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Sixth-Kind Chebyshev Collocation Treatment of Fractional Fredholm-Volterra Integro-Differential Equations

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Abstract: We propose a spectral technique for solving mixed nonlinear and linear Fredholm-Volterra integro-differential equations. Our method involves transforming the problem using Simpson's integration formula and expanding the solution as a series of sixth-kind Chebyshev polynomials. The convergence rate and estimated chopping error are analyzed, and numerical examples are provided to demonstrate the effectiveness and accuracy of the approach."

Keywords: Volterra-Fredholm integral equation; Chebyshev Polynomials of the sixth-kind; collocation spectral method.

1 Introduction

A very important area of mathematics is fractional calculus, which extends the normal integrals and derivatives to any fractional order. Since many physical phenomena may be explained using fractional differential equations, recently it gained appeal in a wide range of fields, including biomathematics, engineering, hydrodynamics, and many more. Fractional differential equations were numerically researched in many different fields of research, including: [1]solving fractional optimum control problems with variational iteration, [2] fractional differential equations can be solved using the Taylor collocation technique, [3] solving fractional Riccati differential equations with ultraspherical wavelets, [4] a nonlinear fractional evolution equation can be worked with using a spectral element approach, [5] when solving the time-fractional modified anomalous sub-diffusion equation using the Legendre spectral element system.
It is well known that solving FDEs theoretically is challenging. Therefore, it is crucial to employ numerical techniques in order to produce efficient and precise solutions for the FDEs. such as: spectral methods, differential transform method, and finite element methods. The most crucial technique for resolving ODES, and FDEs is the use of spectral methods. This is a result of a number of blessings, including the ability of spectral approaches to provide rapid convergence rate of the solutions to differential equations and the accuracy of the results they provide, as well as their efficient and straightforward use.
The spectral method procedure is based on estimating the unknown coefficient expansion in the series to satisfy the DE and its boundary conditions after approximating the solution for a DE by particular basis sets that are often orthogonal. Galerkin, collocation, and tau procedures are three popular types for determining the unknown expansion coefficients for weighted residual techniques. Finding an appropriate orthogonal polynomial (OP) that satisfies the differential equation's beginning and boundary conditions is the first step in the Galerkin method. Next, the residual is forced to be orthogonal to the fundamental functions.
For instance: [6, 7] solving the time-fractional telegraph problem and high even-order differential equations directly using the Galerkin method. The second method, collocation, makes sure that the differential equation's residual disappear at a predetermined set of sites. It is an appropriate technique for handling non-linear equations. For instance: [8–12]researching second-order multi-point

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BVPs, solving nonlinear FDEs, and using the spectral collocation method to analyse nonlinear FDEs, solving multi-term FDEs, and handling 1-D time-fractional convection equation. The final method, tau, reduces the DE's residual and then applies the initial and the boundary conditions. It is regarded as a specific kind of spectral Petrov-Galerkin approach and is typically used to resolve DEs with challenging conditions. For instance: [13, 14] employing Jacobi polynomials to solve a class of fractional optimum control problems using the tau technique to solve a coupled system of FDEs with modified Fibonacci polynomials.

In [15], Jamei offered two half-sin/cos orthogonal Chebyshev polynomials (CPs), and he nominate them the fifth-, sixth-kinds. We report here that the authors in [16–22] offered a series of papers on the use of these two very important kinds of CPs in the numerical analysis of differential and integral equations.

Many areas of practical mathematics rely heavily on the integral equations. It can be found in many practical formulations, including the Volterra-Fredholm integral equations. Parabolic BVPs gave rise to the Fredholm-Volterra integral equations. and The Epidemic Growth From Spatiotemporal Statistical Modeling. Several techniques for solving linear and nonlinear Fredholm-Volterra integral equations have been developed by many researchers. for example: [23] solving the nonlinear Fredholm and Volterra integral equations numerically using the Bernstein's approximation approach, and [24] explored an collocation hp-version technique for handling the first class of nonlinear Volterra integral equations.

CPs are essential for mathematical analysis and its applications. CPs can either have symmetry or not. Since they are ultraspherical, the first and second kinds of CPs are symmetric, whereas the third and fourth kinds are nonsymmetric since they are not. Numerous applications use one or more of the four different types of CPs [25, 26].

The mixed Fredholm-Volterra integro-differential equation is the focus of this paper's numerical solution. [27]

$$\alpha\, D^{\nu}\, u(x) + \beta\, u(x) = \gamma\, f(x) \,+ \lambda \int_0^x \kappa_1(x, y, u(y))\, dy + \mu \int_0^1 \kappa_2(x, y, u(y))\, dy. \tag{1}$$

Here, $\alpha, \beta, \gamma, \lambda$ and μ are given constants, $0 < \nu \leq 1$, $0 \leq x \leq 1$, If $\alpha \neq 0$, we have the condition $u(0) = u_0$ s.t.: $f(x)$, κ_1 and κ_2 are smooth analytic functions, β and α are constants and $u(x)$, is a continuous well-behaved function, is the unknown solution which is needed to be determined.

2 A brief account on sixth-kind CPs (SKCP)

The main target of this part is to report some properties and formulas of SKCP which will be used in this study.

Definition 1. *[15] The SKCPs $\{Y_i(y), i = 0, 1, 2, ...\}$ are a sequence of OPs on $[-1, 1]$, that may be denoted as $Y_j(y) = \tilde{S}_j^{-5,2,-1,1}(y)$, where*

$$\tilde{S}_k^{m,n,p,q}(y) = \left(\prod_{i=0}^{\lfloor \frac{k}{2} \rfloor -1} \frac{(2\,i + (-1)^{k+1} + 2)\, q + n}{(2\,i + (-1)^{k+1} + 2\, \lfloor \frac{k}{2} \rfloor)\, p + m} \right) S_k^{m,n,p,q}(y), \tag{2}$$

and

$$S_k^{m,n,p,q}(y) = \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \left(\binom{\lfloor \frac{k}{2} \rfloor}{r} \left(\prod_{i=0}^{\lfloor \frac{k}{2} \rfloor - r - 1} \frac{(2\,i + (-1)^{k+1} + 2\, \lfloor \frac{k}{2} \rfloor)\, p + m}{(2\,i + (-1)^{k+1} + 2)\, q + n} \right) y^{k-2\,r} \right), \tag{3}$$

The SKCP satisfy the following orthogonality formula ([17]):

$$\int_{-1}^{1} y^2\, \sqrt{1 - y^2}\, Y_i(y)\, Y_j(y)\, dy = h_i\, \delta_{i,j}, \tag{4}$$

where

$$h_i = \frac{\pi}{2^{2\,i+3}} \begin{cases} 1, & \text{if } i \text{ even}, \\ \frac{i+3}{i+1}, & \text{if } i \text{ odd}, \end{cases} \tag{5}$$

and

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{6}$$

$Y_i(y)$ may be obtained with the aid of the following recursive/difference relation:

$$Y_i(y) = y\, Y_{i-1}(y) - \alpha_i\, Y_{i-2}(y) \tag{7}$$

$$,\ Y_0(y) = 1, \quad Y_1(y) = y, \quad i \geq 2,$$

$$\alpha_i = \frac{i\,(i+1) + (-1)^i\,(2\,i+1) + 1}{4\,i\,(i+1)}. \tag{8}$$

The shifted orthogonal SKCPs $Y_i^*(t)$ are defined on $[0, \tau]$ as

$$Y_i^*(y) = Y_i\left(\frac{2\,y}{\tau} - 1\right), \quad \tau > 0. \tag{9}$$

The relation of orthogonality of $Y_i^*(y)$ on $[0, \tau]$ is given by:

$$\int_0^{\tau} \omega(y)\, Y_i^*(y)\, Y_j^*(y)\; dy = h_{\tau,i}\, \delta_{i,j}, \tag{10}$$

where
$$\omega(t) = (2\,y - \tau)^2\,\sqrt{y\,\tau - y^2}, \tag{11}$$
and
$$h_{\tau,i} = \frac{\tau^4}{4}\,h_i. \tag{12}$$

Now we will offer an algorithm for handling the mixed Fredholm-Volterra integral equations.

3 Spectral collocation treatment of Fredholm-Volterra integral equations

Here, we will construct a solution strategy for the Fredholm-Volterra equation approximately via CPs of the sixth-kind.
From Eq.(1), we offer the approximate semi-analytic solution of $u(x)$ as:

$$u(x) \simeq \sum_{i=0}^{M} c_i Y_i^*(x), \tag{13}$$

where,
$$C_i = [c_0, c_1, c_2, ..., c_M]^T,$$
and
$$\Phi(x) = [Y_0^*(x), Y_1^*(x), Y_2^*(x), ..., Y_M^*(x)].$$

The residual of the Fredholm-Volterra integro-differential equation can be paraphrased as:

$$R(x) = \alpha\,(D^\nu u)(x) + \beta\,u(x) - \gamma\,f(x) - \lambda \int_0^x \kappa_1(x,y,u(y))dy - \mu \int_0^1 \kappa_2(x,y,u(y))dy. \tag{14}$$

Now, we set: $y = xz$.
Consequently,

$$R(x) = \alpha\,(D^\nu u)(x) + \beta\,u(x) - \gamma\,f(x) - \lambda x \int_0^1 \kappa_1(x,xz,u(xz))dz - \mu \int_0^1 \kappa_2(x,z,u(z))dz. \tag{15}$$

Or, in other words

$$R(x) = \alpha\,(D^\nu u)(x) + \beta u(x) - \gamma f(x) - \int_0^1 \kappa(x,z,u(z),u(xz))dz, \tag{16}$$

where:
$$\kappa = \lambda x \kappa_1 + \mu \kappa_2.$$

Now we directly apply Romberg's quadrature rule:

$$\int_0^1 g(z)dz = \Gamma_{m,n}(g) + E_{m,n}, \tag{17}$$

where:
$$r_{0,0} = \frac{g(0)+g(1)}{2},$$
$$r_{n,0} = \frac{1}{2} r_{n-1,0} + 2^{-n} \sum_{k=1}^{2^{n-1}} g\left(\frac{2k-1}{2^n}\right),$$
$$r_{n,m}(g) = r_{n,m-1} + \frac{1}{4^m - 1}(r_{n,m-1} - r_{n-1,m}),$$
$$E_{m,n} = \Phi(4^{-n(m+1)}).$$

Consequently,
$$R(x) = \alpha\,(D^\nu u_M)(x) + \beta\,u_M(x) - \gamma\,f(x) - r(k)(x). \tag{18}$$

The final approximation of the Fredholm-Volterra equation will be

$$\alpha\,(D^\nu u_M)(x_j) + \beta\,u_M(x_j) = \gamma f(x_j) + r(k)(x_j) \quad ;\ 0 \le j \le M+1. \tag{19}$$

Subject to the condition, we get $u_M(0) = u_0$.
Where:
$$u_M(x) = \sum_{i=0}^{M} c_i Y_i^*(x). \tag{20}$$

Applying typical tau method for finding the approximate solution for $u(x)$ we get,

$$\int_0^l x^\nu\,R(x)\,Y_i^*(x)\,dx = 0; i = 0,1,2,...,M-1. \tag{21}$$

Rightnow, we get a system of equations of rank $M+1$ that can be solved via Newton's method, and hence, we will obtain the approximate semi-analytic solution of $u(x)$.

Lemma 1. *[17] The inequality is correct, for $Y_j^*(y)$:*

$$|Y_j^*(y)| < \frac{j^2}{2^j}, \quad t \in [0,\tau], \quad \forall\, j > 1, \tag{22}$$

where $|Y_0^*(y)| = |Y_1^*(y)| \leqslant 1$.

Theorem 1. *[17] Let $f(t) \in L_w^2[0,\tau]$, $w = (2\,y - \tau)^2\,\sqrt{y\,\tau - y^2}$, with $|f^{(3)}(y)| \le L$, $L > 0$ and suppose it has the following infinite series expansion:*

$$f(y) = \sum_{i=0}^{\infty} a_i\,Y_i^*(y), \tag{23}$$

this series absolutely converges to $f(y)$ and the unknown coefficients in (23) holds

$$|a_i| \lesssim \frac{1}{i^3}, \quad \forall\, i > 3. \tag{24}$$

Theorem 2.*If $u(z)$ is smooth enough, and if $u_M(z) = \sum_{i=0}^{M} c_i\,Y_j^*(z)$, afterwards, the subsequent truncation error estimate is met:*

$$|u - u_M| \lesssim \frac{1}{2^M}. \tag{25}$$

4 Numerical Results and Comparisons

Here, we debug our algorithm for numerically handling some numerical examples with comparisons to [23,24,28–35]. All of the numerical evaluation were performed via Mathematica 9.

Example 1. [28] We start with the underlying integral problem:

$$x^2 \int_0^1 u(t)\,\sin(x\,t)\,dt - \sin(x) + x\,cos(x) = 0. \quad (26)$$

with exact solution $u(x) = x$.
we applied our method with n=1,we get

$$c_0 + 0.330854\,c_1 = 0.665427$$

$$c_0 + 0.323298\,c_1 = 0.661649$$

and by solving this system we get

$$c_0 = 0.5\,,\ c_1 = 0.5$$

consequently

$$u(x) = x,$$

which is the exact solution.

Example 2. [23] Now, we handle the underlying integral problem:

$$\int_0^x \frac{1}{\sqrt{x-t}}\,u(t)\,dt = f(x), \quad x \in [0,1], \quad (27)$$

where

$$f(x) = \frac{2}{105}\sqrt{x}\,(105 - 56x^2 + 48x^3)\,.$$

With exact solution $u(x) = x^3 - x^2 + 1$.
In Table (1) we compare our results and results in [28]. We discovered that the absolute errors are better than those obtained by the other methods. In Figure (1) we illustrate the absolute error when $N = 6$.

Table 1: Comparison *GFTM with [23] for Example* (2)

M	2	3	4	6
E	$8\,.10^{-14}$	$4\,.10^{-16}$	$2.5\,.10^{-13}$	$4\,.10^{-16}$
[23]	$1\,.10^{-3}$	$1.9\,.10^{-4}$	$2\,.10^{-4}$	$1\,.10^{-5}$

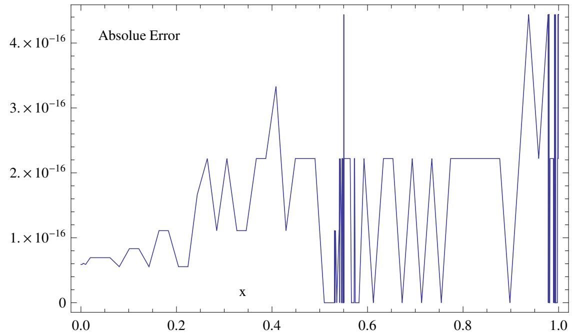

Fig. 1: Absolute Error of Example (2) for $N = 6$

Example 3. [24] Consider the underlying nonlinear integral problem:

$$\int_0^x (1+x-t)^2\,(u(t) - u^3(t))\,dt = x^2, \quad x \in [0,1]. \quad (28)$$

The exact smooth solution for this equation is hard to achieve. We do our best to try to get the exact solution via series method.
We suggest the solution $u(x)$ be a smooth function and apply numerical Taylor method at $x = 0$ as

$$u(x) = \sum_{i=0}^{\infty} a_i x^i\,. \quad (29)$$

Then put in the Eq. (32) as

$$\int_0^x (1+x-t)^2 \left(\sum_{i=0}^{\infty} a_i t^i - \left(\sum_{i=0}^{\infty} a_i t^i \right)^3 \right) dt = T[x^2\,], \quad (30)$$

and $T[x^2\,]$ is the Taylor series for x^2.
Now we obtain,

$$u(x) = \frac{2}{5} - \frac{2}{5}\,x$$

In Table (2) we list the MAE for different values of M. In Figure (2) we illustrate the absolute error when $N = 6$.

Table 2: MAE for Example 3

M	1	2	3	4	6
E	$6\,.10^{-16}$	$1\,.10^{-15}$	$2\,.10^{-15}$	$1\,.10^{-14}$	$6\,.10^{-16}$

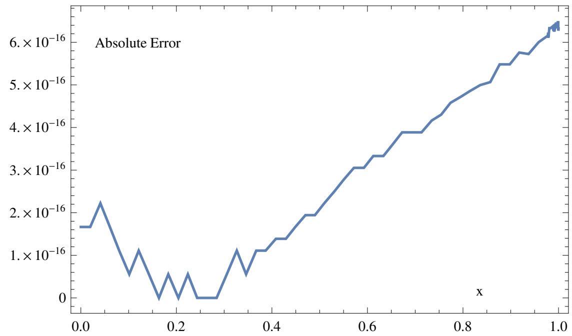

Fig. 2: Absolute Error of Example (3) for $N = 6$

Example 4. [29] We end with the following the underlying nonlinear mixed integral equation problem:

$$u(x) + 3\int_0^x \sin(t-x)\, u^2(t)\, dt - \frac{6}{7-6\cos(1)} \int_0^1 (1-t)\, cos^2(x)\, (t+u(t))\, dt = f(x), \quad (31)$$

where $f(x)$ is chosen to be compatible with the smooth solution $u(x) = \cos(x)$.
In Table (3) we compare our results with the results in [28]. We discovered that the absolute errors (AE) are better than obtained by the other methods, while, Table (3) report the (MAE) of Eq. for numerous values of M.

Table 3: Comparison *GFTM with [29] for Example* (4)

M	2	4	6	8
E	$5\,.10^{-4}$	$5\,.10^{-8}$	$2\,.10^{-9}$	$2.5\,.10^{-11}$
Results in [29]	$2\,.10^{-5}$	–	–	$1.4\,.10^{-8}$

5 Conclusions

Herein, we developed an accurate numerical scheme for handling Fredholm-Volterra integro-differential equation. The offered numerical scheme is based on using the use of the sixth-kind CPs and a suitable spectral method to transform Fredholm-Volterra equation into a system of equations that can be handled by Mathematica software. We also discussed the convergence and error analysis of the sixth-kind CPs.

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