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On a Generalized Fractional Integral and Related Methodological Remarks

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Abstract: In this paper, we define a generalized fractional integral of order α which are the natural extension of the newly defined k-fractional conformable integrals and they can be reduced to other fractional integrals. Later, the existence of such k-generalized integrals is proved. Finally, discuses some future possibilities.

Keywords: Integral operator, Generalized fractional integral, properties

1 Introduction

It is known that the fractional calculus, that is, the calculus with integral and differential operators of non-integer order, is as old as the classical calculus itself. In recent times it has had a theoretical development and its applications have increased in such a way that We have many fractional operators, applied in various fields, from comprehensive inequalities to epidemic modeling. In particular, one of the operators that has had the most development has been the Riemann-Liouville Fractional Integral, on which we will focus our work.

Throughout the work we use the functions Γ (see [1,2,3,4])and Γ_k (cf. defined by [5]):

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \quad \Re(z) > 0, \tag{1}$$

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k}\, \mathrm{d}t, k > 0. \tag{2}$$

It is clear that if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To facilitate understanding of the subject, we present several definitions of fractional integrals, some very recent (eith $0 \le a < t < b \le \infty$). The first is the classic Riemann-Liouville fractional integrals.

One of the first operators that can be called fractional is that of Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) \ge 0$, defined by (see [6]).

Definition 1. *Let $f \in L^1[a,b]; \mathbb{R}, (a,b) \in \mathbb{R}^2, a < b$. The right and life side Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined by*

$$^{RL}J^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, t > a \tag{3}$$

and

$$^{RL}J^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, t < b. \tag{4}$$

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and their corresponding differential operators are given by

$$D^{\alpha}_{a^+}f(t)=\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+}f(t)\right)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{f(t)}{(t-s)^{\alpha}}ds$$

$$D^{\alpha}_{b^-}f(t)=-\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-}f(t)\right)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b\frac{f(t)}{(s-t)^{\alpha}}ds$$

Other definitions of fractional operators are as follows.

The left-sided and right-sided Riemann-Liouville k-fractional integrals are given in [7].

Definition 2.*Let $f\in L^1[a,b]$. Then the Riemann-Liouville k-fractional integrals of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$ and $k>0$ are given by the expressions:*

$$^{\alpha}I^{k}_{a^+}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}f(t)\,dt,\quad x>a,\quad (5)$$

$$^{\alpha}I^{k}_{b^-}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f(t)\,dt,\quad x<b.\quad (6)$$

Another known fractional integral is as follows (see [8] and [9]).

Definition 3.*Let $f\in L^1[a,b];\mathbb{R},(a,b)\in\mathbb{R}^2,a<b$. The right and life side Hadamard fractional integrals of order α with $Re(\alpha)>0$ are defined by*

$$H^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(log\frac{t}{s})^{\alpha-1}\frac{f(s)}{s}ds,\quad a<t<b,\ (7)$$

and

$$H^{\alpha}_{b^-}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(log\frac{s}{t})^{\alpha-1}\frac{f(s)}{s}ds,\quad a<t<b.\ (8)$$

Hadamard differential operators are given by the following expressions.

$$({}^{H}D^{\alpha}_{a^+}f)(t)=t\frac{d}{dt}\left(H^{\alpha}_{a^+}f(t)\right)$$
$$=\frac{-\Gamma(\alpha+1)}{B(\alpha,1-\alpha)}\int_a^t(log\frac{t}{s})^{-\alpha-1}\frac{f(s)}{s}ds,\quad a<t<b$$
$$({}^{H}D^{\alpha}_{b^-}f)(t)=-t\frac{d}{dt}\left(H^{\alpha}_{b^-}f(t)\right)$$
$$=-\frac{\Gamma(\alpha+1)}{B(\alpha,1-\alpha)}\int_t^b(log\frac{s}{t})^{-\alpha-1}\frac{f(s)}{s}ds,\quad a<t<b$$

In [10], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way (also see [11]):

Definition 4.*Let $0<a<b<+\infty$, $f:[a,b]\to\mathbb{R}$ is an integrable function, and $\alpha\in(0,1)$ and $\rho>0$ two fixed real numbers. The right and life side Katugampola fractional integrals of order α are defined by*

$$K^{\alpha,\rho}_{a^+}f(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t\frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}}f(s)ds,\quad a<t\quad (9)$$

and

$$K^{\alpha,\rho}_{b^-}f(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b\frac{t^{\rho-1}}{(s^{\rho}-t^{\rho})^{1-\alpha}}f(s)ds,\quad t<b.\ (10)$$

In [12], it appeared a generalization to the Riemann-Liouville and Hadamard fractional derivatives, as a generalization of the n-integral, called the Katugampola fractional derivatives:

$$(D^{\alpha}_{a^+}f)(t)=\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}t^{1-\rho}\frac{d}{dt}\int_a^t\frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{\alpha}}f(s)ds,\quad a<t,$$

$$(D^{\alpha,\rho}_{b^-}f)(t)=\frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}t^{1-\rho}\frac{d}{dt}\int_a^t\frac{s^{\rho-1}}{(s^{\rho}-t^{\rho})^{\alpha}}f(s)ds,\quad t<b.$$

The relation between these two fractional operators is the following:

$$(D^{\alpha,\rho}_{a^+}f)(t)=t^{1-\rho}\frac{d}{dt}K^{1-\alpha,\rho}_{a^+}f(t),$$
$$(D^{\alpha,\rho}_{b^-}f)(t)=-t^{1-\rho}\frac{d}{dt}K^{1-\alpha,\rho}_{b^-}f(t).$$

In [13] presented the definition of fractional integral of f with respecto to another function g of following way (also see [9]).

Definition 5.*Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function on $(a,b]$ having a continuous derivative $g'(t)$ on (a,b). The left-sided fractional integral of a integrable function f, $f:[a,b]\to\mathbb{R}$, with respect to the function g on $[a,b]$ of order $\alpha>0$ is defined by*

$$I^{\alpha}_{g,a+}(f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{g'(s)f(s)}{[g(t)-g(s)]^{1-\alpha}}ds,\quad t>a,\ (11)$$

similarly the right lateral derivative is defined as well

$$I^{\alpha}_{g,b-}(f)(t)=\frac{1}{\Gamma(\alpha)}\int_t^b\frac{g'(s)f(s)}{[g(s)-g(t)]^{1-\alpha}}ds,\quad t<b.\quad (12)$$

A k-analogue of above definition is defined in [14] (also see [15]), under the same assumptions on function g.

Definition 6.*Consider a certain integrable function $f:[a,b]\to\mathbb{R}$.*

$$I^{\alpha,k}_{g,a+}(f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{g'(s)f(s)}{[g(t)-g(s)]^{1-\frac{\alpha}{k}}}ds,\quad t>a,\quad (13)$$

similarly the right lateral derivative is defined as well

$$I^{\alpha,k}_{g,b-}(f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{[g(s)-g(t)]^{1-\frac{\alpha}{k}}}ds, \quad t<b. \tag{14}$$

In [16] a new integral operator is presented as follows (see also [17]).

Definition 7. *Let's define a function $g:[0,+\infty)\to[0,+\infty)$ satisfying the following assumptions:*

$$\int_0^1 \frac{g(t)}{t}dt < \infty,$$
$$\frac{1}{A} \le \frac{g(s)}{g(r)} \le A, \frac{1}{2} \le \frac{s}{r} \le 2,$$
$$\frac{g(r)}{r^2} \le B\frac{g(s)}{s^2}, s \le r,$$
$$\left|\frac{g(r)}{r^2} - \frac{g(s)}{s^2}\right| \le C|r-s|\frac{g(r)}{r^2}, \frac{1}{2} \le \frac{s}{r} \le 2,$$

with A,B,C real constants independent of $r,s>0$. Therefore, the right and left lateral integrals of an integrable function $f:[a,b]\to\mathbb{R}$ are defined as

$$_{a+}I_g f(x) = \int_a^x \frac{g(x-t)}{x-t}f(t)dt, x>a \tag{15}$$

$$_{b-}I_g f(x) = \int_x^b \frac{g(t-x)}{t-x}f(t)dt, b>x. \tag{16}$$

Remark. If $g(r)r^a$ is increasing for some $a\ge 0$ and $g(r)r^b$ is decreasing for some $b\ge 0$, then g satisfies the above conditions (see [18]).

Starting with (15) - (16), and using an increasing and positive monotone function h on $[a,b]$, with continuous derivative on (a,b), Farid in [19] generalized the above definition in this way.

Definition 8. *Let two functions f,h be such that $f,h:[a,b]\to[0,+\infty)$, with $0<a<b$, f positive and integrable on $[a,b]$ and h be differentiable and increasing. Let g be a positive function satisfying $\frac{g(z)}{z}$ is increasing on $[a,+\infty)$. So, the left and right-sided Farid generalized fractional integral of a function f on $[a,b]$ may be given as follows respectively:*

$$F^{g,h}_{a+}f(x) = \int_a^x \frac{g(h(x)-h(t))}{h(x)-h(t)}g'(t)f(t)dt, x>a \tag{17}$$

$$F^{g,h}_{b-}f(x) = \int_x^b \frac{g(h(t)-h(x))}{h(t)-h(x)}g'(t)f(t)dt, b>x. \tag{18}$$

2 A new integral operator with general kernel

In [20] a generalized fractional derivative was defined in the following way (see also [21,22] and [23]).

Definition 9. *Given a function $f:[0,+\infty)\to\mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$N^\alpha_T f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon T(t,\alpha))-f(t)}{\varepsilon} \tag{19}$$

for all $t>0$, $\alpha\in(0,1)$ being $T(\alpha,t)$ is some function. Here we will use some cases of T defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a),Re(b)>0$. Also we consider $E_{a,b}(.)_k$ is the k-nth term of $E_{a,b}(.)$.

If f is $\alpha-$differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N^\alpha_T f(t)$ exists, then define $N^\alpha_T f(0) = \lim_{t\to 0^+} N^\alpha_T f(t)$, note that if f is differentiable, then $N^\alpha_T f(t) = F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative.

Now, we give the definition of a general fractional integral right and left sided. Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

Definition 10. *Let I be an interval $I\subseteq\mathbb{R}$, $a,t\in I$ and $\alpha\in\mathbb{R}$. The integral operator $J^\alpha_{T,a+}$, right and left, is defined for every locally integrable function f on I as*

$$J^\alpha_{T,a+}(f)(t) = \int_a^t \frac{f(s)}{T(t-s,\alpha)}ds, t>a. \tag{20}$$

$$J^\alpha_{T,b-}(f)(t) = \int_t^b \frac{f(s)}{T(s-t,\alpha)}ds, b>t. \tag{21}$$

Remark. It is easy to see that the case of the J^α_T operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. So, we have

1. Putting $T(t,\alpha)=t^{1-\alpha}$, $T(t,\alpha)=\Gamma(\alpha)F(t-s,\alpha)$, from (20) we have the right side Riemann-Liouville fractional integrals $(R^\alpha_{a+}f)(t)$, similarly from (21) we obtain the left derivative of Riemann-Liouville. Then its corresponding right differential operator is $({}^{RL}D^\alpha_{a+}f)(t) = \frac{d}{dt}(R^{1-\alpha}_{a+}f)(t)$, analogously we obtain the left.
2. With $T(t,\alpha)=t^{1-\alpha}$, $T(t-s,\alpha)=\Gamma(\alpha)F(lnt-lns,\alpha)t$, we obtain the right Hadamard integral from (20), the left Hadamard integral is obtained similarly from (21). The right derivative is

$$({}^H D^\alpha_{a+}f)(t) = t\frac{d}{dt}(H^{1-\alpha}_{a+}f)(t),$$

in a similar way we can obtain the left.

3.The right Katugampola integral is obtained from (20) making

$$T(t,\alpha)=t^{1-\alpha},\quad e(t)=t^{\rho},\quad T(t,\alpha) =\frac{\Gamma(\alpha)}{F(\rho,\alpha)}\frac{F(e(t)-e(s),\alpha)}{e'(s)},$$

analogously for the left fractional integral. In this case, the right derivative is

$$({}^{K}D^{\alpha,\rho}_{a^+}f)(t)=t^{1-\rho}\frac{d}{dt}K^{1-\alpha,\rho}_{a^+}f(t) =F(t,\rho)\frac{d}{dt}K^{1-\alpha,\rho}_{a^+}f(t),$$

and we can obtain the left derivative in the same way.

4.The solution of equation $(-\Delta)^{-\frac{\alpha}{2}}\phi(u)=-f(u)$ called Riesz potential, is given by the expression $\phi=C^{\alpha}_{n}\int_{R^n}\frac{f(v)}{|u-v|^{n-\alpha}}dv$, where C^{α}_{n} is a constant (see [24,25,26]). Obviously, this solution can be expressed in terms of the operator (20) very easily.

5.Obviously, we can define the lateral derivative operators (right and left) in the case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then $N^{\alpha}_{T}f(t)=T(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative. For the right derivative we have $\left(N^{\alpha}_{T,a+}f\right)(t)=N^{\alpha}_{T}\left[J^{\alpha}_{T,a+}(f)(t)\right]=\frac{d}{dx}\left[J^{\alpha}_{T,a+}(f)(t)\right]T(x,\alpha)$, similarly to the left.

6.We can define the function space $L^{p}_{\alpha}[a,b]$ as the set of functions over $[a,b]$ such that $(J^{\alpha}_{T,a+}[f(t)]^{p}(b))<+\infty$.

*Remark.*We will also use the "central" integral operator defined by (see [27] and [23])

$$J^{\alpha}_{T,a}(f)(b)=\int_a^b\frac{f(t)}{T(t,\alpha)}dt,b>a. \tag{22}$$

The following statement is analogous to the one known from the Ordinary Calculus (see [27], and [23]).

Theorem 1.*Let f be N-differentiable function in (t_0,∞) with $\alpha\in(0,1]$. Then for all $t>t_0$ we have*

a)$J^{\alpha}_{T,t_0}(N^{\alpha}_{T}f(t))=f(t)-f(t_0)$.
b)$N^{\alpha}_{T}\left(J^{\alpha}_{T,t_0}f(t)\right)=f(t)$.

An important property, and necessary, in our work is that established in the following result .

Theorem 2.*(Integration by parts) Let u and v be N-differentiable function in (t_0,∞) with $\alpha\in(0,1]$. Then for all $t>t_0$ we have*

$$J^{\alpha}_{T,t_0}((uN^{\alpha}_{T}v)(t))=[uv(t)-uv(t_0)]-J^{\alpha}_{T,t_0}((vN^{\alpha}_{T}u)(t)) \tag{23}$$

One of the current characteristics of classical Fractional Calculus is the appearance of a great variety of integral operators that can be considered successive generalizations of the Riemann-Liouville Fractional Integral knowledge. This work, which can be considered a continuation of [28], aims to provide a certain order in this multiplicity of "versions", providing a particular integral operator, which contains as a particular case, these operators defined in recent years.

3 Aditional results and methodological remarks

Although the general operator defined in the previous section is a generalization of the known fractional integral operators, we would like to give more details in two new directions.

From Definition 7, we are now in a position to define the First Generalized Riemann-Liouville integral.

Definition 11.*Let $f\in L_1[a,b]$, g an increasing and derivable function on $[a,b]$ and T a positive, decreasing and absolut continuous function. Then the k-generalized Riemann-Liouville fractional integrals of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$ and $k>0$ are given by the expressions:*

$${}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt,\quad x>a, \tag{24}$$

$${}_{b-}I^{\frac{\alpha}{k}}_{T,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)f(t)}{T\left(g(t-x),\frac{\alpha}{k}\right)}dt,\quad x<b. \tag{25}$$

*Remark.*If $k=1$, $g(u)=u$ and $T(z,\alpha)=z^{1-\alpha}$ we have the classic Riemann-Liouville of Definition 1. By other hand, if $T(z)=\frac{z}{g(z)}$ we obtain the operator integral of [17]. Similarly, other fractional integral operators reported in the literature can be obtained.

Taking into account the Definition 8 we can present the Second Generalized Riemann-Liouville integral.

Definition 12.*Let $f:[a,b]\to\mathbb{R}$ be an integrable function and T is an absolutely continuous, positive and increasing function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided k-generalized fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\alpha>0$ are defined as:*

$$I^{T,\frac{\alpha}{k}}_{g,a+}(f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(s)f(s)}{T\left(g(x)-g(s),\frac{\alpha}{k}\right)}ds,\quad x>a, \tag{26}$$

and

$$I^{T,\frac{\alpha}{k}}_{g,b-}(f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(s)f(s)}{T\left(g(x)-g(s),\frac{\alpha}{k}\right)}ds,\quad x<b. \tag{27}$$

*Remark.*The essential difference, and that distinguishes, the Definitions 12, 11 and the previous one, is the fact of the composition of the functions T and $z = g(s) - g(t)$ (z=g(s-t)), orz=g(t)-g(s)($z = g(t-s)$). Unless the g function is additive, both definitions give us different integral operators.

*Remark.*There is little problem in including other integral operators in the above definitions. For example, in [29] the authors define a generalized integral operator, with a non-singular kernel that can also be included, without difficulty, in Definition 11.

The following is an essential property to talk about the correction of the operators defined above.

Theorem 3.*Let $f,T \in L_1[a,b]$ positive functions, g an increasing and derivable function on $[a,b]$ and T a decreasing and absolut continuous function. Then, for $x \in [a,b]$, we have*

$$\begin{aligned} &\left|{}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_a^x \frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[a,x]}. \end{aligned} \tag{28}$$

Similarly

$$\begin{aligned} &\left|{}_{b-}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_x^b \frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[x,b]}. \end{aligned} \tag{29}$$

So

$$\begin{aligned} &\left|{}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x) + {}_{b-}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &\qquad\leq \frac{2}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[a,b]}. \end{aligned} \tag{30}$$

*Proof.*Using the properties of functions g and T, we have $\frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)} \leq \frac{g'(t)f(t)}{T\left(g(x-a),\frac{\alpha}{k}\right)}$ for $t \in [a,x]$ and $x \in [a,b]$. From this and Definition 11 we obtain

$$\begin{aligned} &\left|{}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_a^x \frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[a,x]}. \end{aligned} \tag{31}$$

Analogously, we have

$$\begin{aligned} &\left|{}_{b-}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_x^b \frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[x,b]}. \end{aligned} \tag{32}$$

From $\left|{}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x) + {}_{b-}I^{\frac{\alpha}{k}}_{T,g}f(x)\right|$, using the triangular inequality, equations (31) and (32), we have the general bound of (35).

This completes the proof.

*Remark.*The following theorem, can be obtained without difficulty, and is a similar result for the integral operators defined in the Definition 12.

Theorem 4.*Let $f,T \in L_1[a,b]$ positive functions, g an increasing and derivable function on $[a,b]$ and T a decreasing and absolut continuous function. Then, for $x \in [a,b]$, we have*

$$\begin{aligned} &\left|{}_{a+}I^{\frac{\alpha}{k}}_{T,g}f(x)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_a^x \frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[a,x]}. \end{aligned} \tag{33}$$

Similarly

$$\left|{}_{b-}I_{T,g}^{\frac{\alpha}{k}}f(x)\right|$$
$$=\frac{1}{\Gamma(\alpha)}\left|\int_x^b\frac{g'(t)f(t)}{T\left(g(x-t),\frac{\alpha}{k}\right)}dt\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[x,b]}. \tag{34}$$

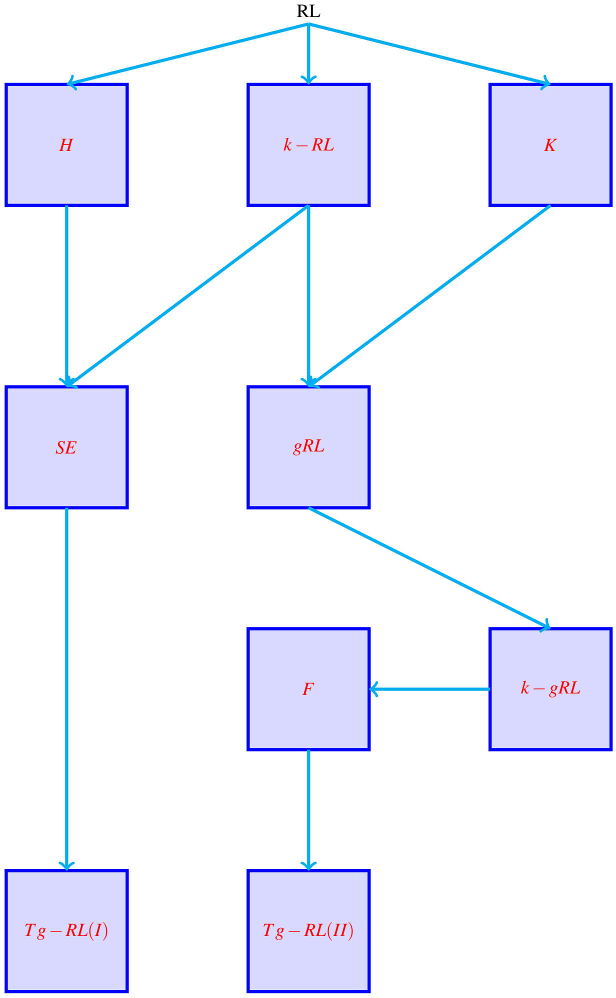
So

$$\left|{}_{a+}I_{T,g}^{\frac{\alpha}{k}}f(x)+{}_{b-}I_{T,g}^{\frac{\alpha}{k}}f(x)\right|\leq\frac{2}{\Gamma(\alpha)}\frac{g(b)-g(a)}{T\left(g(b-a),\frac{\alpha}{k}\right)}\|f\|_{[a,b]}. \tag{35}$$

Remark. If T is additive, then the semigroup law is satisfied as can be easily verified, for the particular cases presented in the Definitions (11 and 12). Due to the generality of the T function, of course, there may be cases of integral operators that do not satisfy this property.

A more complete idea of the place that the previous definitions occupy can be seen in the following scheme, in which we have symbolized the presented integral operators, as follows:

- RL, Riemann-Liouville classic (Definition 1)
- k-RL, k-Riemann-Liouville integral (Definition 2)
- H, Hadamard integral (Definition 3)
- K, Latugampola integral (Definition 4)
- SE, Sarikaya-Ertugral integral (Definition 7)
- gRL, Integral with respect another function (Definition 5)
- k-gRL, k-Integral with respect another function (Definition 6)
- F, Farid integral (Definition 8)
- Tg-RL(I), First Generalized Riemann-Liouville integral (Definition 11)
- Tg-RL(II), Second Generalized Riemann-Liouville integral (Definition 12)



4 Conclusion

Integral inequalities is an area that is gaining more and more followers every day, it is clear then, taking into account the previous diagram, that results obtained within the framework of some of these operators can be generalized using the general formulation of Definitions 11 and 12.

Finally, we would like to draw readers' attention to the following question.

Before making a more general observation, let's return to the operator of the equation (22). In [20] we present said integral operator (independently of [23]) and its study was formalized in [27]. Now we will present a generalized derivative as follows.

Definition 13. *Given a function $f:[0,+\infty)\to\mathbb{R}$. Then the generalized derivative of f of order α is defined by*

$$D_T^{\alpha}f(t)=D_T^{\alpha}\left[J_{T,a}^{\alpha}(f)\right](t)=\frac{d}{dt}\left[\int_a^t\frac{f(s)}{T(s,\alpha)}ds\right],\quad t>a. \tag{36}$$

for all $t > 0$, $\alpha \in (0,1)$ *being* $T(\alpha,t)$ *is the kernel function. If* f *is* $\alpha-$*differentiable in some* $(0,\alpha)$, *and* $\lim_{t\to 0^+} D_T^\alpha f(t)$ *exists, then define* $D_T^\alpha f(0) = \lim_{t\to 0^+} D_T^\alpha f(t)$.

Therefore, the question naturally arises: What relationship exists between the derivative defined by (36) and the derivative (19)? This is not a minor issue, the first is a fractional derivative of the Riemann-Liouville type and the second a local derivative, that is, non-fractional.

Could it be that in the end, the differential operators will admit a single representation?

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