . & & . \\ & . & . & . & & . \\ & . & . & . & & . \\ E_u & \langle \eta_{u1}, \phi_{u1} \rangle & \langle \eta_{u2}, \phi_{u2} \rangle & \langle \eta_{u3}, \phi_{u3} \rangle & --- & \langle \eta_{uv}, \phi_{uv} \rangle \end{bmatrix}$$
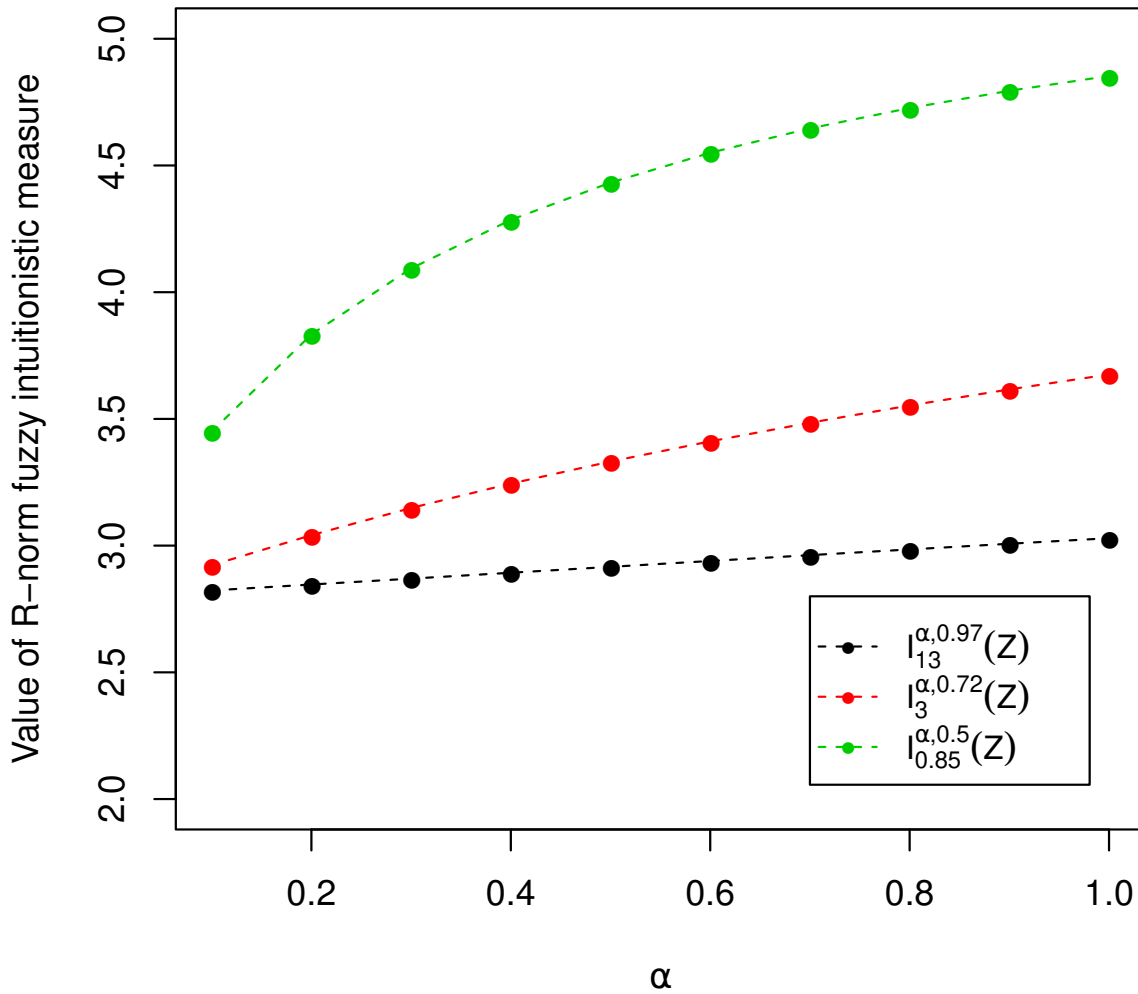
Fig. 1: Graphical Representation of $I_R^{\alpha,\beta}(Z)$ w.r.t α

Hence in order to find the best alternative, we use the following steps to solve the decision-matrix:

Step I: For two types of features i.e., price type and profit type the procedure is to change the rating values of price type into profit type by using normalized form as

$$\tau_{ij} = \begin{cases} \langle \eta_{ij}, \phi_{ij} \rangle & \text{for profit type feature} \\ \langle \phi_{ij}, \eta_{ij} \rangle & \text{for price type feature} \end{cases}$$

Hereafter, we get the normalized decision-matrix as τ_{ij} .

Step II: By making use of this normalized decision matrix, the value of (1) is calculated for different features as $F_1, F_2, ..., F_v$ as

$$[I_R^{\alpha,\beta}(F)]_j = \frac{R+\alpha-\beta}{R-\beta} \sum_{i=1}^{n} \left\{ 1 - \left(\eta_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) + \phi_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) + w_z^{\frac{R+\alpha-\beta}{\alpha}}(y_i) \right)^{\frac{\alpha}{R+\alpha-\beta}} \right\}$$

Step III: Based on Step II i.e., value of IF matrix $[I_R^{\alpha,\beta}(F)]_j$ the weights are calculated as

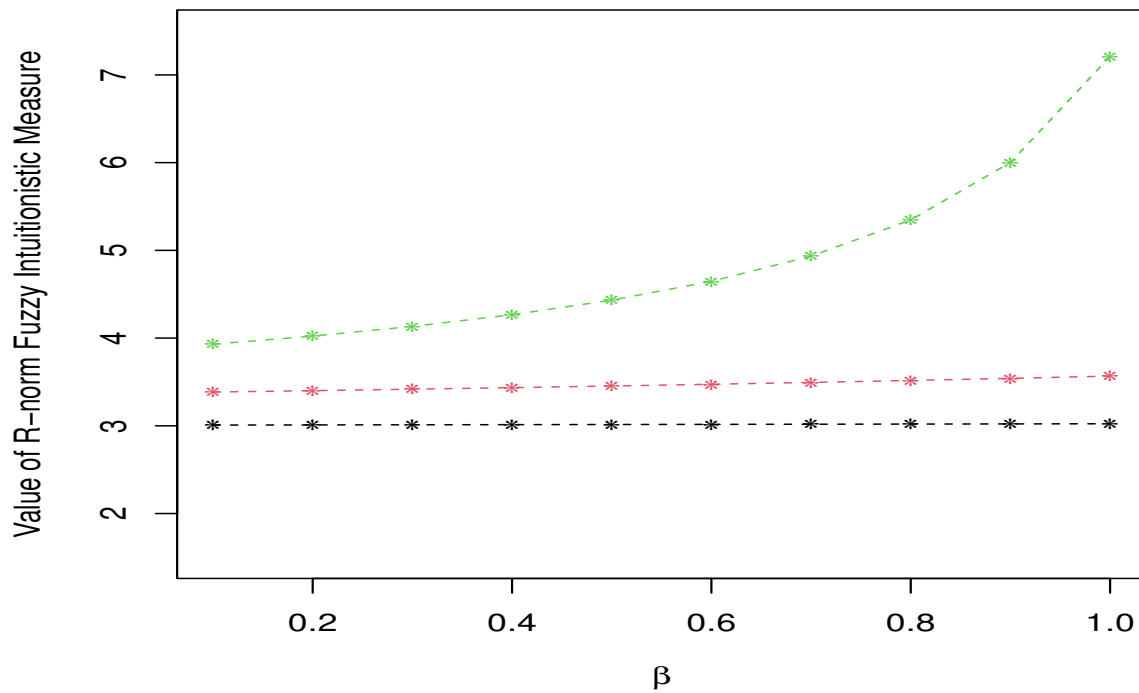$$w_j = \frac{1 - I_R^{\alpha,\beta}(F_j)}{v - \sum_{j=1}^{v} I_R^{\alpha,\beta}(F_j)}$$

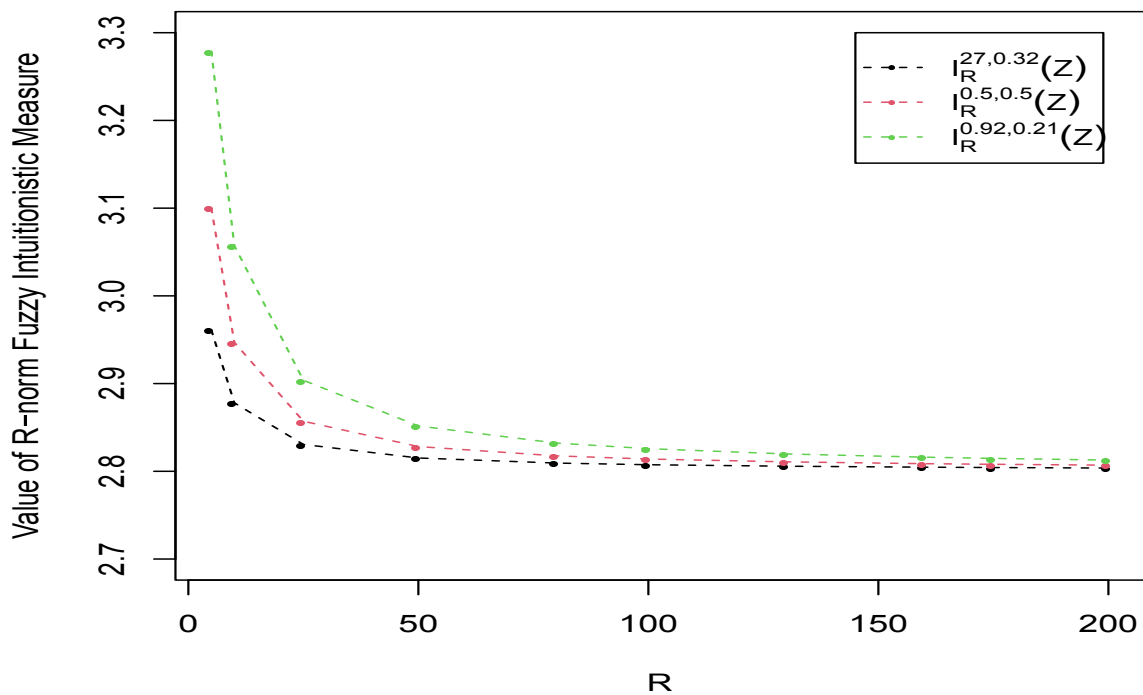**Fig. 2:** Graphical Representation of $I_R^{\alpha,\beta}(Z)$ w.r.t β

Fig. 3: Graphical Representation of $I_R^{\alpha,\beta}(Z)$ w.r.t R

Step IV: Compute the value of alternatives $E_i, i = 1, 2, .., u$ by using score function as follows

$$[H(E)]_i = (\sum_{j=1}^{v} \eta_{ij} \times w_j) - (\sum_{j=1}^{v} \phi_{ij} \times w_j) \forall i = 1, 2, ..., u$$

Step V: Rank all the alternatives $[H(E)]_i$ in ascending orderand choose the alternative with the highest rank.

Now, making use of above procedure, we solve the decision making problem in terms of the proposed measure given in (1)

Example: Let us take into account a decision-making problem in a recruitment drive where a project assistant is to be hired. For this, advertisement is given in the newspaper where different options are available for the selection procedure as F_1 (Required Qualification), F_2 (Academic Record), F_3 (Past Experience), F_4 (Technical Ability), F_5 (Personal Interview). The main aim of the company is to choose the best candidate for the available post among the given alternatives. Based on the above advertisement only five candidates E_1, E_2, E_3, E_4 *and* E_5 are nominated for this post by the panel of interviewers. Following data describes the uncertainties and ambiguity in the IFs environment:

$$M = \begin{bmatrix} & F_1 & F_2 & F_3 & F_4 & F_5 \\ E_1 & \langle 0.1, 0.6 \rangle & \langle 0.3, 0.5 \rangle & \langle 0.7, 0.2 \rangle & \langle 0.5, 0.3 \rangle & \langle 0.2, 0.1 \rangle \\ E_2 & \langle 0.7, 0.2 \rangle & \langle 0.4, 0.3 \rangle & \langle 0.5, 0.1 \rangle & \langle 0.2, 0.6 \rangle & \langle 0.3, 0.1 \rangle \\ E_3 & \langle 0.3, 0.6 \rangle & \langle 0.7, 0.1 \rangle & \langle 0.2, 0.5 \rangle & \langle 0.8, 0.1 \rangle & \langle 0.2, 0.7 \rangle \\ E_4 & \langle 0.4, 0.1 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.1, 0.4 \rangle & \langle 0.5, 0.2 \rangle & \langle 0.3, 0.5 \rangle \\ E_5 & \langle 0.6, 0.3 \rangle & \langle 0.1, 0.6 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.8, 0.1 \rangle \end{bmatrix}$$

The aforementioned procedure is used to solve the MCDM problem as:

Step I: As all the attributes are same, the procedure for normalization is not required.

Step II: For different values of R α and β the value for different features i.e., F_1, F_2, F_3, F_4, F_5 , with the help of equation (1), is obtained in the below table:

Table 5: Values of $[I_R^{\alpha,\beta}(F)]_j$,j=1,2,...,5

R	α	β	$[I_R^{\alpha,\beta}(F)]_j$
			1.6248
			1.6188
0.5	0.2	0.8	1.6215
			1.6283
			1.6257

Step III: Using $[I_R^{\alpha,\beta}(F)]_j$, the weights are calculated as:

w_1	w_2	w_3	w_4	w_5
0.2003	0.1983	0.1992	0.2014	0.2006

Step IV: The values of alternatives $[H(E)]_j$ after computation are given as:

$[H(E)]_1$	$[H(E)]_2$	$[H(E)]_3$	$[H(E)]_4$	$[H(E)]_5$
0.0201	0.1592	0.0397	0.0801	0.0407

Step V: The highest to lowest ranking of alternatives is given as: $[H(E)]_2 > [H(E)]_4 > [H(E)]_5 > [H(E)]_3 > [H(E)]_1$ which inturn gives $E_2 > E_4 > E_5 > E_3 > E_1$ i.e E_2 is best of all alternatives.

On the other hand, to monitor the impact of parameters, same procedure is applied with different values of parameters within their range. With this examination of parameters, the decision maker can have different choices of alternatives and thus help him to reach the goal. These values and ranking are given in the following table:

Table 6: Calculation of $[H(E)]_j$ for different values of parameters

R	α	β	$[H(E)]_1$	$[H(E)]_2$	$[H(E)]_3$	$[H(E)]_4$	$[H(E)]_5$	Ranking
7	0.9	0.1	0.0289	0.1779	0.0585	0.0789	-0.0024	$E_2 > E_4 > E_3 > E_1 > E_5$
10	0.7	0.4	0.0305	0.1788	0.0597	0.0786	-0.0049	$E_2 > E_4 > E_3 > E_1 > E_5$
55	0.5	0.5	0.0312	0.1251	0.0606	0.0784	-0.0074	$E_2 > E_4 > E_3 > E_1 > E_5$

We observe from table 6 that the ranking order remains same for different parametric values and thus gives the same alternative as the best choice.

5 CONCLUSION

In this work, we have developed a new intuitionistic fuzzy measure for R-norm entropy function. The proposed measure has two parameters that can be applied in more complex situations where one parameter gives bounded results. Also, the influence of parameters on the given measure is checked by substituting different values of parameters and observed how this measure behaves in their range. In the end, we have taken real life data and observed the application of the proposed measure with the help of MCDM technique.

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