Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110105

Hybrid Fractional Differential Equations with Impulses and Boundary Conditions Involving Integrals and Derivatives

Mohamed Hannabou[1,*], *Mohamed Bouaouid*[2] *and Khalid Hilal*[2]

[1] Department of Mathematics and Computer Sciences, Multidisciplinary Faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco
[2] Departement of Mathematics, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

Received: 4 Dec. 2023, Revised: 16 Jul. 2024, Accepted: 26 Jul. 2024
Published online: 1 Jan. 2025

Abstract: This research paper focuses on investigating the existence and uniqueness of solutions in fractional boundary value problems related to implicit impulsive fractional differential equations, which encompass both fractional derivatives and integrals. The application of the fixed point theorem is pivotal in exploring the primary outcomes, while the uniqueness of solutions is further explored using the Banach contraction mapping principle. Concrete examples are presented to elucidate and demonstrate the obtained results.

Keywords: Fractional differential equations, impulsive conditions, fractional boundary conditions, existence, fixed point theorems.

1 Introduction

The foundations of fractional calculus and fractional differential equations have emerged as essential models in numerous fields, including engineering, chemistry, physics, economics, signal processing, cancer treatment, mechanics, aerodynamics, complex media electrodynamics, and mathematical biology. For a comprehensive understanding of this theory and its practical applications, interested readers can refer to the following references [1,2,3,4]. These sources delve into the intricacies of the theory and provide in-depth insights into its wide-ranging applications.

Impulsive differential equations have laid the groundwork for understanding the microscopic realm of biology, prompting a reevaluation of nature. They hold significant relevance in various applications in bioinformatics and find practical utility in biotechnologies. For more detailed information on this subject and its applications, the following references can be consulted [5,7]. These sources delve into the significance of impulsive differential equations in advancing our understanding of biological systems and highlight their practical implications in the fields of bioinformatics and biotechnology.

In recent years, fractional differential equations have seen significant progress. Key developments in this area are documented in the monographs cited in the references. For instance, M. Benchohra and J.E. Lazreg have focused on implicit fractional differential equations, as noted in [10]. Moreover, K.D. Kucche, J.J. Nieto, and V. Venktesh have delved into the nonlinear aspects of these equations, with particular attention to their continuous dependence properties, as discussed in [13]. These works offer a thorough understanding of the advancements in fractional differential equations, especially regarding implicit formulations.

The investigation of hybrid fractional differential equations has attracted the attention of various researchers. These equations encompass a combination of a fractional derivative of an unknown function with a nonlinearity that depends on it. Recent advancements in the field of hybrid differential equations can be found in a series of papers referenced as [8]-[11].

This paper focuses on the study of fractional boundary value problems for implicit impulsive hybrid fractional differential

* Corresponding author e-mail: hnnabou@gmail.com

equations of the form:

$$ {}^{c}D^{\nu}\Big(\frac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))}\Big)=\hat{\xi}(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa})),\quad \hat{\varkappa}\in\mathscr{J}:=[0,\mathbf{T}],\quad 1<\nu\leq 2 \tag{1}$$

$$\hat{\vartheta}(\hat{\varkappa}_k^{+})=\hat{\vartheta}(\hat{\varkappa}_k^{-})+y_k,\ k=1,2,...,m\quad y_k\in\mathbb{R},\quad \frac{\hat{\vartheta}(0)}{\varpi(0,\hat{\vartheta}(0))}=0, \tag{2}$$

subject to the boundary conditions

$$\lambda D^{\tau_1}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)+(1-\lambda)D^{\tau_2}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)=\tau_3, \tag{3}$$

$$\mu I^{\sigma_1}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)+(1-\mu)I^{\sigma_2}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)=\sigma_3, \tag{4}$$

$$\mu D^{\tau_1}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)+(1-\mu)I^{\sigma_2}\Big(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\Big)=\tau_3. \tag{5}$$

where D^{ϕ} is the Caputo fractional derivative of order $\phi\in\{\nu,\tau_1,\tau_2\}$ such that $1<\nu\leq 2, 0<\tau_1,\tau_2<\nu, \tau_3,\sigma_3\in\mathbb{R}$, I^{χ} is the Riemann-Liouville fractional integral of order $\chi\in\{\sigma_1,\sigma_2\}$, $0\leq\lambda,\mu\leq 1$ is given constant and $\varpi:\mathscr{J}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $\hat{\xi}:\mathscr{J}\times\mathbb{R}\to\mathbb{R}$ is given continuous functions.
By a solution of the peoblem (1)-(2)-(3) we mean a function $\hat{\vartheta}\in\mathscr{C}(\mathscr{J},\mathbb{R})$ such that
(i) the function $\hat{\varkappa}\mapsto\frac{\hat{\vartheta}}{\varpi(\hat{\varkappa},\hat{\vartheta})}$ is continuous for each $\vartheta\in\mathbb{R}$, and
(ii) ϑ satisfies the equations in (1)-(2)-(3).

The fractional boundary value problem (1)-(2)-(3) is equivalent to the integral equation:

$$\hat{\vartheta}(\hat{\varkappa})=\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))\Big[\sum_{i=1}^{m}y_i+I^{\nu}g(\hat{\varkappa})+\frac{\hat{\varkappa}}{\Lambda_1}\left[\tau_3-I^{\nu-\tau_1}g_1(\hat{\varkappa})-I^{\nu-\tau_2}g_2(\hat{\varkappa})\right]\Big],\text{for}\quad \hat{\varkappa}\in(\hat{\varkappa}_m,\mathbf{T}], \tag{6}$$

where the non zero constant Λ_1 is defined by

$$\Lambda_1=\frac{\lambda T^{1-\tau_1}}{\Gamma(2-\tau_1)}+\frac{(1-\lambda)T^{1-\tau_2}}{\Gamma(2-\tau_2)}. \tag{7}$$

and $g,g_1,g_2\in\mathscr{C}(\mathscr{J},\mathbb{R})$ satisfies the functional equation where

$$\begin{aligned} I^{\nu}g(\hat{\varkappa})&=\frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\hat{\xi}(s,\hat{\vartheta}(s))ds,\\ I^{\nu-\tau_1}g_1(\hat{\varkappa})&=\frac{\lambda}{\Gamma(\nu-\tau_1)}\int_0^{T}(T-s)^{\nu-\tau_1+1}\hat{\xi}(s,\hat{\vartheta}(s))ds,\\ I^{\nu-\tau_2}g_2(\hat{\varkappa})&=\frac{(1-\lambda)}{\Gamma(\nu-\tau_2)}\int_0^{T}(T-s)^{\nu-\tau_2+1}\hat{\xi}(s,\hat{\vartheta}(s))ds.\end{aligned}$$

The remaining sections of this paper are structured as follows. Section 2 provides a concise overview of the fundamental tools associated with fractional calculus. In Section 3, we present the main result of our study. The conclusions are drawn in Section 4. Finally, Section 5 is dedicated to a concrete application.

2 Preliminary Results

In this section, we present the notations, definitions, and preliminary results that will be used throughout the paper
Throughout this paper denotes $\mathscr{J}_0=[0,\hat{\varkappa}_1]$, $\mathscr{J}_1=(\hat{\varkappa}_1,\hat{\varkappa}_2]$, ..., $\mathscr{J}_{n-1}=(\hat{\varkappa}_{n-1},\hat{\varkappa}_n]$, $\mathscr{J}_n=(\hat{\varkappa}_n,1]$, and we introduce the spaces: For $\hat{\varkappa}_i\in(0,1)$ such that $\hat{\varkappa}_1<\hat{\varkappa}_2<\ldots<\hat{\varkappa}_n$, and $I'=I\setminus\{\hat{\varkappa}_1,\hat{\varkappa}_2,...,\hat{\varkappa}_n\}$ and denote by
$\mathscr{C}=\{\vartheta:[0,1]\longrightarrow\mathbb{R}:\vartheta\in C(I')$ and left $\vartheta(\hat{\varkappa}_i^{+})$ and right limit $\vartheta(\hat{\varkappa}_i^{-}))$ exist and $u(\hat{\varkappa}_i^{-})=u(\hat{\varkappa}_i),1\leq i\leq n\}$.
the Banach space with the norm $||\hat{\vartheta}||=\sup_{\hat{\varkappa}\in\mathscr{J}}\{|\hat{\vartheta}(\hat{\varkappa})|,\hat{\vartheta}\in\mathscr{C}(\mathscr{J},\mathbb{R})\}$.

Definition 1. *The derivative of fractional order $\nu > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by*

$$D^{\nu}_{0+}y(\hat{\varkappa}) = \frac{1}{\Gamma(n-\nu)}\left(\frac{d}{dt}\right)^n \int_0^{\hat{\varkappa}} \frac{y(s)}{(\hat{\varkappa}-s)^{\nu-n+1}}ds,$$

where $n = [\nu]+1$, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2. *The fractional order integral of the function $h \in L^1([0,T],\mathbb{R}_+)$ of order $\nu \in \mathbb{R}_+$ is defined by*

$$I^{\nu}h(\hat{\varkappa}) = \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\nu-1}h(s)ds,$$

where Γ is the Euler's gamma function defined by $\Gamma(\nu) = \int_0^{\infty} \hat{\varkappa}^{\nu-1}e^{-\hat{\varkappa}}d\hat{\varkappa}, \nu > 0$.

Definition 3. *For a function $h \in AC^n(\mathscr{J})$, the Caputo's fractional-order derivative of order ν is defined by*

$$^{c}D^{\nu}_{0}h(\hat{\varkappa}) = \frac{1}{\Gamma(n-\nu)}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{n-\nu-1}h^{(n)}(s)ds,$$

where $n = [\nu]+1$ and $[\nu]$ denotes the integer part of the real number ν.

Lemma 1. *For $\nu > 0$, the general solution of the FDE's $^{c}D^{\nu}\hat{\vartheta}(\hat{\varkappa}) = 0$ is given by*

$$\hat{\vartheta}(\hat{\varkappa}) = c_0 + c_1\hat{\varkappa} + \ldots + c_{n-1}\hat{\varkappa}^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [\nu]+1$).

In view of Lemma 1, it follows that

$$I^{\nu}\, {}^{c}D^{\nu}\hat{\vartheta}(\hat{\varkappa}) = \hat{\vartheta}(\hat{\varkappa}) + c_0 + c_1\hat{\varkappa} + \ldots + c_{n-1}\hat{\varkappa}^{n-1}, \tag{8}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [\nu]+1$).

Lemma 2. *The boundary value problem*

$$\begin{gathered} D^{\nu}\left(\frac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))}\right) = \omega(\hat{\varkappa}), \quad \hat{\varkappa} \in (0,\mathbf{T}), \\ \frac{\hat{\vartheta}(0)}{\varpi(0,\hat{\vartheta}(0))} = 0, \quad \lambda D^{\tau_1}\left(\frac{\hat{\vartheta}(T)}{\varpi(\mathbf{T},\hat{\vartheta}(\mathbf{T}))}\right) + (1-\lambda)D^{\tau_2}\left(\frac{\hat{\vartheta}(\mathbf{T})}{\varpi(T,\hat{\vartheta}(\mathbf{T}))}\right) = \tau_3, \end{gathered} \tag{9}$$

is equivalent to the integral equation

$$\begin{aligned} \hat{\vartheta}(\hat{\varkappa}) = \varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))\Big[&\frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\nu-1}\omega(s)ds \\ &+ \frac{\hat{\varkappa}}{\Lambda_1}\Big(\tau_3 - \frac{\lambda}{\Gamma(\nu-\tau_1)}\int_0^T (T-s)^{\nu-\tau_1+1}\omega(s)ds \\ &- \frac{1-\lambda}{\Gamma(\nu-\tau_2)}\int_0^T (T-s)^{\nu-\tau_2+1}\omega(s)ds\Big)\Big], \quad \hat{\varkappa} \in \mathscr{J} := [0,\mathbf{T}], \end{aligned} \tag{10}$$

where the non zero constant Λ_1 is defined by

$$\Lambda_1 = \frac{\lambda T^{1-\tau_1}}{\Gamma(2-\tau_1)} + \frac{(1-\lambda)T^{1-\tau_2}}{\Gamma(2-\tau_2)}. \tag{11}$$

Proof:

From the first equation of (9), we have

$$D^{\nu}\Big(\frac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))}\Big) = \omega(\hat{\varkappa}), \quad \hat{\varkappa} \in \mathscr{J}. \tag{12}$$

we obtain $\dfrac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))} = \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\omega(s)ds + C_1 + C_2\hat{\varkappa}$, for $C_1, C_2 \in \mathbb{R}$.
The first boundary condition of (9) implies that $C_1 = 0$. Hence

$$\frac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))} = \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\omega(s)ds + C_2\hat{\varkappa}. \tag{13}$$

Applying the Caputo fractional derivative of order $\psi \in \{\tau_1, \tau_2\}$ such that $0 < \psi < \nu - \tau$ to (13), we have

$$D^{\psi}\Big(\frac{\hat{\vartheta}(\hat{\varkappa})}{\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))}\Big) = \frac{1}{\Gamma(\nu-\psi)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-\psi+1}\omega(s)ds + C_2\frac{1}{\Gamma(2-\psi)}\hat{\varkappa}^{1-\psi}.$$

Substituting the values $\psi = \tau_1$ and $\psi = \tau_2$ to the above relation and using the second condition of (9), we obtain

$$\begin{aligned}\tau_3 = {} & \frac{\lambda}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}\omega(s)ds + \frac{\lambda\mathbf{T}^{1-\tau_1}}{\Gamma(2-\tau_1)}C_2 \\ & + \frac{1-\lambda}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\omega(s)ds + \frac{(1-\lambda)\mathbf{T}^{1-\tau_2}}{\Gamma(2-\tau_2)}C_2,\end{aligned}$$

which leads to

$$\begin{aligned}C_2 = {} & \frac{1}{\Lambda_1}\Big[\tau_3 - \frac{\lambda}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}\omega(s)ds \\ & - \frac{1-\lambda}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\omega(s)ds\Big].\end{aligned}$$

Substituting the value of the constant C_2 in (13), we deduce the integral equation (10). The converse follows by direct computation. This completes the proof.

Theorem 1.[9] *Let $\hat{\mathscr{S}}$ be a nonempty, closed convex and bounded subset of a Banach algebra $\hat{\mathscr{X}}$ and let $\hat{\mathscr{A}}, \hat{\mathscr{C}} : \hat{\mathscr{X}} \longrightarrow \hat{\mathscr{X}}$ and $\hat{\mathscr{B}} : \hat{\mathscr{S}} \longrightarrow \hat{\mathscr{X}}$ be three operators satisfying:*

(a_1) *$\hat{\mathscr{A}}$ is Lipschitzian with Lipschitz constants δ,*
(b_1) *$\hat{\mathscr{B}}$ is compact and continuous,*
(c_1) *$\mathbf{x} = \hat{\mathscr{A}}\mathbf{x}\hat{\mathscr{B}}\mathbf{y} \Longrightarrow \mathbf{x} \in \hat{\mathscr{S}}$ for all $\mathbf{y} \in, \hat{\mathscr{S}}$*
(d_1) *$\delta\hat{\mathscr{M}} + \rho < 1$, where $\hat{\mathscr{M}} = \|\hat{\mathscr{B}}(\hat{\mathscr{S}})\|$.*

Then the operator equation $\mathbf{x} = \hat{\mathscr{A}}\mathbf{x}\hat{\mathscr{B}}\mathbf{y}$ has a solution.

3 Main Results

In the following sections of this paper, we will base our analysis on the following assumptions:
(A_1) The function $\varpi : \mathscr{J} = [0,\mathbf{T}] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a continuous function.
(A_2) There exists constants $q_0, q_1 > 0$ such that

$$\begin{aligned}|\varpi(\hat{\varkappa},\hat{u}) - \varpi(\hat{\varkappa},\hat{v})| &\le q_0|\hat{u}-\hat{v}|, \quad \text{for any} \quad \hat{u},\hat{v} \in \mathbb{R}, \hat{\varkappa} \in \mathscr{J}, \\ |\hat{\xi}(\hat{\varkappa},\hat{u}) - \hat{\xi}(\hat{\varkappa},\hat{v})| &\le q_1|\hat{u}-\hat{v}|, \quad \text{for any} \quad \hat{u},\hat{v} \in \mathbb{R}, \hat{\varkappa} \in \mathscr{J}.\end{aligned}$$

(A_3) There exists a continuous nondecreasing functions m, φ on $[0,\infty) \to (0,\infty)$ such that

$$|\varpi(\hat{\varkappa},u)| \le m(\hat{\varkappa}),$$

$$|\xi(\hat{\varkappa},u)| \leq \varphi(\hat{\varkappa}).$$

(A_4) There exist a function $\mathrm{p}(\hat{\varkappa}) \in \mathscr{C}^1([0,\mathbf{T}],\mathbb{R}^+)$ such that

$$|\varpi(\hat{\varkappa},0)| < \mathrm{p}(\hat{\varkappa}), \quad \hat{\varkappa} \in [0,\mathbf{T}].$$

Theorem 2. *Assume that $(A_1)-(A_3)$ are holds. If*

$$q_0\Big[q_1\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} - \frac{T^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \frac{T^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big) + \sum_{i=1}^{m}|y_i| + \frac{T^{\nu+1}}{\Lambda_1}|\tau_3|\Big] < 1,$$

then there exists a on solution for (1) − (2) − (3) on $\mathscr{J}$.

Proof:

We defined a subset $\hat{\mathscr{S}}$ of $\hat{\mathscr{C}}$ by be a closed bounded and convex subset of $\hat{\mathscr{S}}$ of $\hat{\mathscr{X}}$ where r is a fixed constant by:

$\hat{\mathscr{S}} = \{\vartheta \in \hat{\mathscr{C}} / \|\vartheta\| \leq r\}$,

where,

$$r \geq \frac{M_0\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)} - \frac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \frac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big] + \sum_{i=1}^{m}|y_i| + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big]}{1 - q_1\Big[\frac{T^{\nu}}{\Gamma(\nu+1)} - \frac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \frac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big] + \sum_{i=1}^{m}|y_i| + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big]},$$

and therefor $\hat{\mathscr{S}}$ satisfies hypothesis of theorem 1 . Define three operators $\hat{\mathscr{A}} : \hat{\mathscr{C}} \longrightarrow \hat{\mathscr{C}}$, $\hat{\mathscr{B}} : \hat{\mathscr{S}} \longrightarrow \hat{\mathscr{C}}$ by:

$$\hat{\mathscr{A}}\hat{\vartheta}(\hat{\varkappa}) = \varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa})), \quad \hat{\varkappa} \in \mathscr{J} \tag{14}$$

and

$$\hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}) = I^{\nu}\hat{\xi}(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa})) + \sum_{i=1}^{m} y_i + \frac{\hat{\varkappa}}{\Lambda_1}\Big[\gamma_3 - I^{\nu-\tau_1}\hat{\xi}(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa})) - I^{\nu-\tau_2}\hat{\xi}(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big], \text{for} \quad \hat{\varkappa} \in (\hat{\varkappa}_m,\mathbf{T}] \tag{15}$$

Now with this data we can transform the hybrid integral equation 10 into the operator equation as

$$\hat{\vartheta}(\hat{\varkappa}) = \mathscr{A}\hat{\vartheta}(\hat{\varkappa})\mathscr{B}\hat{\vartheta}(\hat{\varkappa}) \quad , \quad \hat{\varkappa} \in [0,\mathbf{T}] \tag{16}$$

Clearly, the fixed points of operator 16 is solution of problem (1) − (2) − (3).
First let us show that the operators $\hat{\mathscr{A}}, \hat{\mathscr{B}}$ satisfy all the conditions of Theorem 1.
Claim 1: let $\hat{\vartheta}, \varepsilon \in \hat{\mathscr{C}}$ then by hypothesis (H_1),

$$|\hat{\mathscr{A}}\hat{\vartheta}(\hat{\varkappa}) - \hat{\mathscr{A}}\varepsilon(\hat{\varkappa})| = |\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa})) - \varpi(\hat{\varkappa},\varepsilon(\hat{\varkappa}))| \leq q_0|\hat{\vartheta}(\hat{\varkappa}) - \varepsilon(\hat{\varkappa})| \leq q_0\|\hat{\vartheta} - \varepsilon\|,$$

for all $\hat{\varkappa} \in \mathscr{J}$.
Taking supremum over $\hat{\varkappa}$, we obtain

$$\|\hat{\mathscr{A}}\hat{\vartheta} - \hat{\mathscr{A}}\varepsilon\| \leq q_0\|\hat{\vartheta} - \varepsilon\| \quad \textit{for all} \quad \hat{\vartheta}, \varepsilon \in \hat{\mathscr{C}}.$$

Claim 2: second let us show that $\hat{\mathscr{B}}$ is continuous in $\hat{\mathscr{S}}$.
Let $(\hat{\vartheta}_n)_n$ be a sequence in $\hat{\mathscr{S}}$ converging to a point $\hat{\vartheta} \in \hat{\mathscr{S}}$. Then by Lebesgue dominated convergence theorem, Then

$$\begin{aligned}\lim_{n\to\infty}\hat{\mathscr{B}}\hat{\vartheta}_n(\hat{\varkappa}) &= \lim_{n\to\infty}\Big[\sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\\ &+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\Big]\\ &-\tfrac{|(1-\lambda)|}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\Big)\\ &= \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\lim_{n\to\infty}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\\ &+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}\lim_{n\to\infty}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\\ &-\tfrac{|(1-\lambda)|}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\lim_{n\to\infty}\hat{\xi}(s,\hat{\vartheta}_n(s))ds\Big)\\ &= \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\hat{\xi}(s,\hat{\vartheta}(s))ds\\ &+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\tau-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}\hat{\xi}(s,\hat{\vartheta}(s))ds\\ &-\tfrac{|(1-\lambda)|}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\hat{\xi}(s,\hat{\vartheta}(s))ds\Big)\\ &= \hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}),\end{aligned}$$

for all $\hat{\varkappa}\in[0,\mathbf{T}]$. Therefore the $\hat{\mathscr{B}}$ is a continuous operator on $\hat{\mathscr{S}}$.

Claim 3: To show that the $\hat{\mathscr{B}}$ is compact operator on $\hat{\mathscr{S}}$.

Let $\hat{\vartheta}\in\hat{\mathscr{S}}$, for all $\hat{\varkappa}\in\mathscr{J}$ define

$$\sup_{0\le\hat{\varkappa}\le 1}|\hat{\xi}(\hat{\varkappa},0)| = M_0,$$

where

$$r = |m|\Big[|\varphi|\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)} - \tfrac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \tfrac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big] + \sum_{i=1}^{m}|y_i| + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big],$$

First, we show that $\hat{\mathscr{B}}(\hat{\mathscr{S}})$ is a uniformly bounded set in $\hat{\mathscr{S}}$.

For $\hat{\vartheta}\in\hat{\mathscr{S}}$, $\hat{\varkappa}\in[0,\mathbf{T}]$, we have:

$$\begin{aligned}|\hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa})| &\le \sum_{i=1}^{m}|y_i| + \frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}|\hat{\xi}(s,\hat{\vartheta}(s)) - \hat{\xi}(s,0) + \hat{\xi}(s,0)|ds\\ &+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\natural-\tau_1+1}|\xi(s,\vartheta(s)) - \hat{\xi}(s,0) + \hat{\xi}(s,0)|ds\\ &-\tfrac{|(1-\lambda)|}{\Gamma(\nu-\tau_2)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}|\xi(s,\vartheta(s)) - \hat{\xi}(s,0) + \hat{\xi}(s,0)|ds\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\|\hat{\mathscr{B}}\hat{\vartheta}\| &\le \sum_{i=1}^{m}|y_i| + \frac{T^{\nu}}{\Gamma(\nu+1)}(q_1\|\hat{\vartheta}\| + M_0) + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}\Big(|\tau_3| - \frac{|\lambda|}{\Gamma(\nu+\tau_1+1)}(q_1\|\hat{\vartheta}\| + M_0)\\ &+\frac{|1-\lambda|}{\Gamma(\nu+\tau_2+1)}(q_1\|\hat{\vartheta}\| + M_0)\Big)\\ &\le (q_1\|\hat{\vartheta}\| + M_0)\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)} - \tfrac{T^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \tfrac{T^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big] + \sum_{i=1}^{m}|y_i| + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|.\end{aligned}$$

Thus

$$\|\hat{\mathscr{B}}\hat{\vartheta}\| \le (q_1\|\hat{\vartheta}\| + M_0)\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)} - \tfrac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \tfrac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big] + \sum_{i=1}^{m}|y_i| + \frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big].$$

for all $\hat{\vartheta}\in\hat{\mathscr{S}}$.

This shows that $\hat{\mathscr{B}}$ is uniformly bounded on $\hat{\mathscr{S}}$.

Next, we show that $\hat{\mathscr{B}}(\hat{\mathscr{S}})$ is an equi-continuous set on $\hat{\mathscr{S}}$.

Let $\hat{\varkappa}_1,\hat{\varkappa}_2\in\mathscr{J}$, then for any $\hat{\vartheta}\in\hat{\mathscr{S}}$

$$\begin{aligned}|\hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}_2) - \hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}_1)| &\le \sum_{i=1}^{m}|y_i| + \frac{1}{\Gamma(\nu)}\Big(\int_0^{\hat{\varkappa}_2}(\hat{\varkappa}_2-s)^{\nu-1}|\hat{\xi}(s,\hat{\vartheta}(s))| - \int_0^{\hat{\varkappa}_1}(\hat{\varkappa}_1-s)^{\nu-1}|\hat{\xi}(s,\hat{\vartheta}(s))|ds\Big)\\ &+\tfrac{(\hat{\varkappa}_2-\hat{\varkappa}_1)}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}|\hat{\xi}(s,\hat{\vartheta}(s)|ds\\ &\le \sum_{i=1}^{m}|y_i| + \frac{1}{\Gamma(\nu)}\int_{\hat{\varkappa}_1}^{\hat{\varkappa}_2}|\hat{\xi}(s,\hat{\vartheta}(s))|ds\\ &+\tfrac{(\hat{\varkappa}_2-\hat{\varkappa}_1)}{\Lambda_1}\Big(|\tau_3| - \tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}|\hat{\xi}(s,\hat{\vartheta}(s)|ds.\end{aligned}$$

Hence

$$\forall\varepsilon>0,\quad \exists\eta>0\quad : \quad |\hat{\varkappa}_1-\hat{\varkappa}_2|<\eta\Longrightarrow|\hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}_1)-\hat{\mathscr{B}}\hat{\vartheta}(\hat{\varkappa}_2)|<\varepsilon$$

for all $\hat{\varkappa}_1,\hat{\varkappa}_2\in\mathscr{J}$ and for all $\hat{\vartheta}\in\hat{\mathscr{C}}$.
Therefore the operator $\hat{\mathscr{B}}(\hat{\mathscr{S}})$ is equicontinuous set in $\hat{\mathscr{C}}$.
Then by Arzelá-Ascoli theorem, $\hat{\mathscr{B}}$ is a continuous and compact operator on $\hat{\mathscr{S}}$.
Claim 4: Next we prove that (c) of theorem 1 .
Let $\hat{\vartheta}\in\mathscr{C}$ and $\varepsilon\in\hat{\mathscr{S}}$ be arbitrary such that $\hat{\vartheta}=\hat{\mathscr{A}}\hat{\vartheta}\hat{\mathscr{B}}\varepsilon$.
Then,

$$\begin{aligned}
|\hat{\vartheta}(\hat{\varkappa})| &= |\hat{\mathscr{A}}\hat{\vartheta}(\hat{\varkappa})||\hat{\mathscr{B}}\varepsilon(\hat{\varkappa})|\\
&\le|\varpi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))|\Big[\sum_{i=1}^{m}|y_i|+\frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}|\hat{\xi}(s,\varepsilon(s))|ds\\
&\quad+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3|-\tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\textstyle\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}|\hat{\xi}(s,\varepsilon(s))|ds\\
&\quad-\tfrac{|(1-\lambda)|}{\Gamma(\hat{\vartheta}-\tau_2)}\textstyle\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}|\hat{\xi}(s,\hat{\vartheta}(s))|ds\Big)\Big]\\
&\le|m(\hat{\varkappa})|\Big[\sum_{i=1}^{m}|y_i|+\frac{1}{\Gamma(\nu)}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\nu-1}\phi(\hat{\varkappa})ds\\
&\quad+\tfrac{\hat{\varkappa}}{\Lambda_1}\Big(|\tau_3|-\tfrac{|\lambda|}{\Gamma(\nu-\tau_1)}\textstyle\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_1+1}|\phi(\hat{\varkappa})ds\\
&\quad-\tfrac{|(1-\lambda)|}{\Gamma(\vartheta-\tau_2)}\textstyle\int_0^{\mathbf{T}}(\mathbf{T}-s)^{\nu-\tau_2+1}\phi(\hat{\varkappa})ds\Big)\Big]\\
&\le|m|\Big[|\phi|\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)}-\tfrac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)}+\tfrac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big]+\sum_{i=1}^{m}|y_i|+\frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big].
\end{aligned}$$

Further, we obtain

$$\|\hat{\vartheta}\|\le r.$$

Then $\hat{\vartheta}\in\hat{\mathscr{S}}$ and hence the hypothesis (c_1) of theorem 1 is satisfied.
Finally, we have

$$M=\|\hat{\mathscr{B}}(\hat{\mathscr{S}})\|=\sup\{|\hat{\mathscr{B}}\hat{\vartheta}|:\hat{\vartheta}\in\hat{\mathscr{S}}\}\ \le q_1\Big[\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)}-\tfrac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)}+\tfrac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big]+\sum_{i=1}^{m}|y_i|+\frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big].$$

and so,

$$q_0M\le q_0\Big[q_1\Big(\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)}-\tfrac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)}+\tfrac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big)+\sum_{i=1}^{m}|y_i|+\frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big]<1.$$

Ttherefore by Theorem of Theorem 1 are satisfied and hence the operator equation $\mathscr{A}\vartheta\mathscr{B}\vartheta=\vartheta$ has a solution in $\mathscr{S}$. As a result, the problem (1)-(2)-(3) has a solution defined on $\mathscr{J}$. This completes the proof. The following theorems concerning the Boundary value problems $(1)-(2)-(4)$ and $(1)-(2)-(5)$, are similar to that of theorem 2, we omit the proofs.

Theorem 3. *Assume that $(A_1)-A_3)$ are holds. If*

$$q_0\Big[q_1\Big(\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)}-\frac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_2\Gamma(\nu+\sigma_1+1)}+\frac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\sigma_2+1)}\Big)+\sum_{i=1}^{m}|y_i|+\frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big]<1$$

then there exists a on solution for $(1)-(2)-(4)$ on $\mathscr{J}$.

Theorem 4. *Assume that $(A_1)-(A_3)$ are holds. If*

$$q_0\Big[q_1\Big(\frac{\mathbf{T}^{\nu}}{\Gamma(\nu+1)}-\frac{\mathbf{T}^{\nu+1}|\lambda|}{\Lambda_3\Gamma(\nu+\tau_1+1)}+\frac{\mathbf{T}^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\sigma_2+1)}\Big)+\sum_{i=1}^{m}|y_i|+\frac{\mathbf{T}^{\nu+1}}{\Lambda_1}|\tau_3|\Big]<1$$

then there exists a on solution for $(1)-(2)-(5)$ on $\mathscr{J}$.

4 Uniqueness Results via Lipschitz Integral Conditions

In this section, to prove the uniqueness results of solution for problem $(1)-(2)-(3)$, we define the operator $\ominus:\hat{\mathscr{C}}\to\hat{\mathscr{C}}$, we have

$$\ominus\hat{\vartheta}(\hat{\varkappa})=\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big[I^{\nu}(\hat{\xi}_{\hat{\vartheta}})(\hat{\varkappa})+\sum_{i=1}^{m}y_i+\frac{\hat{\varkappa}}{\Lambda_1}\Big(\gamma_3-I^{\nu-\tau_1}(\xi_x)(T)-I^{\nu-\tau_2}(\hat{\xi}_{\hat{\vartheta}})(\mathbf{T})\Big)\Big],\tag{17}$$

where $\hat{\xi}_{\hat{\vartheta}} = \hat{\xi}(s, \hat{\vartheta}(s))$

$$I^{\nu}(\hat{\xi}_{\hat{\vartheta}})(\hat{\varkappa}) = \frac{1}{\Gamma(\nu)} \int_0^{\mathbf{T}} (\hat{\varkappa}-s)^{\nu-1} \hat{\xi}_{\hat{\vartheta}} ds,$$
$$I^{\nu-\tau_1}(\hat{\xi}_{\hat{\vartheta}})(\mathbf{T}) = \frac{\lambda}{\Gamma(\nu-\tau_1)} \int_0^{\mathbf{T}} (\mathbf{T}-s)^{\mathfrak{h}-\tau_1+1} \hat{\xi}_{\hat{\vartheta}} ds,$$
$$I^{\nu-\tau_2}(\hat{\xi}_{\hat{\vartheta}})(\mathbf{T}) = \frac{(1-\lambda)}{\Gamma(\nu-\tau_2)} \int_0^{\mathbf{T}} (\mathbf{T}-s)^{\nu-\tau_2+1} \hat{\xi}_{\hat{\vartheta}} ds.$$

Let

$$\Lambda_3 = I^{\nu}(1)(\mathbf{T}) + \sum_{i=1}^{m} |y_i| + \frac{\hat{\varkappa}}{\Lambda_1}\Big[\gamma_3 - I^{\nu-\tau_1}(1)(\mathbf{T}) - I^{\nu-\tau_2}(1)(\mathbf{T})\Big].$$

Theorem 5.*Assume that (A_4) and (A_5) are holds. If $|m|\Lambda_3 q_1 < 1$, then there exists a unique solution for $(1)-(2)-(3)$ on $\mathscr{J}$.*

Proof:

Let $B_r = \{\vartheta \in \mathscr{C} : \|\vartheta\| \leq r\}$ be a closed bounded and convex subset of $\mathscr{C}$, where the fixed constant r satisfies

$$r \geq \frac{\mathrm{p}\Lambda_3}{1-|m|q_1\Lambda_3}. \tag{18}$$

First we will prove that $\ominus B_r \subset B_r$ and by using the triangle inequality $|\hat{\xi}_{\hat{\vartheta}}| \leq |\hat{\xi}_{\hat{\vartheta}} - \hat{\xi}_0| + |\hat{\xi}_0|$, where $\hat{\xi}_0 = (\hat{\varkappa}, 0)$ for $\hat{\varkappa} \in (\hat{\varkappa}_m, \mathbf{T}]$.

$$\begin{aligned}
|\ominus\hat{\vartheta}(\hat{\varkappa})| &\leq |m|\Big[I^{\nu}(|\hat{\xi}_{\hat{\vartheta}}|)(\hat{\varkappa}) + \sum_{i=1}^{m}|y_i| + \frac{|\hat{\varkappa}|}{\Lambda_1}\Big(\tau_3 - I^{\nu-\tau_1}(|\hat{\xi}_{\hat{\vartheta}}|)(\mathbf{T}) - I^{\nu-\tau_2}(|\hat{\xi}_{\hat{\vartheta}}|)(\mathbf{T})\Big)\Big] \\
&\leq |m|\Big[I^{\nu}\big[|\hat{\xi}_{\hat{\vartheta}} - \hat{\xi}_0| + |\hat{\xi}_0|\big](\hat{\varkappa}) + \sum_{i=1}^{m}|y_i| \\
&\quad + \frac{|\hat{\varkappa}|}{\Lambda_1}\Big(\tau_3 - I^{\nu-\beta_1}\big[|\hat{\xi}_{\hat{\vartheta}} - \hat{\xi}_0| + |\xi_0|\big](T) - I^{\nu-\tau_2}\big[|\hat{\xi}_{\hat{\vartheta}} - \hat{\xi}_0| + |\hat{\xi}_0|\big](\mathbf{T})\Big)\Big] \\
&\leq |m|\Big[I^{\mathfrak{h}}\big(q_1 r + \mathrm{p}\big)(\mathbf{T}) + \sum_{i=1}^{m}|y_i| \\
&\quad + \frac{\hat{\varkappa}}{\Lambda_1}\Big(\tau_3 - I^{\nu-\tau_1}\big[q_1 r + \mathrm{p}\big](\mathbf{T}) - I^{\nu-\tau_2}\big[q_1 r + \mathrm{p}\big](\mathbf{T})\Big)\Big] \\
&= |m|\big(q_1 r\Lambda_3 + \mathrm{p}\Lambda_3\big) \\
&\leq r.
\end{aligned}$$

Therefore, $\ominus B_r \subset B_r$. Let $\hat{\vartheta}_1, \hat{\vartheta}_2 \in B_r$, we have

$$\begin{aligned}
|\ominus\hat{\vartheta}_1(\hat{\varkappa}) - \ominus\hat{\vartheta}_2(\hat{\varkappa})| &\leq |m|\Big[I^{\nu}(|\hat{\xi}_{\hat{\vartheta}_{11}} - \hat{\xi}_{\hat{\vartheta}_2}|)(\hat{\varkappa}) + \sum_{i=1}^{m}|y_i| \\
&\quad + \frac{\hat{\varkappa}}{\Lambda_1}\Big(\gamma_3 - I^{\nu-\tau_1}(|\hat{\xi}_{\hat{\vartheta}_{11}} - \hat{\xi}_{\hat{\vartheta}_{12}}|)(\mathbf{T}) - I^{\nu-\tau_2}(|\hat{\xi}_{\hat{\vartheta}_{11}} - \hat{\xi}_{\hat{\vartheta}_{12}}|)(\mathbf{T})\Big)\Big] \\
&\leq |m|\Big[I^{\nu}(q_1||\hat{\vartheta}_1 - \hat{\vartheta}_2||)(\mathbf{T}) + \sum_{i=1}^{m}|y_i| \\
&\quad + \frac{\hat{\varkappa}}{\Lambda_1}\Big(\gamma_3 - I^{\nu-\tau_1}(q_1||\hat{\vartheta}_1 - \hat{\vartheta}_2||)(\mathbf{T}) - I^{\nu-\tau_2}(q_1||\hat{\vartheta}_1 - \hat{\vartheta}_2||)(\mathbf{T})\Big)\Big] \\
&= |m|q_1\Lambda_3||\hat{\vartheta}_1 - \hat{\vartheta}_2||,
\end{aligned}$$

since $|m|q_1\Lambda_3 < 1$, the operator $\ominus$ is a contraction. By Banach contraction mapping principle the operator $\ominus$ has unique fixed point, which leads that problem (1)-(2)-(3) has a unique solution on $\mathscr{J}$.

5 Examples

5.1 Example

Consider the impulsive FDE's with boundary conditions involving two fractional derivatives of the form

$$ {}^cD^{\frac{10}{7}}\left(\frac{\hat{\vartheta}(\hat{\varkappa})}{\frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\hat{\vartheta}(\hat{\varkappa})+\frac{|\hat{\vartheta}(\hat{\varkappa})|}{1+|\hat{\hat{\vartheta}}(\hat{\varkappa})|}+3\right)}\right)=\frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa}))},\quad \hat{\varkappa}\in(0,\mathbf{T}),\quad 1<\nu\leq 2 \tag{19}$$

$$\hat{\vartheta}(\hat{\varkappa}_k^+)=\hat{\vartheta}(\hat{\varkappa}_k^-)+\frac{1}{6}, \tag{20}$$

$$\frac{\hat{\vartheta}(0)}{\varpi(0,\hat{\vartheta}(0))},\quad \frac{8}{20}D^{\frac{6}{14}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\vartheta}(\pi))}\right)+\frac{3}{5}D^{\frac{4}{17}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\hat{\vartheta}}(\pi))}\right)=\frac{1}{11}, \tag{21}$$

here $\nu=\frac{10}{7}$, $\hat{\xi}(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))=\frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa})|)}$, $\lambda=\frac{8}{20},\tau_1=\frac{6}{14},\tau_2=\frac{4}{17},\tau_3=\frac{1}{11},T=\pi$, observe that $0<\tau_1,\tau_2<\frac{10}{7}$. Hence the hypothesis $(Assumption:2)$ is hold with $q_0=\frac{1}{100},q_1=\frac{1}{10}$ and we shall check that

$$q_0\Big[q_1\Big(\frac{T^\nu}{\Gamma(\nu+1)}-\frac{T^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)}+\frac{T^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big)+\sum_{i=1}^{m}|y_i|+\frac{T^{\nu+1}}{\Lambda_1}|\tau_3|\Big]\approx 0.12758496<1.$$

Thus, the theorem 2, the fractional boundary value problem (19)-(21) has a on solution on $\mathscr{J}$.

5.2 Example

Consider the implicit impulsive FDE's with boundary conditions involving two fractional integrals of the form

$$ {}^cD^{\frac{10}{7}}\left(\frac{\hat{\vartheta}(\hat{\varkappa})}{\frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\hat{\vartheta}(\hat{\varkappa})+\frac{|\hat{\vartheta}(\hat{\varkappa})|}{1+|\hat{\vartheta}(\hat{\varkappa})|}+3\right)}\right)=\frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa}))},\quad \hat{\varkappa}\in(0,T),\quad 1<\nu\leq 2 \tag{22}$$

$$\hat{\vartheta}(\hat{\varkappa}_k^+)=\hat{\vartheta}(t_k^-)+\frac{1}{3}, \tag{23}$$

$$\frac{\hat{\vartheta}(0)}{\varpi(0,\hat{\vartheta}(0))},\quad \frac{8}{20}I^{\frac{6}{14}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\vartheta}(\pi))}\right)+\frac{3}{5}I^{\frac{4}{17}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\vartheta}(\pi))}\right)=\frac{1}{11}, \tag{24}$$

here $\nu=\frac{10}{7},\xi(\hat{\varkappa},\hat{\vartheta}(\hat{\varkappa}))=\frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa})|)},\mu=\frac{8}{20},\tau_1=\frac{6}{14},\tau_2=\frac{4}{17},\tau_3=\frac{1}{11},T=\pi$, observe that $0<\tau_1,\tau_2<\frac{10}{7}$. Hence the hypothesis (A_2) is hold with $q_1=\frac{1}{100},q_0=\frac{1}{10}$ and we shall check that

$$q_0\Big[q_1\Big(\frac{T^\nu}{\Gamma(\nu+1)}-\frac{T^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)}+\frac{T^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big)+\sum_{i=1}^{m}|y_i|+\frac{T^{\nu+1}}{\Lambda_1}|\tau_3|\Big]\approx 0.1457896451<1$$

Thus, the theorem 3, the fractional boundary value problem (22)-(24) has a on solution on $\mathscr{J}$.

5.3 Example

Consider the implicit impulsive fractional differential equations with boundary conditions involving one fractional derivative and one fractional integral of the following form:

$$ {}^cD^{\frac{10}{7}}\left(\frac{\hat{\vartheta}(\hat{\varkappa})}{\frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\hat{\vartheta}(\hat{\varkappa})+\frac{|\hat{\vartheta}(\hat{\varkappa})|}{1+|\hat{\vartheta}(\hat{\varkappa})|}+3\right)}\right)=\frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa}))},\quad \hat{\varkappa}\in(0,T),\quad 1<\nu\leq 2 \tag{25}$$

$$\hat{\vartheta}(\hat{\varkappa}_k^+)=\hat{\vartheta}(\hat{\varkappa}_k^-)+\frac{1}{4} \tag{26}$$

$$\frac{\hat{\vartheta}(0)}{\varpi(0,\hat{\vartheta}(0))},\quad \frac{8}{20}D^{\frac{6}{14}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\vartheta}(\pi))}\right)+\frac{3}{5}I^{\frac{4}{17}}\left(\frac{\hat{\vartheta}(\pi)}{\varpi(\pi,\hat{\vartheta}(\pi))}\right)=\frac{1}{11}, \tag{27}$$

here $\nu = \frac{10}{7}, \hat{\xi}(\hat{\varkappa}, \hat{\vartheta}(\hat{\varkappa})) = \frac{1}{10(1+|\hat{\vartheta}(\hat{\varkappa})|)}, \mu = \frac{8}{20}, \sigma_1 = \frac{6}{14}, \sigma_2 = \frac{4}{17}, \sigma_3 = \frac{1}{11}, T = 1$, observe that $0 < \tau_1, \tau_2 < \frac{10}{7}$. Hence the hypothesis (A_2) is hold with $q_0 = \frac{1}{100}, q_1 = \frac{1}{10}$ and we shall check that

$$q_0\Big[q_1\Big(\frac{T^\nu}{\Gamma(\nu+1)} - \frac{T^{\nu+1}|\lambda|}{\Lambda_1\Gamma(\nu+\tau_1+1)} + \frac{T^{\nu+1}|1-\lambda|}{\Lambda_1\Gamma(\nu+\tau_2+1)}\Big) + \sum_{i=1}^{m}|y_i| + \frac{T^{\nu+1}}{\Lambda_1}|\tau_3|\Big] \approx 0.31245678 < 1.$$

Thus, according to Theorem 4, the fractional boundary value problem (25)-(27) has one solution on $\mathscr{J}$.

6 Conclusion

In this paper, we have investigated the existence and uniqueness results concerning fractional boundary value problems related to implicit impulsive fractional differential equations, which involve fractional derivatives and integrals. We have also explored uniqueness results using the Banach contraction mapping principle and Lipschitz integral conditions. Furthermore, we have extended these results to include new categories of fractional boundary conditions, specifically the Caputo-Hadamard and Hadamard-Caputo sequential fractional differential equations. This extension was accomplished by employing fixed-point theorems and Lipschitz integral conditions.

References

[1] W. G. Glockle and T. F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics*, *Biophysical J.* **68**, 46–53 (1995).

[2] R. Hilfer, *Applications of fractional calculus in physics*, *World Scientific.* Singapore, 2000.

[3] I. Podlubny, *Fractional differential equation*, *Academic Press,* San Diego, 1999.

[4] K. B. Oldham and J. Spanier, *The fractional calculus*, *Academic Press,* New York, London, 1974.

[5] D. N. Chalishajar and F. S. Acharya, *Controllability of second order semi-linear neutral impulsive differential inclusion on unbounded domain with infinite delay in Banach spaces*, *Bull. Korean Math. Soc.* **48(4)**,813–838 (2011).

[6] I. Podlubny, *Fractional differential equations, Academic Press*, New York, 1993.

[7] A. M. Samoilenko and N. A. Perestyuk, *Impulsive differential equations*, *World Scientific, Singapore*, 1995.

[8] S. Sun, Y. Zhao, Z. Han and Y. Li, The existence of solutions for boundary value problem of fractional hybrid differential equations, *Communications in Nonlinear Science and Numerical Simulation*, **17**(12), 4961–4967, (2012).

[9] B.C.Dhage, A fixed point theorem in Banach algebras with applications to functional integral equations, *Kyungpook Math. J.* **44**, 145–155 (2004).

[10] *M. Benchohra and J. E. Lazreg*, *Exsitence results for nonlinear implicit fractional differential equations*, *Surv. Math. Appl.* **9**,79–92 (2014).

[11] K. Hilal, and A.Kajouni, Boundary value problems for hybrid differential equations with fractional order, *Adv. Differ. Equ.* **1**(2015).

[12] K. D. Kucche, J. J. Nieto and V. Venktesh, *Theory of nonlinear implicit fractional differential equations*, *Differ. Equ. Dynam. Syst.* 2016, 1–17 (2016).

[13] Melliani, S.; HilaL, K.; Hannabou, M, *Existence results in the theory of hybrid fractional integro-differential equations*, *J. Univ. Math.* **(2018)**, 1 166–179.

[14] M. Hannabou and K. Hilal, *Existence results for a system of coupled hybrid differential equations with fractional order*, *Int. J. Differ. Equ.* **2020**, (2020).

[15] M. Awadalla, M. Hannabou, K. Abuasbeh and K. Hilal, *A novel implementation of Dhage's fixed point theorem to nonlinear sequential hybrid fractional differential equation*, *Fractal Fract.* **7(2)**, 144 (2023) .