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On Stochastic Comparisons of Concomitants of Generalized Order Statistics

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Abstract: In this article, the problem of comparing concomitants of generalized order statistics (GOSs) in terms of different types of stochastic orders is considered. Some stochastic ordering results for compound random variables in the one-sample problems are recalled and extended. Analogous results are obtained in the two-sample setup. The derived results are used to compare concomitants of GOSs in both one-sample problems and two-sample problems. We also introduce some new joint stochastic orders (namely, the joint reversed hazard order and the joint convex order) and compare concomitants in terms of these orders.

Keywords: Stochastic orders, compound distributions (mixture), one-sample comparisons, two-sample comparisons.

1 Introduction

Kamps[1] developed the idea of generalized order statistics (GOSs) as a unified strategy for studying ordered random variables like order statistics, records, and type II progressive censoring with two stages, which appear as submodels of the GOSs. Let $n \in \mathbb{N}, k > 0$ and $m_i \in \mathbb{R}, 1 \leq i \leq n-1$ be parameters such that $\gamma_r = k + (n-r) + \sum_{i=r}^{n-1} m_i \geq 1$ for $1 \leq r \leq n-1$ and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m}$ arbitrary if $n = 1$). The random variables (RVs), $W_{(r,n,\tilde{m},k)}, 1 \leq r \leq n$ are called GOSs based on an absolutely continuous distribution function (DF), F, with density (PDF), f, and survival function, $\bar{F}$, if the joint PDF is given by

$$f_{1,2,\ldots,n:n,\tilde{m},k}(w_1,\ldots,w_n) = k\left(\prod_{i=1}^{n-1}\gamma_i\right)\left(\prod_{i=1}^{n-1} f(w_i)(\bar{F}(w_i))^{m_i}\right) f(w_n)[\bar{F}(w_n)]^{k-1}, \tag{1}$$

where $F^{-1}(0) < w_1 \leq w_2 \leq \cdots \leq w_n < F^{-1}(1)$. More specially if $m_1 = \cdots = m_{n-1} = m$ the GOSs are called the m-GOSs. For more details on GOSs and its submodels see [1] and [2].

Let $(W_1,V_1),(W_2,V_2),\ldots,(W_n,V_n)$ be n independent copies from continuous DF, $F(w,v)$, with marginals F_W and F_V. Suppose that the sample is arranged in accordance with the variable W such that $W_{(r,n,\tilde{m},k)}$ represents the rth GOS based on F_W. The corresponding variable V is denoted by $V_{[r,n,\tilde{m},k]}$ and is called the concomitant of the rth GOS (see [2], p.224). It is known in the literature (cf. [3],[4]) that

$$[V_{[r,n,\tilde{m},k]}|W_{(r,n,\tilde{m},k)} = w] =_{st} [V|W = w], \tag{2}$$

where $=_{st}$ means that both random variables have the same PDF. See also [5] for the rigorous proof of (2) in the case of order statistics, records, and m-GOSs, while [1] for the proof in the case of Pfeifer records. Using (2) the PDF of $V_{[r,n,\tilde{m},k]}$ is expressed as

$$f_{[r,n,\tilde{m},k]}(v) = \int_{-\infty}^{\infty} f_{V|W}(v|w) f_{(r,n,\tilde{m},k)}(w)dw, \tag{3}$$

where $f_{V|W}(v|w)$ is the PDF of V given $W = w$ and $f_{(r,n,\tilde{m},k)}(w)$ is the density function of the rth GOS.

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Similarly, the joint PDF of the concomitants of the rth and the sth GOSs, $1 \le r \le s \le n$, indicated as $f_{[r,s:n,\tilde{m},k]}(v_1,v_2)$, is expressed by

$$f_{[r,s:n,\tilde{m},k]}(v_1,v_2) = \int_{-\infty}^{\infty}\int_{w_1}^{\infty} f(v_1|w_1)f(v_2|w_2)f_{(r,s:n,\tilde{m},k)}(w_1,w_2)dw_2dw_1, \tag{4}$$

where $f_{(r,s:n,\tilde{m},k)}(w_1,w_2)$ is the joint PDF of the rth and the sth GOSs.

Concomitants are essential in selection problems, parameter estimation, ranked set sampling, characterization and determination of the parent bivariate distributions. For a brief discussion of the applications of the concomitants of ordered random variables, see[6] and the references therein.

Motivated by the role of concomitants in many applications, we consider in this paper the problem of comparing concomitants of GOSs stochastically in terms of stochastic orders. Stochastic ordering is a method of comparing random quantities in magnitude, variability, dependence, and skewness based on information gathered from the random quantities' distributions. For a comprehensive treatise of the theory of stochastic orders and its application, see [7].

Many authors have drawn attention to the stochastic comparisons of ordered random variables and their concomitants in both univariate and multivariate cases. For a detailed description of some stochastic ordering findings on order statistics and spacing, see [7] and [8]. Stochastic ordering of GOSs was considered by many authors, see, for example, [9], [10], and [11] and the references therein. The stochastic comparisons of order statistics' concomitants were considered by khaledie and Kochar[12], Blessinger[13],[14], Bairamov et al.[15], Amini-Seresht et al. [16] and Erylimaz [17]. Stochastic comparisons of concomitants of type II progressive censored ordered statistics was considered by Izadi and Khaledi[4].

To the best of our knowledge, there are no previous results regarding the stochastic comparisons of concomitants of GOSs except for a remark mentioned by Izadi and khaledie[4], who pointed out that most of their one-sample comparisons in [4] can be valid for the concomitants of GOSs. Their remark is considered as a starting point in our one-sample comparisons results. In fact, in this article, we are concerned with studying three different types of stochastic orderings. Precisely, let (W_i,V_i) $,1 \le i \le n$ be n independent copies of (W,V) based on continuous bivariate DF. Let $W_{(r,n,\tilde{m},k)}$ and $V_{[r,n,\tilde{m},k]}$ be the rth GOS and the corresponding concomitant, respectively. The types of stochastic comparisons considered are as follows:

1. The one-sample comparisons: We discuss the following scenarios.
   (a) Comparison of the GOSs' concomitants based on the same distribution with different parameters in terms of some univariate stochastic orders.
   (b) Comparison of the rth and sth GOSs' concomitants with the same parameters $n,\tilde{m}$ and k based on the same distribution in terms of some joint stochastic orders introduced by Shanthikumar and Yao[18]. Also, we introduce some new definitions of joint stochastic orders, and perform comparison of GOSs' concomitants in terms of these definitions.
2. The two-sample comparisons: We compare, in terms of different stochastic orders, the concomitants of GOSs based on the same parameters and different parent distributions.

The remaining sections of this article are structured as follows: The preliminaries used in deriving our results are introduced in Section 2. Our main results are derived in sections 3, 4, and 5. In Section 6, some illustrative examples of stochastic ordering of concomitants of GOSs, based on some specific bivariate distributions, are introduced.

We will assume in the following, for a continuous RV, Z, having DF, F_Z, and PDF, f_Z, that the hazard, the reversed hazard, and the mean residual life functions are respectively defined by $h_Z(s) = f_Z(s)/\bar{F}_Z(s)$, $r_Z(s) = f_Z(s)/F_Z(s)$ and $m_Z(s) = (\int_s^\infty \bar{F}_Z(u)du)/\bar{F}_Z(s)$. We denote $[V|W=w]$ to the conditional random variable V given $W=w$. We say that the RV, Z, is increasing [decreasing] failure rate IFR[DFR] if $h_Z(s)$ increases [decreases] in s. The increasing [decreasing] reversed failure rate (IRFR [DRFR]) and the increasing [decreasing] mean residual life (IMRL [DMRL]) distributions are defined analogously.

2 Preliminaries

The following definitions and theorems are used to derive our major findings. Most of the preliminaries presented in this section can be found in [7].

Definition 1. *Assume that W and V are two RVs with DFs F_W and F_V, respectively. W is called less than V in the*

1. *usual stochastic order, symbolized as $W \le_{st} V$, if $\bar{F}_W(s) \le \bar{F}_V(s)$ $\forall s \in \mathbb{R}$ (or alternatively, if for all increasing functions $\psi : \mathbb{R} \to \mathbb{R}$, we have $E(\psi(W)) \le E(\psi(V))$),*
2. *hazard rate order, symbolized as $W \le_{hr} V$, if $\bar{F}_V(s)/\bar{F}_W(s)$ increases in $s \in \mathbb{R}$ (or alternatively, if $h_W(s) \ge h_V(s)$ for all $s \in \mathbb{R}$),*

3.*reversed hazard order, symbolized as* $W \leq_{rh} V$, *if* $F_V(s)/F_W(s)$ *is increasing in* $s \in \mathbb{R}$ *(or if* $r_W(s) \leq r_V(s)$ $\forall s \in \mathbb{R}$*) (or alternatively, if* $f_W(v)F_V(w) \leq f_V(v)F_W(w)$ *for all* $w \leq v$*),*
4.*mean residual life order, symbolized as* $W \leq_{mrl} V$, *if* $\int_s^\infty \bar{F}_V(v)dv / \int_s^\infty \bar{F}_w(w)dw$ *is increasing in* s *(or alternatively, if* $m_W(s) \leq m_V(s)$, $\forall s$*),*
5.*harmonic mean residual life order, symbolized as* $W \leq_{hmrl} V$, *if* $\frac{\int_s^\infty \bar{F}_w(w)dw}{E(w)} \leq \frac{\int_s^\infty \bar{F}_V(w)dw}{E(V)}$ *for all* $s \geq 0$,
6.*likelihood ratio order, symbolized as* $W \leq_{lr} V$, *if* $f_V(s)/f_W(s)$ *increases for all* $s \in \mathbb{R}$,
7.*dispersive order, symbolized as* $W \leq_{disp} V$, *if* $F_V^{-1}(w) - F_W^{-1}(w)$ *is increasing in* $w \in (0,1)$,
8.*excess wealth order, symbolized as* $W \leq_{ew} V$, *if* $\int_{F_W^{-1}(q)}^\infty \bar{F}_W(w)dw \leq \int_{F_V^{-1}(q)}^\infty \bar{F}_V(w)dw$ *for all* $q \in (0,1)$,
9.*convex [concave] [increasing convex] [increasing concave] order, (symbolized as* $W \leq_{cx} V[W \leq_{cv} V][W \leq_{icx} V]$ $[W \leq_{icv} V]$*), if* $E[\psi(W)] \leq E[\psi(V)]$, *for all convex [concave][increasing convex] [increasing concave] functions* $\psi : \mathbb{R} \to \mathbb{R}$.

The following set of implications are well known (see[7]).

$$W \leq_{lr} V \Rightarrow W \leq_{hr} V[W \leq_{rh} V] \Rightarrow W \leq_{mrl} V \Rightarrow W \leq_{st} V[W \leq_{hmrl} V] \Rightarrow W \leq_{icx} V \tag{5}$$

$$W \leq_{disp} V \Rightarrow W \leq_{ew} V. \tag{6}$$

Theorem 1(see [7] (p. 156)). *Assume that* W *and* V *are two nonnegative RVs. If* $W \leq_{hr} V$ *and either* W *or* V *are DFR, then* $W \leq_{disp} V$.

Theorem 2(see[7] (p. 168)). *Assume that* W *and* V *are two RVs with* $E(W) \leq \infty, E(V) \leq \infty$ *and* $-\infty < l_W \leq l_V < \infty$. *If* $W \leq_{mrl} V$, *and if either* W *or* V *is IMRL, then* $W \leq_{ew} V$.

Some stochastic orderings of compound random variables are stated in the next theorem.

Theorem 3(cf. [7]). *Consider a family of distributions* $\{F_\varphi, \varphi \in \mathscr{X}\}$. *Let* $W(\varphi)$ *indicates a RV with DF* $F_{W(\varphi)}$. *Let* Φ_1 *and* Φ_2 *be two RVs with support in* $\mathscr{X}$ *and DFs* G_1 *and* G_2, *respectively. Let the PDF of the compound RV* $W(\Phi_i), i = 1,2$ *is given as*

$$h_i(w) = \int_{\mathscr{X}} f_{W(\varphi)}(w) dG_i(\varphi), \quad w \in \mathbb{R}, \tag{7}$$

where $f_{W(\varphi)}$ *is the corresponding PDF of* $F_{W(\varphi)}$. *Then, for* $\varphi_1 \leq \varphi_2$

1.$W(\varphi_1) \leq_{lr} W(\varphi_2)$ *and* $\Phi_1 \leq_{lr} \Phi_2 \Rightarrow W(\Phi_1) \leq_{lr} W(\Phi_2)$,
2.$W(\varphi_1) \leq_{hr} W(\varphi_2)$ *and* $\Phi_1 \leq_{hr} \Phi_2 \Rightarrow W(\Phi_1) \leq_{hr} W(\Phi_2)$,
3.$W(\varphi_1) \leq_{mrl} W(\varphi_2)$ *and* $\Phi_1 \leq_{hr} \Phi_2 \Rightarrow W(\Phi_1) \leq_{mrl} W(\Phi_2)$,
4.$W(\varphi_1) \leq_{hmrl} W(\varphi_2)$ *and* $\Phi_1 \leq_{hr} \Phi_2 \Rightarrow W(\Phi_1) \leq_{hmrl} W(\Phi_2)$,
5.$W(\varphi_1) \leq_{rh} W(\varphi_2)$ *and* $\Phi_1 \leq_{rh} \Phi_2 \Rightarrow W(\Phi_1) \leq_{rh} W(\Phi_2)$,
6.$W(\varphi_1) \leq_{st} W(\varphi_2)$ *and* $\Phi_1 \leq_{st} \Phi_2 \Rightarrow W(\Phi_1) \leq_{st} W(\Phi_2)$.

The stochastic orderings mentioned in Definition 1 are concerned with comparing the marginals of F_W and F_V without considering the dependence between W and V. To take the dependence into consideration, a bivariate criterion was proposed by [18], known as the joint stochastic ordering, which is outlined below.

Definition 2.*Consider the following classes of functions*

$$\mathscr{G}_{lr} := \{g(w,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}; \Delta g(w,v) = g(w,v) - g(v,w) \geq 0, \forall w \geq v\},$$
$$\mathscr{G}_{hr} := \{g(w,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}; \Delta g(w,v) \text{ increases in } w, \forall w \geq v\},$$
$$\text{and } \mathscr{G}_{st} := \{g(w,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}; \Delta g(w,v) \text{ is increasing in } w, \forall v\}.$$

We say that

1.*W is greater than V with respect to the joint likelihood ratio order, denoted as* $W \geq_{lr:j} V$, *if* $Eg(W,V) \geq Eg(V,W)$, $\forall g \in \mathscr{G}_{lr}$,
2.*W is greater than V with respect to the joint hazard rate order, symbolized as* $W \geq_{hr:j} V$, *if* $Eg(W,V) \geq Eg(V,W)$, $\forall g \in \mathscr{G}_{hr}$,
3.*W is greater than V in the joint stochastic order, denoted as* $W \geq_{st:j} V$, *if* $Eg(W,V) \geq Eg(V,W)$, $\forall g \in \mathscr{G}_{st}$.

Necessary and sufficient conditions for orderings $\leq_{lr:j}$ and $\leq_{hr:j}$ are given below (see [18]).

Theorem 4.*Suppose that* W *and* V *are two RVs with joint PDF* $f_{WV}(w,v)$ *and joint survival function* $\bar{F}_{WV}(w,v)$. *Then,*

1. $V \leq_{lr:j} W \Leftrightarrow f_{WV}(w,v) \in \mathscr{G}_{lr}$
2. $V \leq_{hr:j} W \Leftrightarrow \frac{\partial}{\partial v}\bar{F}_{WV}(w,v) \leq \frac{\partial}{\partial v}\bar{F}_{WV}(v,w)\ \forall w \geq v$

Closely related to the definitions of stochastic orders is the concept of sign regular of order 2, SR_2, functions defined as follows (see [19]).

Definition 3. *A function $f(w,v)$ is said to be SR_2 if $\varepsilon_1 f(w,v) \geq 0$ and*

$$\varepsilon_2[f(w_1,v_1)f(w_2,v_2) - f(w_1,v_2)f(w_2,v_1)] \geq 0 \tag{8}$$

for $w_1 \leq w_2, v_1 \leq v_2$, and $\varepsilon_i \in \{-1,1\}$ for $i = 1,2$.

If the relation in (8) holds with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$ then f is claimed to be totally positive of order 2 (TP_2), while if the relation holds with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$then f is said to be reverse regular of order 2 (RR_2).

Theorem 5(see [19], p.99). *Assume that $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are subsets of $\mathbb{R}$, $Q(u,w)$ is SR_2 for $u \in \mathscr{A}, w \in \mathscr{B}$ and $R(w,v)$ is SR_2 for $w \in \mathscr{B}, v \in \mathscr{C}$. Then $P(u,v) = \int Q(u,w)R(w,v)d\mu(w)$ is SR_2 for $u \in \mathscr{A}, v \in \mathscr{C}$ and $\varepsilon_i(P) = \varepsilon_i(Q) \times \varepsilon_i(R), \forall i = 1,2$, where μ is a σ-finite measure.*

Clearly, from Definition 3 and Theorem 5, we see that the composition of two TP_2 functions or two RR_2 functions is TP_2 and the composition of a TP_2 function and a RR_2 function is RR_2.

Two RVs, W and V, with a joint PDF f are said to be TP_2 $[RR_2]$ dependent (see[20]) if the function $f(w,v)$ is TP_2 $[RR_2]$ in u and v; that is if

$$f(w_1,v_1)f(w_2,v_2) \geq [\leq] f(w_1,v_2)f(w_2,v_1) \text{ for all } w_1 \leq w_2,\ v_1 \leq v_2. \tag{9}$$

To end this part, we present some key results that establish the stochastic ordering of GOSs.

Theorem 6(Esna-Ashari et al. [9]). *Assume that $W_{(r,n,\tilde{m},k)}, 1 \leq r \leq n$ and $W_{(r',n',\tilde{m}',k')}, 1 \leq r' \leq n'$ are GOSs based on absolutely continuous DF. Then, $W_{(r,n,\tilde{m},k)} \leq_{lr} W_{(r',n',\tilde{m}',k')}$ whenever $r' \geq r, m_{r-i} \geq m'_{r'-i}, i \in \{1,2,\ldots,r-1\}$ and $\gamma_r \geq \gamma'_{r'}$.*

The following theorem uses the concept of "p-larger than"(see [21]). We say that the vector $\mathbf{u} \in \mathbb{R}^{n^+}$ is p larger than the vector $\mathbf{v} \in \mathbb{R}^{n^+}$(written $\mathbf{u} \succcurlyeq^p \mathbf{v}$) if $\prod_{i=1}^{j} u_{(i)} \leq \prod_{i=1}^{j} v_{(i)}, 1 \leq j \leq n$, where $\{z_{(1)} \leq \cdots \leq z_{(n)}\}$ denotes the increasing components arrangement of any vector $\mathbf{z} = \{z_1,\ldots,z_n\}$.

Theorem 7(Khaledi [22]). *For $r \leq r'$, $W_{(r:n,\tilde{m},k)} \leq_{hr} W_{(r',n',\tilde{m}',k')}$ whenever*

$$(\gamma'_{l_1},\ldots,\gamma'_{l_r}) \succcurlyeq^p (\gamma_1,\ldots,\gamma_r) \text{ for some set } \{l_1,\ldots,l_r\} \subseteq \{1,\ldots,r'\} \tag{10}$$

3 One-Sample Comparisons

In this section, we are concerned with the comparison of the GOSs' concomitants, $V_{[r,n,\tilde{m},k]}$ and $V_{[r',n',\tilde{m}',k']}$, assuming the same parent bivariate DF and different GOSs' parameters. Comparing Equations (3) and (7), one can see that $V_{[r,n,\tilde{m},k]}$ is a compound random variable of $[V|W = w]$ and $W_{(r,n,\tilde{m},k)}$. So, initially, we extend several stochastic comparison findings for compound random variables, given in Theorem 3, considering alternative assumptions on the stochastic increase of the RV $W(\varphi)$ in φ, and on the stochastic ordering of Φ_1 and Φ_2, and then we use these results in comparing the concomitants.

3.1 One-Sample comparison of compound random variables

Now we seek the conditions under which the stochastic orderings of $W(\Phi_1)$ and $W(\Phi_2)$, stated in Theorem 3, are inverted. We can easily see, in all parts of Theorem 3, that whenever the ordering of Φ_1 and Φ_2 is inverted, keeping the ordering of $W(\varphi_1)$ and $W(\varphi_2)$ unchanged, the ordering of $W(\Phi_1)$ and $W(\Phi_2)$ will be inverted. However, another case to be considered in each order is stated below. Through Lemmas 1, 2, and 3, we assume that $W(\varphi)$, Φ_i and $W(\Phi_i)$, $i = 1,2$, are defined as in Theorem 3.

Lemma 1. *For $\varphi_1 \leq \varphi_2$, we have*

1. $W(\varphi_1) \geq_{lr} W(\varphi_2)$ *and* $\Phi_1 \leq_{lr} \Phi_2 \Rightarrow W(\Phi_1) \geq_{lr} W(\Phi_2)$.
2. $W(\varphi_1) \geq_{st} W(\varphi_2)$ *and* $\Phi_1 \leq_{st} \Phi_2 \Rightarrow W(\Phi_1) \geq_{st} W(\Phi_2)$.

Proof. 1.If $W(\varphi_1) \geq_{lr} W(\varphi_2)$ for $\varphi_1 \leq \varphi_2$, then the function $f_{W(\varphi_i)}(w)$ is RR_2 in w and $i \in \{1,2\}$. If $\Phi_1 \leq_{lr} \Phi_2$, then the function $g_i(\varphi)$ is TP_2 in φ and i. Therefore, from Theorem 5, the function $h_i(w) = \int_{-\infty}^{\infty} f_{W(\varphi)}(w) g_i(\varphi) d\varphi$ is RR_2 in w and i. Thus $W(\Phi_1) \geq_{lr} W(\Phi_2)$.

2.If $\Phi_1 \leq_{st} \Phi_2$, then, by Definition 1, $\int_{-\infty}^{\infty} k(\varphi) g_1(\varphi) d\varphi \leq \int_{-\infty}^{\infty} k(\varphi) g_2(\varphi) d\varphi$ for any increasing function k. Setting $k(\varphi) = -\psi(\varphi)$ for any decreasing function $\psi(\varphi)$ we have

$$\int_{-\infty}^{\infty} \psi(\varphi) g_1(\varphi) d\varphi \geq \int_{-\infty}^{\infty} \psi(\varphi) g_2(\varphi) d\varphi. \tag{11}$$

Additionally, if $W(\varphi_1) \geq_{st} W(\varphi_2)$ for $\varphi_1 \leq \varphi_2$, then $\bar{F}_{W(\varphi_1)}(w) \geq \bar{F}_{W(\varphi_2)}(w)$; that is the function $\bar{F}_{W(\varphi)}(w)$ is decreasing function in φ. If we choose $\psi(\varphi) = \bar{F}_{W(\varphi)}(w)$, it follows from (11) that

$$\bar{H}_1(w) = \int_{-\infty}^{\infty} \bar{F}_{W(\varphi)}(w) g_1(\varphi) d\varphi \geq \int_{-\infty}^{\infty} \bar{F}_{W(\varphi)}(w) g_2(\varphi) d\varphi = \bar{H}_2(w).$$

Thus, $W(\Phi_1) \geq_{st} W(\Phi_2)$

For similar results for the orders $\leq_{hr}$, $\leq_{rh}$, and $\leq_{mrl}$, we introduce the following lemma, which is an extension of Theorem 2.1 in [23].

Lemma 2.*Let $G_i(\phi)$, $i = 1,2$ be two DFs. Let $f_{W(\phi)}(w)$ be any function of w and ϕ. Suppose that the integral $\int_{-\infty}^{\infty} f_{W(\phi)}(w) dG_i(\phi)$ exists and is finite. Then the function $h_i(w) = \int_{-\infty}^{\infty} f_{W(\phi)}(w) dG_i(\phi)$ is TP_2 [RR_2] in i and w if one of the following cases occurs.*

1.The function $\bar{G}_i(\phi)$ is TP_2 in ϕ and i, and the function $f_{W(\phi)}(w)$ is TP_2 [RR_2] in ϕ and w and increasing in ϕ.
2.The function $\bar{G}_i(\phi)$ is RR_2 in ϕ and i, and the function $f_{W(\phi)}(w)$ is RR_2 [TP_2] in ϕ and w and increasing in ϕ.
3.The function $G_i(\phi)$ is TP_2 in i and ϕ, and the function $f_{W(\phi)}(w)$ is TP_2 [RR_2] in ϕ and w and decreasing in ϕ.
4.The function $G_i(\phi)$ is RR_2 in i and ϕ, and the function $f_{W(\phi)}(w)$ is RR_2 [TP_2] in ϕ and w and decreasing in ϕ .

*Proof.*The function $h_i(w)$ is TP_2[RR_2] whenever, for $w_1 \leq w_2$,

$$D = \int_{-\infty}^{\infty} f_{W(\phi_2)}(w_2) g_2(\phi_2) d\phi_2 \int_{-\infty}^{\infty} f_{W(\phi_1)}(w_1) g_1(\phi_1) d\phi_1 - \int_{-\infty}^{\infty} f_{W(\phi_2)}(w_1) g_2(\phi_2) d\phi_2 \int_{-\infty}^{\infty} f_{W(\phi_1)}(w_2) g_1(\phi_1) d\phi_1 \geq [\leq] 0.$$

Using the basic composition theorem in [19] (p.16) and by means of integration by parts with respect to ϕ_2, one can have (see theorem 2.1 in [23])

$$D = \iint\limits_{\phi_2 \geq \phi_1} \left(f_{W(\phi_1)}(w_1) \frac{\partial}{\partial \phi_2} f_{W(\phi_2)}(w_2) - f_{W(\phi_1)}(w_2) \frac{\partial}{\partial \phi_2} f_{W(\phi_2)}(w_1) \right) \left(g_1(\phi_1) \bar{G}_2(\phi_2) - g_2(\phi_1) \bar{G}_1(\phi_2) \right) d\phi_1 d\phi_2. \tag{12}$$

By a similar argument, if we integrate by parts with respect to ϕ_1, one can have

$$D = \iint\limits_{\phi_2 \geq \phi_1} \left(f_{W(\phi_2)}(w_1) \frac{\partial}{\partial \phi_1} f_{W(\phi_1)}(w_2) - f_{W(\phi_2)}(w_2) \frac{\partial}{\partial \phi_1} f_{W(\phi_1)}(w_1) \right) \left(g_2(\phi_2) G_1(\phi_1) - g_1(\phi_2) G_2(\phi_1) \right) d\phi_1 d\phi_2. \tag{13}$$

Now, it can be seen that each of the conditions in part 1 and part 2 ensures the positivity [negativity] of D in (12), whereas each of the conditions in part 3 and part 4 ensures the positivity [negativity] of D in (13).

Lemma 3.*For $\varphi_1 \leq \varphi_2$ we have*

1.$W(\varphi_1) \geq_{hr} W(\varphi_2)$ and $\Phi_1 \leq_{rh} \Phi_2 \Rightarrow W(\Phi_1) \geq_{hr} W(\Phi_2)$.
2.$W(\varphi_1) \geq_{rh} W(\varphi_2)$ and $\Phi_1 \leq_{hr} \Phi_2 \Rightarrow W(\Phi_1) \geq_{rh} W(\Phi_2)$.
3.$W(\varphi_1) \geq_{mrl} W(\varphi_2)$ and $\Phi_1 \leq_{rh} \Phi_2 \Rightarrow W(\Phi_1) \geq_{mrl} W(\Phi_2)$.
4.$W(\varphi_1) \geq_{hmrl} W(\varphi_2)$ and $\Phi_1 \leq_{rh} \Phi_2 \Rightarrow W(\Phi_1) \geq_{hmrl} W(\Phi_2)$.

*Proof.*1. First, we observe that $W(\Phi_1) \geq_{hr} W(\Phi_2) \iff \bar{H}_1(w)/\bar{H}_2(w)$ is an increasing function in w, which is the case if the function $\bar{H}_i(w) = \int_{-\infty}^{\infty} \bar{F}_{W(\varphi)}(w) g_i(\varphi) d\varphi$ is RR_2 in i and w. If, for $\varphi_1 \leq \varphi_2$, $W(\varphi_1) \geq_{hr} W(\varphi_2)$, then $\bar{F}_{W(\varphi_i)}(w)$ is RR_2 in $i \in \{1,2\}$ and φ. Moreover, it follows that $W(\varphi_1) \geq_{st} W(\varphi_2)$. Thus $\bar{F}_{W(\varphi)}(w)$ decreases in φ. On the other hand, if $\Phi_1 \leq_{rh} \Phi_2$, then the function $G_i(\varphi)$ is TP_2 in i and φ. It follows now from Lemma 2 (part 3) that $\bar{H}_i(w)$ is RR_2 function in w and $i \in \{1,2\}$.

2. For this part, it is sufficient to have the function $H_1(w)/H_2(w)$ an increasing function in w; this means, to have $H_i(w) = \int_{-\infty}^{\infty} F_{W(\varphi)}(w) g_i(w) dw$ an RR$_2$ function in w and i, $i = 1,2$. In fact, for $\varphi_1 \le \varphi_2$, if $W(\varphi_1) \ge_{rh} W(\varphi_2)$, then the function $F_{W(\varphi)}(w)$ is RR$_2$ in w and φ. Moreover, for $\varphi_1 \le \varphi_2$, we have $W(\varphi_1) \ge_{st} W(\varphi_2)$. Therefore the function $F_{W(\varphi)}(w)$ is increasing in φ. Since $\Phi_1 \le_{hr} \Phi_2$ implies that the function $G_i(\varphi)$ is TP$_2$ in φ and i, it follows, from Lemma 2, Part 1, that $H_i(w)$ is RR$_2$ function in i and w.

3. For this part, $W(\Phi_1) \ge_{mrl} W(\Phi_2)$ whenever the function

$$\int_w^{\infty} \bar{H}_i(t)dt = \int_{-\infty}^{\infty} \left(\int_w^{\infty} \bar{F}_{W(\varphi)}(t)dt \right) g_i(\varphi) d\varphi$$

is RR$_2$ in w and i. In fact, if $W(\varphi_1) \ge_{mrl} W(\varphi_2)$ then the function $\int_w^{\infty} \bar{F}_{W(\varphi)}(t)dt$ is RR$_2$ in w and φ. Moreover, we have, for $\varphi_1 \le \varphi_2$, $W(\varphi_1) \ge_{icx} W(\varphi_2)$, implying that the function $\int_w^{\infty} \bar{F}_{W(\varphi)}(w)dw$ is a decreasing function in φ (see Equation (4.A.5) in[7]). Since $\Phi_1 \le_{hr} \Phi_2$ ensures that the function $G_i(\varphi)$ is TP$_2$ in i and φ. The required now is immediate using Lemma 2 (part 1).

4. The proof of this part follows in the same manner as in Part 3, realizing that $W(\Phi_1) \ge_{hmrl} W(\Phi_2)$ is similar to having

$$\frac{\int_{-\infty}^{\infty} \left(\int_w^{\infty} \bar{F}_{W(\varphi)}(s)ds \right) g_1(\varphi) d\varphi}{\int_{-\infty}^{\infty} \left(\int_0^{\infty} \bar{F}_{W(\varphi)}(s)ds \right) g_1(\varphi) d\varphi} \ge \frac{\int_{-\infty}^{\infty} \left(\int_w^{\infty} \bar{F}_{W(\varphi)}(s)ds \right) g_2(\varphi) d\varphi}{\int_{-\infty}^{\infty} \left(\int_0^{\infty} \bar{F}_{W(\varphi)}(s)ds \right) g_2(\varphi) d\varphi}, \ \forall w \ge 0.$$

Or equivalently that the function the $h_{ij} = \int_{-\infty}^{\infty} \left(\int_j^{\infty} \bar{F}_{W(\varphi)}(s)ds \right) g_i(\varphi) d\varphi$ is RR$_2$ in $i \in \{1,2\}$ and $j \in \{0,w\}$ for $0 < w$.

3.2 One-Sample Comparisons of Concomitants

Now we turn our attention to stochastic orderings of concomitants of GOSs. It is worth noting that most of the results obtained in this subsection are direct consequences of the one-sample comparisons of compound random variables, replacing $W(\varphi)$ by $[V|W=w]$, Φ_1 by $W_{(r,n,\tilde{m},k)}$, and Φ_2 by $W_{(r',n',\tilde{m}',k')}$.

Theorem 8. *Assume that $W_{(r,n,\tilde{m},k)}, 1 \le r \le n$ and $W_{(r',n',\tilde{m}',k')}, 1 \le r' \le n'$ are GOSs based on an absolutely continuous DF, F. If W and V are TP$_2$ [RR$_2$] dependent, then $V_{[r,n,\tilde{m},k]} \le_{lr} [\ge_{lr}] V_{[r',n',\tilde{m}',k']}$ whenever $r' \ge r, m'_{r'-i} \le m_{r-i}$ for $i = 1,2,\dots r-1$, and $\gamma_r \ge \gamma'_{r'}$.*

Proof. Since W and V are TP$_2$ [RR$_2$] dependent, it follows from (9) that $f(v|w_2)/f(v|w_1)$ is an increasing [decreasing] function in v $\forall w_1 \le w_2$. Consequently, from Definition 1 (part 6), $[V|W=w_1] \le_{lr} [\ge_{lr}][V|W=w_2]$. Now, using Theorem 6, Theorem 3 (part 1) and Lemma 1 (part 1), we have $V_{[r,n,\tilde{m},k]} \le_{lr} [\ge_{lr}] V_{[r',n',\tilde{m}',k']}$.

Theorem 9. *Let $(\gamma'_{l_1}, \dots, \gamma'_{l_r}) \succcurlyeq^p (\gamma_1, \dots, \gamma_r)$ for some set $\{l_1, \dots, l_r\} \subseteq \{1, \dots, r'\}$ for $r' \ge r$.*

1. *If $h_{V|W}(v|w)$ is decreasing [increasing] in w, then $V_{[r,n,\tilde{m},k]} \le_{hr} [\ge_{hr}] V_{[r',n',\tilde{m}',k']}$.*
2. *If $r_{V|W}(v|w)$ is decreasing in w, then $V_{[r,n,\tilde{m},k]} \ge_{rh} V_{[r',n',\tilde{m}',k']}$.*
3. *If $m_{V|W}(v|w)$ is increasing [decreasing] in w, then $V_{[r,n,\tilde{m},k]} \le_{mrl} [\ge_{mrl}] V_{[r',n',\tilde{m}',k']}$.*
4. *If, for non negative RVs, W and V, $\int_j^{\infty} \bar{F}_{V|W}(v|w)dy$ is TP$_2$ [RR$_2$] in w and $j \in \{0,t\}; 0 \le t$, then $V_{[r,n,\tilde{m},k]} \le_{hmrl} [\ge_{hmrl}] V_{[r',n',\tilde{m}',k']}$.*

Proof. Under the given condition, Theorem 7 implies that $W_{(r,n,\tilde{m},k)} \le_{hr} W_{(r',n',\tilde{m}',k')}$. So,

1. If $h_{V|W}(v|w)$ decreases [increases] in w, then $[V|W=w_1] \le_{hr} [\ge_{hr}][V|W=w_2]$ for all $w_1 \le w_2$, and the result in part 1 follows from Theorem 3 (part 2) [Lemma 3 (part 1)].
2. If $r_{V|W}(v|w)$ decreases in w, then $[V|W=w_1] \ge_{rh} [V|W=w_2]$ for all $w_1 \le w_2$. Lemma 3 (part 2) leads now to the conclusion in part 2.
3. If $m_{V|W}(v|w)$ increases [decreases] in w then $[V|W=w_1] \le_{mrl} [\ge_{mrl}][V|W=w_2]$ for all $w_1 \le w_2$. The result in part 3 is immediate now from Theorem 3 (part 3) [Lemma 3 (part 3)].

4.If $\int_j^\infty \bar{F}_{V|W}(v|w)dv$ is TP$_2$ [RR$_2$] in $j \in \{0,t\}$ and w, then

$$\int_t^\infty \bar{F}_{V|W}(v|w_1)dv \int_0^\infty \bar{F}_{V|W}(v|w_2)dv \le [\ge] \int_t^\infty \bar{F}_{V|W}(v|w_2)dv \int_0^\infty \bar{F}_{V|W}(v|w_1)dv$$

for all $w_2 \ge w_1$. Or equivalently we have

$$\frac{\int_t^\infty \bar{F}_{V|W}(v|w_1)dv}{E(V|W=w_1)} \le [\ge] \frac{\int_t^\infty \bar{F}_{V|W}(v|w_2)dv}{E(V|W=w_2)}, \ \forall w_1 \le w_2.$$

Or, $[V|W=w_1] \le_{hmrl} [\ge_{hmrl}][V|W=w_2]$ for all $w_1 \le w_2$. Theorem 3 (part 4) [Lemma 3 (part 4)] ensures the result in part 4.

The following lemma is used in deriving the excess wealth ordering among concomitants. The lemma demonstrates that the mixture of IMRL distributions is an IMRL distribution.

Lemma 4. *If F_α is IMRL distribution for all $0 < \alpha < \infty$, and G is a distribution function on $(0,\infty)$, then F is IMRL, where $F(t) = \int_0^\infty F_\alpha(t)dG(\alpha)$.*

Proof. First we note that if F_α is IMRL $\forall\alpha$ then $d/dw[(\int_w^\infty \bar{F}_\alpha(u)du)/\bar{F}_\alpha(w)] \ge 0 \ \forall\alpha$. Therefore

$$(\bar{F}_\alpha(w))^2 \le f_\alpha(w)\int_w^\infty \bar{F}_\alpha(u)du, \ \forall\alpha, \forall w. \tag{14}$$

It follows that

$$\begin{aligned}[\bar{F}(w)]^2 = \left[\int_0^\infty \bar{F}_\alpha(w)dG(\alpha)\right]^2 &\le \left[\int_0^\infty (f_\alpha(w))^{\frac{1}{2}}\left(\int_w^\infty \bar{F}_\alpha(u)du\right)^{\frac{1}{2}} dG(\alpha)\right]^2 \\ &\le \left(\int_0^\infty f_\alpha(w)dG(\alpha)\right)\left(\int_0^\infty \left(\int_w^\infty \bar{F}_\alpha(u)du\right)dG(\alpha)\right) \\ &= f(w)\int_w^\infty \bar{F}(u)du, \end{aligned} \tag{15}$$

where the first inequality is due to (14), and the second inequality results from Cauchy-Schwartz inequality (see e.g.[24]). Equation (15) ensures that the function $\int_w^\infty \bar{F}(u)du/\bar{F}(u)$ increases in w. Consequently, F is IMRL.

Corollary 1. *Let $(\gamma'_{l_1},\ldots,\gamma'_{l_r}) \succcurlyeq^p (\gamma_1,\ldots,\gamma_r)$ for some set $\{l_1,\ldots,l_r\} \subseteq \{1,\ldots,r'\}$ for $r \le r'$.*

1.If $h_{V|W}(v|w)$ decreases in both w and v, then $V_{[r,n,\tilde{m},k]} \le_{disp} V_{[r',n',\tilde{m}',k']}$.
2.If $m_{V|W}(v|w)$ increases in both w and v, then $V_{[r,n,\tilde{m},k]} \le_{ew} V_{[r',n',\tilde{m}',k']}$.

Proof. 1.The assumption that $h_{V|W}(v|w)$ decreases in w along with the given condition imply that $V_{[r,n,\tilde{m},k]} \le_{hr} V_{[r',n',\tilde{m}',k']}$ by Theorem 9 (part 1). Moreover, if $h_{V|W}(v|w)$ is decreasing in v, then the distribution of $V_{[r,n,\tilde{m},k]}$ is a mixture of DFR distribution and hence it is a DFR distribution (see Ross[24], Proposition 9.1.5). Now the result is immediate by Theorem 1.
2.The given conditions in part 2 imply $V_{[r,n,\tilde{m},k]} \le_{mrl} V_{[r',n',\tilde{m}',k']}$ by Theorem 9 (part 2). Moreover, if $m_{V|W}(v|w)$ increases in v, then $V_{[r,n,\tilde{m},k]}$ has a DF that is a mixture of IMRL distribution functions, and hence, by using Lemma 4, it is an IMRL distribution. Now, Theorem 2 ensures our conclusion.

4 The Joint Stochastic Ordering

In this section, we identify the conditions for constructing joint stochastic orders of concomitants of GOSs in terms of the joint stochastic orders introduced in [18]. Namely, we investigate the orders $\le_{lr:j}$,$\le_{hr:j}$ and $\le_{st:j}$. Furthermore, based on some bivariate characterization for the orders $\le_{rh}$ and $\le_{cx}$, we introduce a "joint reversed hazard ordering," denoted by $\le_{rh:j}$, and a "joint convex [concave] ordering," denoted by $\le_{cx:j}$ [$\le_{cv:j}$], and compare the concomitants of GOSs using these new orders.

Theorem 10. *Assume that W and V are two RVs with joint PDF $f_{WV}(w,v)$. If W and V are RR_2 [TP_2] dependent, then $V_{[r:n,\tilde{m},k]} \ge_{lr:j} [\le_{lr:j}]V_{[s:n,\tilde{m},k]}$ for $r \le s$.*

Proof. If W and V are RR_2 $[TP_2]$ dependent, then, from (9), we have $f(v_1|w_1)f(v_2|w_2) - f(v_2|w_1) \times f(v_1|w_2) \geq 0$ for $w_1 \leq w_2$ and $v_2 \leq [\geq] v_1$. Consequently, for $r \leq s$,

$$f_{[r,s:n,\tilde{m},k]}(v_1,v_2) - f_{[r,s:n,\tilde{m},k]}(v_2,v_1) = \iint\limits_{w_2 \geq w_1} [f(v_1|w_1)f(v_2|w_2) - f(v_2|w_1)f(v_1|w_2)] f_{s,r,n,\tilde{m},k}(w_2,w_1)dw_2dw_1 \geq 0,$$

for all $v_2 \leq [\geq] v_1$. Thus, for $r \leq s$, $f_{[r,s:n,\tilde{m},k]}(v_1,v_2) \in \mathscr{G}_{lr}$ $[f_{[s,r:n,\tilde{m},k]}(v_2,v_1) \in \mathscr{G}_{lr}]$, where the class $\mathscr{G}_{lr}$ is given as in Definition 2. Using Theorem 4, Part 1, results directly that $V_{[r:n,\tilde{m},k]} \geq_{lr:j} [\leq_{lr:j}] V_{[s:n,\tilde{m},k]}$ for $r \leq s$.

Theorem 11. *Let W and V be two RVs with conditional hazard function $h_{V|W}(v|w)$. If $h_{V|W}(v|w)$ increases [decreases] in w, then $V_{[r:n,\tilde{m},k]} \geq_{hr:j} [\leq_{hr:j}] V_{[s:n,\tilde{m},k]}$ for $s \geq r$.*

Proof. If $h_{V|W}(v|w)$ increases in w, then $h_{V|W}(v|w_2) \geq h_{V|W}(v|w_1)$ for all $w_2 \geq w_1$. Consequently, using Definition 1, Part 3, we get $f(v_2|w_2)\bar{F}(v_1|w_1) \geq \bar{F}(v_1|w_2)f(v_2|w_1)$ for all $v_1 \geq v_2$ and $w_2 \geq w_1$. It follows that, for $r \leq s$,

$$\frac{\partial}{\partial v_2}\bar{F}_{[r,s:n,\tilde{m},k]}(v_1,v_2) - \frac{\partial}{\partial v_2}\bar{F}_{[r,s:n,\tilde{m},k]}(v_2,v_1) = \iint\limits_{w_2 \geq w_1} [-\bar{F}(v_1|w_1)f(v_2|w_2) + f(v_2|w_1)\bar{F}(v_1|w_2)] f_{r,s:n,\tilde{m},k}(w_2,w_1)dw_1dw_2$$

is ≤ 0 for $v_1 \geq v_2$. Thus, using Theorem 4, Part 2, it follows that $V_{[r:n,\tilde{m},k]} \geq_{hr:j} V_{[s:n,\tilde{m},k]}$ for $s \geq r$. We can prove in the same manner that the ordering is reversed if $h_{V|W}(v|w)$ is decreasing in w.

Theorem 12. *Consider two RVs, W and V, with joint DF, $F(w,v)$. If $[V|W=w_1] \geq_{st} [\leq_{st}][V|W=w_2]$ for $w_1 \leq w_2$, then $V_{[r,n,\tilde{m},k]} \geq_{st:j} [\leq_{st:j}] V_{[s,n,\tilde{m},k]}$ for $r \leq s$.*

Proof. Let $g \in \mathscr{G}_{st}$, where $\mathscr{G}_{st}$ is defined in Definition 2. Since, for $r \leq s$,

$$\begin{aligned}
&E(g(V_{[r,n,\tilde{m},k]}, V_{[s,n,\tilde{m},k]})) - E(g(V_{[s,n,\tilde{m},k]}, V_{[r,n,\tilde{m},k]})) \\
&= \iint\limits_{w_2 \geq w_1} \left(\iint_{-\infty}^{\infty} \Delta g(v_1,v_2) dF(v_1|w_1) dF(v_2|w_2) \right) f_{r,s:n,\tilde{m},k}(w_1,w_2)dw_1dw_2 \\
&= \iint\limits_{w_2 \geq w_1} \left(\int_0^1 \int_0^1 \Delta g(F^{-1}_{V|W=w_1}(u_1), F^{-1}_{V|W=w_2}(u_2)) du_1 du_2 \right) f_{r,s:n,\tilde{m},k}(w_1,w_2)dw_1dw_2 \\
&= \iint\limits_{w_2 \geq w_1} \left(\iint_{u_1 \geq u_2} \Delta g(F^{-1}_{V|W=w_1}(u_1), F^{-1}_{V|W=w_2}(u_2)) - \right. \\
&\qquad \left. \Delta g(F^{-1}_{V|W=w_2}(u_1), F^{-1}_{V|W=w_1}(u_2)) du_1 du_2 \right) f_{r,s:n\tilde{m},k}(w_1,w_2)dw_1dw_2 \geq [\leq] 0,
\end{aligned}$$

where, by setting $v_1 = F^{-1}_{V|W=w_1}(u_1)$ and $v_2 = F^{-1}_{V|W=w_2}(u_2)$ the second equality results. The inequality follows from the fact that if V decreases stochastically [increases] in W, then, using Definition 1 part (1), we have $F^{-1}_{V|w_1}(u_1) \geq [\leq] F^{-1}_{V|w_2}(u_1)$ and $F^{-1}_{V|w_2}(u_2) \leq [\geq] F^{-1}_{V|w_1}(u_2)$. Further, if $g \in \mathscr{G}_{st}$, then $\Delta g(u,v)$ decreases in v $\forall u$ and increases in u $\forall v$. Thus, $\Delta g(F^{-1}_{V|w_1}(u_1), F^{-1}_{V|w_2}(u_2)) \geq [\leq] \Delta g(F^{-1}_{V|w_2}(u_1), F^{-1}_{V|w_1}(u_2))$. Now the result follows using Definition 2, Part 3.

A bivariate characterization for the order $\leq_{rh}$ is given in the next theorem (cf. Theorem 3.2 in [25]).

Theorem 13. *For two independent RVs, W and V, $W \geq_{rh} V \Leftrightarrow E(g(W,V)) \geq E(g(V,W))$, $\forall g \in \mathscr{G}_{rh}$, where*

$$\mathscr{G}_{rh} := \{g : \mathbb{R}^2 \to \mathbb{R}; \Delta g(w,v) = g(w,v) - g(v,w) \text{ is decrasing in } v \ \forall w \geq v\}. \tag{16}$$

Based on this characterization, we introduce the definition of the order $\leq_{rh:j}$ as follows.

Definition 4. *Let W and V be two RVs, we say that W is greater than V in the joint reversed hazard order, symbolized as $W \geq_{rh:j} V$, if $E(g(W,V) \geq E(g(V,W))$, $\forall g \in \mathscr{G}_{rh}$; where $\mathscr{G}_{rh}$ is defined in (16).*

Next, we introduce a sufficient and necessary condition for the joint reversed hazard ordering.

Lemma 5. *Let W and V be two RVs having joint DF $F_{WV}(w,v)$. $W \geq_{rh:j} V$ if and only if $\frac{\partial}{\partial w}F_{WV}(w,v) \geq \frac{\partial}{\partial w}F_{WV}(v,w), \forall w \geq v$.*

*Proof.*Define the class $\mathcal{G}^*_{rh}$ of bivariate functions as

$$\mathcal{G}^*_{rh} = \{g : \mathbb{R}^2 \to \mathbb{R} : g(w,v) \text{ decreases in } v,\ \forall w \geq v, \text{ and increases in } w,\ \forall v \geq w\}.$$

Clearly, $\mathcal{G}^*_{rh} \subseteq \mathcal{G}_{rh}$. If $W \geq_{rh;j} V$, then, from Definition 4, for all $g \in \mathcal{G}^*_{rh}$,

$$\iint_{-\infty}^{\infty} g(t,s) f_{WV}(t,s)dtds \geq \iint_{-\infty}^{\infty} g(s,t) f_{WV}(t,s)dtds. \tag{17}$$

Let, for $w \geq v$ and $\Delta \geq 0$, $g(t,s) = I_{\{w+\Delta \geq t \geq w, s \leq v\}}$, where $I_{\mathscr{A}}$ symbolizes the indicator function on a set $\mathscr{A}$. It is clear that $g(t,s)$ is decreasing in s and $g(t,s) = 0$ for $s \geq t$. Therefore $g(t,s) \in \mathcal{G}^*_{rh}$. It follows from (17) that

$$F(w+\Delta, v) - F(w,v) \geq F(v, w+\Delta) - F(v,w), \tag{18}$$

for $w \geq v$. Dividing the two sides of (18) by Δ and applying limit as $\Delta \to 0$, we have $\frac{\partial}{\partial u} F_{WV}(w,v) \geq \frac{\partial}{\partial w} F_{WV}(v,w)$, $\forall w \geq v$.

Conversely, let $\frac{\partial}{\partial w} F_{WV}(w,v) \geq \frac{\partial}{\partial w} F_{WV}(v,w), \forall w \geq v$, and let $g \in \mathcal{G}_{rh}$. Then

$$\begin{aligned} E(g(W,V)) - E(g(V,W)) &= \iint_{-\infty}^{\infty} \Delta g(w,v) f_{WV}(w,v)dwdv \\ &= \iint_{w \geq v} \Delta g(w,v)[f_{WV}(w,v) - f_{WV}(v,w)]dwdv \\ &= \iint_{w \geq v} \left(-\frac{\partial}{\partial v}\Delta g(w,v)\right)\left(\frac{\partial}{\partial w} F_{WV}(w,v) - \frac{\partial}{\partial w} F_{WV}(v,w)\right) dwdv \geq 0, \end{aligned}$$

where integrating by parts with respect to v results in the final equality. The inequality is due to the fact that $g \in \mathcal{G}_{rh}$ implies that $-\frac{\partial}{\partial v}\Delta g(w,v) \geq 0$. Using Definition 4, and the above inequality we see that $W \geq_{hr:j} V$.

Theorem 14.*Assume that W and V are two RVs with joint DF, $F_{WV}(w,v)$. If $r_{V|W}(v|w)$ decreases [increases] in w, then, for $s \geq r$, $V_{[r,n,\tilde{m},k]} \geq_{rh:j} [\leq_{rh:j}] V_{[s,n,\tilde{m},k]}$.*

*Proof.*If $r_{V|W}(v|w)$ decreases in w, then $r_{V|W}(v|w_1) \geq r_{V|W}(v|w_2)$ for all $w_2 \geq w_1$. Consequently, using Definition 1, Part 3, we get $F(v_2|w_2)f(v_1|w_1) \geq F(v_2|w_1)f(v_1|w_2)$ for all $v_1 \geq v_2$ and $w_2 \geq w_1$. It follows that, for $r \leq s$,

$$\frac{\partial}{\partial v_1} F_{[r,s:n,\tilde{m},k]}(v_1,v_2) - \frac{\partial}{\partial v_1} F_{[r,s:n,\tilde{m},k]}(v_2,v_1) = \iint_{w_2 \geq w_1} [f(v_1|w_1)F(v_2|w_2) - F(v_2|w_1)f(v_1|w_2)] f_{r,s:n,\tilde{m},k}(w_1,w_2)dw_1dw_2 \geq 0,$$

for $v_1 \geq v_2$. Thus, using Lemma 5, it follows that $V_{[r:n,\tilde{m},k]} \geq_{rh:j} V_{[s:n,\tilde{m},k]}$ for $s \geq r$. We can prove in the same manner that if $r(v|w)$ increases in w, then $V_{[r:n,\tilde{m},k]} \leq_{rh:j} V_{[s:n,\tilde{m},k]}$ for $s \geq r$.

In order to define the joint convex [increasing convex][concave][increasing concave] orders, we recall the next bivariate characterizations for the orders $\leq_{cx}$ $[\leq_{icx}][\leq_{cv}][\leq_{icv}]$. Consider first the next classes of functions.

$$\begin{aligned} \mathcal{G}_{cx}[\mathcal{G}_{cv}][\mathcal{G}_{icx}][\mathcal{G}_{icv}] := \{ g : \mathbb{R}^2 \to \mathbb{R} : \Delta g(w,v) = g(w,v) - g(v,w) \\ \text{is convex [increasing and convex] [concave][increasing and concave] in } w\ \forall v\} \end{aligned}$$

Theorem 15(see [7]. P. 115, 185). *Let W and V be two independent RVs. Then $W \geq_{cx} [\geq_{cv}][\geq_{icx}][\geq_{icv}] V \Leftrightarrow Eg(W,V) \geq Eg(V,W)$, $\forall g \in \mathcal{G}_{cx}[\mathcal{G}_{cv}][\mathcal{G}_{icx}][\mathcal{G}_{icv}]$.*

Based on the characterization in Theorem 15 we introduce the "joint convex order" below. The joint orders, $\leq_{cv:j}, \leq_{icx:j}$ and $\leq_{icv:j}$ can be defined analogously.

Definition 5.*Let W and V be two RVs, we claim that W is greater than V in the joint convex order, symbolized as $W \geq_{cx:j} V$, if $Eg(W,V) \geq Eg(V,W)$, $\forall g \in \mathcal{G}_{cx}$.*

Theorem 16. *Assume that W and V are two RVs. If $[V|W=w_1] \geq_{cx} [\leq_{cx}][V|W=w_2]$, $w_1 \leq w_2$, then $V_{[r,n,\tilde{m},k]} \geq_{cx:j} [\leq_{cx:j}]V_{[s,n,\tilde{m},k]}$ for $s \geq r$.*

Proof. If $[V|W=w_1] \geq_{cx} [\leq_{cx}][V|W=w_2]$ for $w_1 \leq w_2$, then

$$\int_{-\infty}^{\infty} \phi(v_1) f_{V|W}(v_1|w_1)dv_1 \geq [\leq] \int_{-\infty}^{\infty} \phi(v_2) f_{V|W}(v_2|w_2)dv_2, \tag{19}$$

for all $w_1 \leq w_2$ and for any convex functions ϕ. Define $\phi(v_1) = \Delta g(v_1, v_2)$ to be a convex function in v_1, we have

$$\begin{aligned}
&Eg(V_{[r,n,\tilde{m},k]}, V_{[s,n,\tilde{m},k]}) - Eg(V_{[s,n,\tilde{m},k]}, V_{[r,n,\tilde{m},k]}) \\
&= \iint_{w_2 \geq w_1} \left(\iint_{-\infty}^{\infty} \Delta g(v_1,v_2) f(v_2|w_2) f(v_1|w_1) dv_1 dv_2 \right) f_{(r,s,n,\tilde{m},k)}(w_1,w_2)dw_1dw_2 \\
&= \iint_{w_2 \geq w_1} \left(\int_{\infty}^{\infty} \left[\int_{-\infty}^{\infty} \phi(v_1) f(v_1|w_1)dv_1 \right] f(v_2|w_2)dv_2 \right) f_{(r,s,n,\tilde{m},k)}(w_1,w_2)dw_1dw_2 \\
&\geq [\leq] \iint_{w_2 \geq w_1} \left(\int_{\infty}^{\infty} \left[\int_{-\infty}^{\infty} \phi(v_2) f(v_2|w_2)dv_2 \right] f(v_2|w_2)dv_2 \right) f_{(r,s,n,\tilde{m},k)}(w_1,w_2)dw_1dw_2 = 0
\end{aligned}$$

where the inequality follows from (19), and the final equality is obtained by recognizing that $\phi(v_2) = \Delta g(v_2, v_2) = 0$. The result now follows from Definition 5.

5 Two-Sample Comparisons

Let $(W_i, V_i), 1 \leq i \leq n$ be n independent copies of (W, V). Let these tuples be arranged according to the W variate so that $W_{(r,n,\tilde{m},k)}$ and $V_{[r,n,\tilde{m},k]}$ are the rth GOS and the corresponding concomitant, respectively. Let $(S_i, T_i), 1 \leq i \leq n$ be another bivariate sample of size n from another bivariate variables (S, T). Assume further that these tuples are arranged according to their S variable, so that $S_{(r,n,\tilde{m},k)}$ and $T_{[r,n,\tilde{m},n]}$ are the rth GOS and the corresponding concomitant, respectively. Here we are interested to compare $V_{[r,n,\tilde{m},k]}$ and $T_{[r,n,\tilde{m},n]}$ in different types of stochastic orders. Such comparisons are called two-sample comparisons.

The next lemma is used to drive the fundamental results of the concomitants' two-sample comparisons.

Lemma 6. *Let $W(\phi)$ be a RV with DF, $F_{W(\phi)}(.)$,$\phi \in \mathscr{X} \subseteq \mathbb{R}$. Let Φ be a RV having support $\mathscr{X}$ and DF, $G_\Phi(.)$. Let the compound random variable $W(\Phi)$ have a PDF, $h_{W(\Phi)}$ given as*

$$h_{W(\Phi)}(u) = \int_{\mathscr{X}} f_{W(\phi)}(u) g_\Phi(\phi) d\phi. \tag{20}$$

Let $V(\omega)$ have a distribution function, $F_{V(\omega)}(.)$, $\omega \in \mathscr{X} \subseteq \mathbb{R}$. Let Ω be a RV having support $\mathscr{X}$ and having a DF, $G_\Omega(.)$. Let the compound RV, $V(\Omega)$ have a PDF $h_{V(\Omega)}(.)$ given as

$$h_{V(\Omega)}(u) = \int_{\chi} f_{V(\omega)}(u) g_\Omega(\omega) d\omega. \tag{21}$$

1. *If $W(\phi) \leq_{lr} V(\omega)$, then $W(\Phi) \leq_{lr} V(\Omega)$.*
2. *If $W(\phi) \leq_{hr} V(\omega)$, then $W(\Phi) \leq_{hr} V(\Omega)$.*
3. *If $W(\phi) \leq_{rh} V(\omega)$, then $W(\Phi) \leq_{rh} V(\Omega)$.*
4. *If $W(\phi) \leq_{mrl} V(\omega)$, then $W(\Phi) \leq_{mrl} V(\Omega)$.*
5. *If $W(\phi) \leq_{hmrl} V(\omega)$, then $W(\Phi) \leq_{hmrl} V(\Omega)$.*
6. *For non-negative RVs, $W(\phi)$ and $V(\omega)$, if $W(\phi) \leq_{hr} V(\omega)$, and if either $W(\phi)$ is DFR $\forall \phi \in \mathscr{X}$ or, $V(\omega)$ is DFR $\forall \omega \in \mathscr{X}$, then $W(\Phi) \leq_{disp} V(\Omega)$.*
7. *If $W(\phi) \leq_{mrl} V(\omega)$ and either $W(\phi)$ is IMRL $\forall \phi \in \mathscr{X}$ or $V(\omega)$ is IMRL $\forall \omega \in \mathscr{X}$, then $W(\Phi) \leq_{ew} V(\Omega)$.*
8. *If $W(\phi) \leq_{st} V(\omega)$ and $\Phi \leq_{st} \Omega$, then $W(\Phi) \leq_{st} V(\Omega)$.*
9. *If $W(\phi) \leq_{icx} V(\omega), \Phi \leq_{st} \Omega$ and either $\int_t^\infty \bar{F}_{W(\phi)}(u)du$ is increasing in ϕ, or $\int_t^\infty \bar{F}_{V(\omega)}(u)du$ is increasing in ω, then $W(\Phi) \leq_{icx} V(\Omega)$.*

Proof. 1.If $W(\phi) \leq_{lr} V(\omega)$, then, by Definition 1, Part 6, $f_{V(\omega)}(w_2)f_{W(\phi)}(w_1) - f_{V(\omega)}(w_1)f_{W(\phi)}(w_2) \geq 0$ for $w_1 \leq w_2$. It follows, for $w_1 \leq w_2$ that

$$\begin{aligned}&h_{V(\Omega)}(w_2)h_{W(\Phi)}(w_1) - h_{V(\Omega)}(w_2)h_{W(\Phi)}(w_1)\\&= \int_{\mathscr{X}}\int_{\mathscr{X}} [f_{V(\omega)}(w_2)f_{W(\phi)}(w_1) - f_{V(\omega)}(w_1)f_{W(\phi)}(w_2)]g_{\Phi}(\phi)g_{\Omega}(\omega)d\phi d\omega \geq 0.\end{aligned}$$

Therefore $h_{V(\Omega)}(w)/h_{W(\Phi)}(w)$ is an increasing function in $w \in \mathbb{R}$. Thus $W(\Phi) \leq_{lr} V(\Omega)$.

2.If $W(\phi) \leq_{hr} V(\omega)$, then, by Definition 1, Part 2, $\bar{F}_{V(\omega)}(w_2)\bar{F}_{W(\phi)}(w_1) - \bar{F}_{V(\omega)}(w_1)\bar{F}_{W(\phi)}(w_2) \geq 0$ for $w_2 \geq w_1$. It follows, for $w_2 \geq w_1$, that

$$\begin{aligned}&\bar{H}_{V(\Omega)}(w_2)\bar{H}_{W(\Phi)}(w_1) - \bar{H}_{V(\Omega)}(w_2)\bar{H}_{W(\Phi)}(w_1)\\&= \int_{\mathscr{X}}\int_{\mathscr{X}} \left[\bar{F}_{V(\omega)}(w_2)\bar{F}_{W(\phi)}(w_1) - \bar{F}_{V(\omega)}(w_1)\bar{F}_{W(\phi)}(w_2)\right] g_{\Omega}(\omega)g_{\Phi}(\phi)d\omega d\phi \geq 0.\end{aligned}$$

Therefore, the function $\bar{H}_{V(\Omega)}(w)/\bar{H}_{W(\Phi)}(w)$ is increasing in $w \in \mathbb{R}$. Thus, $W(\Phi) \leq_{hr} V(\Omega)$.

3.The proof of this part is parallel to the proof of part 2, replacing every survival function by the corresponding distribution function. The result follows using Definition 1, Part 3.

4.If $W(\phi) \leq_{mrl} V(\omega)$, we have $\int_{w_2}^{\infty}\bar{F}_{V(\omega)}(t)dt \int_{w_1}^{\infty}\bar{F}_{W(\phi)}(t)dt - \int_{w_2}^{\infty}\bar{F}_{V(\omega)}(t)dt \int_{w_1}^{\infty}\bar{F}_{W(\phi)}(t)dt \geq 0$ for $w_2 \geq w_1$. Then,

$$\begin{aligned}&\int_{w_2}^{\infty}\bar{H}_{V(\Omega)}(t)dt\int_{w_1}^{\infty}\bar{H}_{W(\Phi)}(t)dt - \int_{w_1}^{\infty}\bar{H}_{V(\Omega)}(t)dt\int_{w_2}^{\infty}\bar{H}_{W(\Phi)}(t)dt\\&= \int_{\mathscr{X}}\int_{\mathscr{X}}\left(\int_{w_2}^{\infty}\bar{F}_{V(\omega)}(t)dt\int_{w_1}^{\infty}\bar{F}_{W(\phi)}(t)dt - \int_{w_2}^{\infty}\bar{F}_{V(\omega)}(t)dt\int_{w_1}^{\infty}\bar{F}_{W(\phi)}(t)dt\right)g_{\Omega}(\omega)g_{\Phi}(\phi)d\phi d\omega \geq 0,\end{aligned}$$

for $w_2 \geq w_1$. Therefore, the function $\int_{w}^{\infty}\bar{H}_{V(\Omega)}(t)dt/(\int_{w}^{\infty}\bar{H}_{W(\Phi)}(t)dt)$ is increasing in w. Thus, $W(\Phi) \leq_{mrl} V(\Omega)$.

5.For proving part 5, it suffices to demonstrate that

$$\delta = \left(\int_{w}^{\infty}\bar{H}_{V(\Omega)}(t)dt\right)E(W(\Phi)) - \left(\int_{w}^{\infty}\bar{H}_{W(\Phi)}(t)dt\right)E(V(\Omega)) \geq 0$$

Since $E(W(\Phi)) = \int_{\mathscr{X}}\left(\int_{-\infty}^{\infty} t f_{W(\phi)}(t)dt\right)g_{\phi}(\phi)d\phi$, therefore

$$\delta = \int_{\mathscr{X}}\int_{\mathscr{X}}\left[\left(\int_{w}^{\infty}\bar{F}_{V(\omega)}(t)dt\right)\left(\int_{-\infty}^{\infty} t f_{W(\phi)}(t)dt\right) - \left(\int_{w}^{\infty}\bar{F}_{W(\Phi)}(t)dt\right)\left(\int_{-\infty}^{\infty} t f_{V(\omega)}(t)dt\right)\right]g_{\Phi}(\phi)g_{\Omega}(\omega)d\phi d\omega.$$

Now, for $W(\phi) \leq_{hmrl} V(\omega)$, it is clear to observe that the quantity between the square brackets is ≥ 0, ensuring that $\delta \geq 0$.

6.If $W(\phi) \leq_{hr} V(\omega)$, then, by Part 2, $W(\Phi) \leq_{hr} V(\Omega)$. If $W(\phi)$ is DFR, $\forall\phi \in \mathscr{X}$, it follows that the mixture $W(\Phi)$ is DFR (see [24] P. 408). Now, the result is immediate using Theorem 1 .

7.If $W(\phi) \leq_{mrl} V(\omega)$, then, by Part 4, $W(\Phi) \leq_{mrl} V(\Omega)$. If $W(\phi)$ is IMRL, $\forall\phi \in \mathscr{X}$, it follows by Lemma 4 that the mixture $W(\Phi)$ is IMRL. Now, the result is immediate using Theorem 2.

8.The proof of this part depends on the multi-variable stochastic order concept (see [7]). since

$$[W(\Phi)|\Phi = \phi] =_{st} W(\phi) \leq_{st} V(\omega) =_{st} [V(\Omega)|\Omega = \omega], \tag{22}$$

and if further $\Phi \leq_{st} \Omega$, then it results from Theorem 6.B.3 in [7] that $(\Phi, W(\Phi))$ less than $(\Omega, V(\Omega))$ in multivariate stochastic order. Since the joint stochastic order is closed under marginalization (Theorem 6.B.16 (Part c) in [7]). It results that $W(\Phi) \leq_{st} V(\Omega)$.

9.For proving part 9, we have to prove that $\int_x^{\infty}\bar{H}_{W(\Phi)}(t)dt \leq \int_x^{\infty}\bar{H}_{V(\Omega)}(t)dt, x \in \mathbb{R}$. or equivalently to prove, $\forall\, x \in \mathbb{R}$, that

$$\int_{\mathscr{X}}\left(\int_{x}^{\infty}\bar{F}_{W(\phi)}(t)dt\right)g_{\Phi}(\phi)d\phi \leq \int_{\mathscr{X}}\left(\int_{x}^{\infty}\bar{F}_{V(\omega)}(t)dt\right)g_{\Omega}(\omega)d\omega. \tag{23}$$

If $W(\phi) \leq_{icx} V(\omega)$, therefore $\int_x^{\infty}\bar{F}_{W(\phi)}(t)dt \leq \int_x^{\infty}\bar{F}_{V(\omega)}(t)dt$, for all $x \in \mathbb{R}$, ϕ and $\omega \in \mathscr{X}$. In particular, for all $x \in \mathbb{R}$, we have

$$\int_{\mathscr{X}}\left(\int_{x}^{\infty}\bar{F}_{W(\omega)}(t)dt\right)g_{\Omega}(\omega)d\omega \leq \int_{\mathscr{X}}\left(\int_{x}^{\infty}\bar{F}_{V(\omega)}(t)dt\right)g_{\Omega}(\omega)d\omega. \tag{24}$$

Now, if $\Phi \leq_{st} \Omega$, therefore $\int_{\mathscr{X}} \psi(\phi) g_{\Phi}(\phi) d\phi \leq \int_{\mathscr{X}} \psi(\omega) g_{\Omega}(\omega) d\omega$ for all increasing functions $\psi(.)$. If further $\psi(\phi) = \int_x^{\infty} \bar{F}_{W(\phi)}(t)dt$ is increasing function in ϕ, therefore

$$\int_{\mathscr{X}} \left(\int_x^{\infty} \bar{F}_{W(\phi)}(t)dt \right) g_{\Phi}(\phi) d\phi \leq \int_{\mathscr{X}} \left(\int_x^{\infty} \bar{F}_{W(\omega)}(t)dt \right) g_{\Omega}(\omega) d\omega \;\; \forall x \in \mathbb{R} \tag{25}$$

from (24) and(25) we get the required in (23).

Comparing (20) and (21) with (3), the results in 1 through 9 in the following theorem follow directly from their counterparts in Lemma 6, replacing $W(\phi)$ by $[V|W=w]$, $V(\omega)$ by $[T|S=w]$, Φ by $W_{(r,n,\tilde{m},k)}$, and Ω by $S_{(r,n,\tilde{m},k)}$, taking into consideration that $W \leq_{st} S$ implies, from Corollary 3.2 in [26], that $W_{(r,n,\tilde{m},k)} \leq_{st} S_{(r,n,\tilde{m},k)}$.

Theorem 17. *Let $V_{[r,n,\tilde{m},k]}$ and $T_{[r,n,\tilde{m},k]}$ be the concomitants of GOSs based on bivariate samples $(W_i,V_i), 1 \leq i \leq n$ and $(S_i,T_i), 1 \leq i \leq n$, respectively.*

1. *If $[V|W=w] \leq_{lr} [T|S=w]$, then $V_{[r,n,\tilde{m},k]} \leq_{lr} T_{[r,n,\tilde{m},k]}$.*
2. *If $[V|W=w] \leq_{hr} [T|S=w]$, then $V_{[r,n,\tilde{m},k]} \leq_{hr} T_{[r,n,\tilde{m},k]}$.*
3. *If $[V|W=w] \leq_{rh} [T|S=w]$, then $V_{[r,n,\tilde{m},k]} \leq_{rh} T_{[r,n,\tilde{m},k]}$.*
4. *If $[V|W=w] \leq_{mrl} [T|S=w]$, then $V_{[r,n,\tilde{m},k]} \leq_{mrl} T_{[r,n,\tilde{m},k]}$.*
5. *If $[V|W=w] \leq_{hmrl} [T|S=w]$, then $V_{[r,n,\tilde{m},k]} \leq_{hmrl} T_{[r,n,\tilde{m},k]}$.*
6. *For non-negative RVs $[V|W=w]$ and $[T|S=w]$, if $[V|W=w] \leq_{hr} [T|S=w]$ and if either $[V|W=w]$ is DFR $\forall w$ or $[T|S=s]$ is DFR, $\forall s$, then $V_{[r,n,\tilde{m},k]} \leq_{disp} T_{[r,n,\tilde{m},k]}$.*
7. *If $[V|W=w] \leq_{mrl} [T|S=w]$ and if either $[V|W=w]$ is IMRL, $\forall w$, or $[T|S=s]$ is IMRL, $\forall s$, then $V_{[r,n,\tilde{m},k]} \leq_{ew} T_{[r,n,\tilde{m},k]}$.*
8. *If $[V|W=w] \leq_{st} [T|S=w]$ and $W \leq_{st} S$, then $V_{[r,n,\tilde{m},k]} \leq_{st} T_{[r,n,\tilde{m},k]}$.*
9. *If $[V|W=w] \leq_{icx} [T|S=w]$, $W \leq_{st} S$ and if either $\bar{F}_{V|W=w}(v|w)$ increases in w or $\bar{F}_{T|S=s}(t|s)$ increases in s, then $V_{[r,n,\tilde{m},k]} \leq_{icx} T_{[r,n,\tilde{m},k]}$.*

6 Illustrative Examples

Some illustrative examples of stochastic ordering of GOSs' concomitants are provided. These examples are based on certain particular bivariate distributions, which can be found in [20].

Example 1. Let $V_{[r,n,\tilde{m},k]}$ and $V_{[r',n',\tilde{m}',k']}$ be the concomitants corresponding to the GOSs $W_{(r,n,\tilde{m},k)}$ and $W_{(r',n',\tilde{m}',k')}$, based on the bivariate Lomax PDF

$$f(w,v) = \frac{c[ab + c(a+\phi v)(b+\phi w) - \phi]}{(1+aw+bv+\phi wv)^{c+2}}, \tag{26}$$

where $0 \leq \phi \leq ab(c+1), 0 \leq a,b,c$. If $0 \leq \phi \leq ab$, then:

1. $V_{[r,n,\tilde{m},k]} \leq_{lr} V_{[r',n',\tilde{m}',k']}$ if $r \leq r', m_{r-i} \geq m'_{r'-i}$ for $i = 1,\ldots,r-1$ and $\gamma_r \geq \gamma'_{r'}$
2. $V_{[r,n,\tilde{m},k]} \leq_{disp} V_{[r',n',\tilde{m}',k']}$ if $(\gamma'_{l_1},\ldots,\gamma'_{l_r}) \succeq^{p} (\gamma_1,\ldots,\gamma_r)$ for some set $\{l_1,\ldots,l_r\} \subseteq \{1,\ldots,r'\}$ for $r \leq r'$.

In fact, It follows, from (26), that $\frac{\partial}{\partial v}\left(\frac{f(w_2,v)}{f(w_1,v)}\right) \geq 0$ for $w_1 \leq w_2$ if

$$\begin{aligned}&(1+aw_1+bv+\phi w_1 v)(1+aw_2+bv+\phi w_2 v)\; c\phi(ab-\phi)\phi(w_2-w_1)\\&\qquad +(c+2)(c(b+\phi w_2)(a+\phi v)+ab-\phi)(c(b+\phi w_1)(a+\phi v)+ab-\phi)(w_2-w_1)(ba-\phi) \geq 0,\end{aligned}$$

which is satisfied whenever $0 \leq \phi \leq ab$, for all $w_1 \leq w_2$, and for $a,b,c \geq 0$. Thus W and V are TP_2 dependent. The order $\leq_{lr}$ in Part 1 follows now using Theorem 8.

Moreover, since $f(w,v)$ is TP_2, it results that $[V|W=w_1] \leq_{lr} [V|W=w_2]$ for $w_1 \leq w_2$. Thus, $[V|W=w_1] \leq_{hr} [V|W=w_2]$. Hence, the function $h(v|w)$ decreases in w. On the other hand, the conditional hazard of V given $W=w$, given by

$$h(v|w) = \frac{c(b+\phi w)(a+\phi v)+ab-\phi}{(1+aw+bv+\phi wv)(a+\phi v)},$$

decreases in v whenever $\frac{\partial}{\partial v}[h(v|w)] \leq 0$. Or equivalently if $-[c(b+aw)^2(a+\phi v)^2 + (ab-\phi)[\phi(1+aw+bv+\phi wv) + (b+\phi w)(a+\phi v)]] \leq 0$, which is the case whenever $ab \geq \phi$. Corollary 1, Part 1, now forms the basis for the dispersive ordering in Part 2.

An example of bivariate DF with negative dependence is given below.

*Example 2.*Let $V_{[r,n,\tilde{m},k]}$ and $V_{[r',n',\tilde{m}',k']}$ be the concomitants corresponding to the GOSs $W_{(r,n,\tilde{m},k)}$ and $W_{(r',n',\tilde{m}',k')}$ based on the bivariate finite range survival function given by

$$\bar{F}(w,v) = (1-\theta_1 w - \theta_2 v - \theta_3 wv)^p, \tag{27}$$

where $\theta_1, \theta_2 > 0$, $-1 \leq \frac{\theta_3}{\theta_1\theta_2} \leq p-1$, $0 \leq w \leq \theta_1^{-1}$, $0 \leq v \leq \frac{1-\theta_1 w}{\theta_2+\theta_3 w}$. If $p \geq 2$, then $V_{[r,n,\tilde{m},k]} \geq_{lr} V_{[r',n',\tilde{m}',k']}$ if $r' \geq r, m'_{r'-l} \leq m_{r-l}$ for $1 \leq l \leq r-1$, and $\gamma_r \geq \gamma'_{r'}$.

In fact, the PDF corresponding to (27) is

$$f(w,v) = p(1-\theta_1 w - \theta_2 v - \theta_3 wv)^{p-2}[p(\theta_1+\theta_3)v(\theta_2+\theta_3 w) - \theta_1\theta_2 - \theta_3].$$

It follows that, for $w_1 \leq w_2$,

$$\begin{aligned}\frac{\partial}{\partial v}\left[\frac{f(w_2,v)}{f(w_1,v)}\right] &= (\theta_1\theta_2+\theta_3)(w_1-w_2)\big[(p-2)(p(\theta_1+\theta_3 v)(\theta_2+\theta_3 w_2)-\theta_1\theta_2-\theta_3)\\ &\times(p(\theta_1+\theta_3 v)(\theta_2+\theta_3 w_1)-\theta_1\theta_2-\theta_3)+p\theta_3^2(1-\theta_1 w_2-\theta_2 v-\theta_3 w_2 v)(1-\theta_1 w_1-\theta_2 v-\theta_3 w_2 v)\big].\end{aligned}$$

It can be seen that, if $p \geq 2$, then $\partial/\partial v[f(w_2,v)/f(w_1,v)] \leq 0$, $\forall\ w_1 \leq w_2$. Consequently W and V are RR_2 dependent. Theorem 8 then dictates the result.

An example of the ordering $\leq_{lr}$ of concomitants in the two-sample setup is given by Example 3.

*Example 3.*Suppose that $V_{[r,n,\tilde{m},k]}$ is the concomitant of the rth GOS based on the Marshal-Olkin bivariate exponential distribution (MOBVE) with parameters $(\lambda_1,\lambda_2,\lambda_{12})$ (see [20] p.413) with conditional pdf given by

$$f_{V|W}(v|w) = \begin{cases}\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_{12}}e^{-\lambda_2 v-\lambda_{12}(v-w)}; & \text{for } v>w\\ \lambda_2 e^{-\lambda_2 v}; & \text{for } v<w\end{cases}.$$

Suppose that $T_{[r,n,\tilde{m},k]}$ is the concomitant of the rth GOS of the MOBVE with parameters $(\lambda_1',\lambda_2',\lambda_{12}')$. If $\lambda_2 \geq \lambda_2'$ and $\lambda_{12} \geq \lambda_{12}'$, then $V_{[r,n,\tilde{m},k]} \leq_{lr} T_{[r,n,\tilde{m},k]}$.

This follows from the fact that, for $w \geq t$, we have

$$\frac{f_{T|S}(t|w)}{f_{V|W}(t|w)} = \frac{\lambda_2'}{\lambda_2}e^{-(\lambda_2'-\lambda_2)t},$$

which increases in t if $\lambda_2' \leq \lambda_2$.
For $w \leq t$, we have

$$\frac{f_{T|S}(t|w)}{f_{V|W}(t|w)} = \left(\frac{(\lambda_1+\lambda_{12})(\lambda_2'+\lambda_{12}')\lambda_1'}{(\lambda_1'+\lambda_{12}')\lambda_1(\lambda_2+\lambda_{12})}\right)e^{(\lambda_2-\lambda_2')t+(\lambda_{12}-\lambda_{12}')(t-w)}. \tag{28}$$

It is clear that the ratio in (28) increases in t, for $w \leq t$, if $\lambda_2 \geq \lambda_2'$ and $\lambda_{12} \geq \lambda_{12}'$. Therefore $[V|W=w] \leq_{lr} [T|S=w]$ whenever $\lambda_2 \geq \lambda_2'$ and $\lambda_{12} \geq \lambda_{12}'$. Theorem 17(part 1) leads to the conclusion.

7 Concluding Remarks

- In the one sample problem, we have noticed that the stochastic comparisons of the GOSs' concomitants, based on a continuous bivariate DF $F_{WV}(w,v)$, depend on the stochastic ordering of GOSs and on how V is stochastically monotone in W (dependence between W and V).
- It has been proven that if V increases stochastically in W with regard to the orders $\leq_{cx}, \leq_{rh}, \leq_{hr}$, and $\leq_{lr}$, then, for $r = 1,\ldots,n$ the concomitants $V_{[r,n,\tilde{m},k]}$ are stochastically increasing in r in sense of the orders $\leq_{cx:j}, \leq_{rh:j}, \leq_{hr:j}$, and $\leq_{lr:j}$, respectively.
- In the two-sample setup, based on any two continuous bivariate distribution $F_{WV}(w,v)$ and $F_{ST}(s,t)$, the stochastic ordering of concomitants have been seen to be dependent on the stochastic ordering of$[V|W=w]$ and $[T|S=w]$ in various senses.
- For some specific bivariate distributions, some illustrative examples have been provided for the concomitants' comparisons with regard to the orders $\leq_{lr}$ and $\leq_{disp}$. The other weaker orders will follow immediately, using the implications in (5) and (6).

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