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Nonlocal Problems for Retarted Fractional Differential Equations via Generalization of Darbo's Fixed Point Theorem

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Abstract: In the present paper, we study the existence of mild solutions for retarted semilinear fractional differential equations subject to nonlocal in separable Banach space. The result is proved by means of the theory of a generalization of Darbo's fixed point theorem, upon making some suitable assumptions.

Keywords: Fractional differential equation, mild solutions, fixed point.

1 Introduction

We investigate the following nonlocal initial value problem:

$$\begin{cases} {}^cD^\alpha u(t) = Au(t) + f(t,u(t),u_t), & t \in (0,b], \\ u(s) = \varphi(s) + g(u), & s \in [-h,0] \end{cases} \tag{1}$$

where the state u takes values in a separable Banach space E, u_t stands for the history of the state function up to the time t, i.e., $u_t(s) = u(t+s)$, ${}^cD^\alpha u(t)$ is the fractional Caputo derivative of order $\alpha \in (0,1)$, A is the infinitesimal generator of a C_0-semigroup $T(t)$ in E, f and g are functions which will be specified later.

The topic of fractional differential equations has garnered significant interest in recent times because of its crucial role in modeling various phenomena in the fields of science and engineering.Employing differential equations with fractional order enables the resolution of a wide range of challenges across various domains including fluid flows, rheology, electrical networks, viscoelasticity, electrochemistry, and more. For more details, we refer the reader to the monographs of Miller and Ross [1], Podlubny [2], and Kilbas et. al. [3]. Recently, the theory of fractional differential equations in Banach spaces has been studied extensively by several authors [4,5,6,7,8,9,10,11,12].

Byszewski [13] was the first to investigate the nonlocal problem for first order differential equations. Extensive research has been conducted on this topic because the nonlocal condition provides a more accurate description for Cauchy problems compared to the classical initial condition. Without being exhaustive with the references, let us quote some remarkable solvability results in [14,15,16,17,18,19,20]. Many of these authors use fixed point theorems in $C([a,b],X)$ to prove existence.

Lastly, Junfei Cao et al. [21] use Darbo's fixed point theorem in $L^p([a,b],X)$ to study the fractional functional differential with nonlocal initial condition. Due to the role played by Darbo's theorem in proving the existence of solutions for a lot of classes of nonlinear equations, many researchers work to generalize this important theorem, see for example [22,23,24,25,26,27,28,29,30]. Motivated by this, in this paper, we establish a result concerning the existence of mild solution to the Problem (1) by virtue of the theory of measure of noncompactness associated with a generalization of Darbo's fixed point theorem [25] in $L^p([a,b],X)$ under some suitable conditions, which extend the result in [21].

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2 Preliminaries

2.1 Fixed point theory

Let Y be a Banach space. If B is a subset of Y then the symbols $\overline{B}$ and $\text{conv}(B)$ stand for the closure and the convex hull of B, respectively. Moreover, let $\mathfrak{M}_Y$ be the family of all nonempty and bounded subsets of Y and $\mathfrak{N}_Y$ be its subfamily consisting of all relatively compact sets.

We mention the following definition of the measure of noncompactness, given in [31].

Definition 1. *A function $\mu : \mathfrak{M}_Y \to [0,\infty)$ is called a measure of noncompactness in Y if it satisfies the following conditions:*

(i) The family $\ker\mu = \{A \in \mathfrak{M}_Y : \mu(A) = 0\}$ *is nonempty and* $\ker\mu \subseteq \mathfrak{N}_Y$.
(ii) $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{A}) = \mu(A) = \mu(\text{conv}(A))$.
(iv) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$, *for* $\lambda \in [0,1]$.
(v) If (A_n) *is a sequence of closed sets from* $\mathfrak{M}_Y$ *such that* $A_{n+1} \subseteq A_n$ *for* $(n = 1,2,...)$ *and if* $\lim\limits_{n\to+\infty} \mu(A_n) = 0$, *then* $A_\infty = \cap_{n=1}^{\infty} A_n \neq \emptyset$.

The family $\ker\mu$ *defined in axiom (i) is called the kernel of the measure of noncompactness.*

An important example of measure of noncompactness is the Hausdorff's measure of noncompactness $\chi(.)$, which is defined as follows [32]

$$\chi(A) = \inf\{\varepsilon > 0; A \text{ admits a finite cover by balls of radius} \leq \varepsilon\}.$$

It should be mentioned that the the Hausdorff's measure of noncompactness has also the following additional properties. If A, B are bounded subsets of Y, then

(1) $\chi(\lambda A) = |\lambda|\chi(A)$ for every $\lambda \in \mathbb{R}$.
(2) $\chi(A+B) \leq \chi(A) + \chi(B)$.
(3) If $\{A_n\}_{n\geq 1}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim\limits_{n\to+\infty} \chi(A_n) = 0$, then $\cap_{n=1}^{\infty} A_n$ is a compact subset of Y.
(4) If $T : Y \to Y$ is a bounded linear operator, then $\chi(TA) \leq \|T\|_{\mathscr{L}(Y)}\chi(A)$ [32].

We need the following assertions.

Lemma 1([32]). *Let $\{u_n; n \geq 1\}$ be a subset in $L^1([0,b],E)$ for which there exists $m(.) \in L^1([0,b],\mathbb{R}^+)$ such that $\|u_n(t)\| \leq m(t)$ for each $n \geq 1$ and for a.e. $t \in [0,b]$. Then the function $t \to \chi(t) := \chi(\{u_n(t); n \geq 1\})$ is integrable on $[0,b]$ and, for each $t \in [0,b]$, we have*

$$\chi\left(\left\{\int_0^t u_n(s)\,ds; n \geq 1\right\}\right) \leq \int_0^t \chi(s)\,ds.$$

Let J be a compact interval of $\mathbb{R}$ and χ_p be the Hausdorff's measure of noncompactness in $L^p(J;Y)$. We recall the following fact (see [33]), wich will be used later: for each bounded set $B \subset C(J;Y)$, one has

If B is an equicontinuous set, then

$$\chi_p(B) = \left(\int_J \chi^p(B(t))\,dt\right)^{\frac{1}{p}}$$

where $B(t) = \{u(t) : u \in B\} \subset Y$.

Lemma 2([25]). *Let Ω be a nonempty, bounded, closed and convex subset of Y and let $T : \Omega \to \Omega$ be a continuous mapping such that*

$$\mu(TA) \leq \eta(\mu(A)) \tag{2}$$

for any nonempty subset A of Ω, where μ is a measure of noncompactness defined in X and $\eta : [0,+\infty) \to [0,+\infty)$ is a mapping such that $\eta(t) < t$, for each $t > 0$ and $\frac{\eta(t)}{t}$ is non-decreasing. Then T has a fixed point in Ω.

2.2 Fractional calculus

Let E be a real separable Banach space endowed with the norm $\|.\|$ and $C([a,b],E)$ the Banach space of continuous functions from $[a,b]$ into E endowed with the norm $\|u\|_C = \sup_{a\le t\le b}\|u(t)\|$. For $1\le p<\infty$, we denote by $L^p([a,b],E)$ the space of E-valued measurable functions $u\colon [a,b]\to E$ such that

$$\|u\|_p = \left(\int_a^b \|u(t)\|^p\right)^{\frac{1}{p}} < \infty$$

Lemma 3([33]). *The space $L^p([a,b],E)$ is a Banach space with respect to the norm $\|u\|_p$.*

Lemma 4([33]). *If $u\in L^p([0,b],E)$ the fractional Bochner-Liouville integral of order $\alpha>0$, defined by*

$$I^\alpha u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds,$$

exists for a.e. $t\in[0,b]$, and I^α is a bounded linear operator from $L^p([0,b],E)$ to itself. Also, if $u\in L^p([0,b],E)$ is bounded, then $I^\alpha u(t)$ exists for every $t\in[0,b]$.

Lemma 5([33]). *If $u\colon [0,b]\to E$ is differentiable a.e on $[a,b]$ and $u'\in L^p([0,b],E)$. Then the fractional Caputo derivative of order $\alpha\in(0,1)$ defined by*

$$^cD^\alpha u(t) := I^{1-\alpha}u'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}u'(s)\,ds,\, t\in[0,b],$$

exists for a.e.$t\in[0,b]$.

We denote the space of all bounded linear operators acting on a Banach space E by $\mathscr{L}(E)$.

Definition 2([34]). *A family $\{T(t);t\ge 0\}\subset\mathscr{L}(E)$ is called a C_0-semigroup if the following three properties are satisfied:*

(i) $T(0)=I$, the identity operator on E.
(ii) $T(t)T(s)=T(t+s)$ for all $t,s\ge 0$.
(iii) $\lim_{t\to 0} T(t)u = u$ for all $u\in E$.

The infintesimal generator of the C_0-semigroup $\{T(t);t\ge 0\}$ is the operator $A\colon D(A)\subset E\to E$, defined by

$$D(A) = \left\{u\in E;\ \lim_{h\to 0}\frac{T(h)u-u}{h} \text{ exists}\right\}$$

and

$$Au = \lim_{h\to 0}\frac{T(h)u-u}{h}, \quad u\in D(A),$$

the generator is always a closed, densely defined operator in E.

Lemma 6([35]). *The C_0-semigroup $\{T(t);t\ge 0\}$ is equicontinuous if the function $t\to T(t)$ is continuous from $[0,\infty)$ to $\mathscr{L}(E)$ endowed with the uniform operator norm $\|.\|_{\mathscr{L}(E)}$. In particular, if A is the generator of an uniformly continuous semigroup, a differentiable semigroup, a compact semigroup or an analytic semigroup $\{T(t);t\ge 0\}$, then $\{T(t);t\ge 0\}$ is an equicontinuous C_0-semigroup.*

A function $u\colon [-h,b]\to E$, is a mild solution of (1) if

$$u(t) = \begin{cases} T_\alpha(t)(\varphi(0)+g(u)) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,u(s),u_s)\,ds, & t\in(0,b],\\ \varphi(t)+g(u), \quad t\in[-h,0] \end{cases}$$

here

$$T_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)\,d\theta,$$

$$S_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)\,d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{w}_\alpha(\theta^{-\frac{1}{\alpha}}),$$

$$\overline{w}_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ \theta\in(0,\infty)$$

and ξ_α is a probability density function defined on $(0,\infty)$, that is, $\xi_\alpha(\theta)\ge 0$ for $(0,\infty)$ and $\int_0^\infty \xi_\alpha(\theta)\,d\theta = 1$.

Lemma 7([36]). *Let $\{T(t); t\ge 0\}$ be a C_0-semigroup.*

(i) For any fixed $t\ge 0$, the operators $T_\alpha(t)$ and $S_\alpha(t)$ are linear and bounded operators, that is, for any $x\in E$,

$$\|T_\alpha(t)x\|\le M\|x\| \quad \textit{and} \quad \|S_\alpha(t)x\|\le \frac{M}{\Gamma(\alpha)}\|x\|,$$

where $M := \sup\{\|T(t)\|; t\in[0,b]\}$

(ii) The operators $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous for all $t\ge 0$.

(iii) If $\{T(t); t\ge 0\}$ is equicontinuous, then $\{T_\alpha(t); t\ge 0\}$ and $\{S_\alpha(t); t\ge 0\}$ are equicontinuous, that is, the function $T_\alpha\colon (0,\infty)\to\mathscr{L}(E)$ and $S_\alpha\colon(0,\infty)\to\mathscr{L}(E)$ are continuous.

Lemma 8([34]). *If B is bounded set in $L^p([0,b],E)$, $p>\frac{1}{\alpha}$, then the set*

$$W = \left\{w(.); w(t) := \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)v(s)\,ds, t\in[0,b], v\in B\right\}$$

is uniformly equicontinuous.

3 Main result

Concerning problem (1), we give the following assumptions.

($\mathbf{H_1}$) The C_0-semigroup $\{T(t)\}_{t\ge 0}$ generated by A is equicontinuous.

($\mathbf{H_2}$)

(1) The function $g\colon L^p([-h,b],E)\to E$ is continuous.

(2) For any $u\in L^p([-h,b],E)$,

$$\|g(u)\|\le \psi_g(\|u\|_p),$$

where ψ_g is a real- valued, continuous and non-decreasing function.

(3) There exists $c_1>0$ such that for any bounded $B\subset L^p([-h,b],E)$,

$$\chi(g(B))\le c_1\eta_1(\chi_p(B)),$$

where χ and χ_p are the Hausdorff measures of noncompactness in E and $L^p([-h,b],E)$, respectively and $\eta_1\colon[0,+\infty)\to[0,+\infty)$ is a mapping such that $\frac{\eta_1(t)}{t}$ be a non-decreasing.

($\mathbf{H_3}$)

(1) $f\colon[0,b]\times E\times L^p([-h,0],E)\to E$ satisfies the Carathéodory condition, i.e, $f(.,u,v)$ is measurable for all $(u,v)\in E\times L^p([-h,0],E)$ and $f(t,.,.)$ is continuous for all $t\in[0,b]$.

(2) There exist $d_2, e_2\in L^p([0,b],\mathbb{R}^+)$ such that

$$\|f(t,u,v)\|\le d_2(t)\psi_f(\|u\|+\|v\|_p)+e_2(t),$$

for $(t,u,v)\in[0,b]\times E\times L^p([-h,0],E)$, where ψ_f is a real- valued, continuous and nondecreasing function.

(3) There exists $L>0$ such that for all bounded subsets $D_1\subset E$, $D_2\subset L^p([-h,0],E)$

$$\chi\Big(f(t,D_1,D_2)\Big)\le L\Big(\chi(D_1)+\sup_{\theta\in[-h,0]}\chi\big(D_2(\theta)\big)\Big)$$

for a.e $t\in[0,b]$, where $D_2(\theta)=\{v(\theta): v\in D_2\}$.

(H₄)

(1) $\liminf_{k\to\infty}\frac{1}{k}\left[\|\varphi\|_C+M\|\varphi(0)\|+2\psi_g\big(k(h+b)^{\frac{1}{p}}\big)\right]$
$+\frac{1}{k}\left[+\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\left(\psi_f\big(k+k(b+h)^{\frac{1}{p}}\big)\|d_2\|_p+\|e_2\|_p\right)\right]<1.$

(2) $\big((h+b)^{\frac{1}{p}}+b^{\frac{1}{p}}M\big)c_1\eta_1(t)<(1-\frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}b^{\alpha})t$ for all $t>0$.

Theorem 1. *If the hypotheses (H_1), (H_2), (H_3) and (H_4) hold, then (1) has at least one mild solution.*

*Proof.*Consider the operator $\Lambda: L^p([-h,b],E)\to L^p([-h,b],E)$ given by $\Lambda=\Lambda_1+\Lambda_2$, where

$$\Lambda_1 u(t)=\begin{cases}0, & t\in(0,b],\\ \varphi(t)+g(u), & t\in[-h,0]\end{cases}$$

$$\Lambda_2 u(t)=\begin{cases}T_\alpha(t)(\varphi(0)+g(u))+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,u(s),u_s)\,ds, & t\in(0,b],\\ 0, & t\in[-h,0]\end{cases}$$

Let B_k be the set defined by

$$B_k=\left\{u\in L^p([-h,b],E):\|u(s)\|\le k,\ s\in[-h,b]\right\}$$

where $k>0$. Obviously $B_k\subset L^p([-h,b],E)$ is uniformly integrable, bounded, closed and convex.

First, we show that Λ is well defined on $L^p([-h,b],E)$ and $\Lambda(B_k)\subseteq B_k$.
For any $u\in L^p([-h,b],E)$, we get

$$\|(\Lambda_1 u)(t)\|\le\|\varphi\|_C+\psi_g(\|u\|_p)\le\|\varphi\|_C+\psi_g\big(k(h+b)^{\frac{1}{p}}\big),\quad t\in[-h,0],\tag{3}$$

and for $t\in(0,b]$

$$\begin{aligned}\|(\Lambda_2 u)(t)\|&\le\|T_\alpha(t)(\varphi(0)+g(u))\|+\int_0^t(t-s)^{\alpha-1}\|S_\alpha(t-s)f(s,u(s),u_s)\|\,ds\\&\le M(\|\varphi(0)\|+\psi_g(\|u\|_p))+\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[d_2(s)\psi_f(\|u(s)\|+\|u_s\|_p)+e_2(s)\right]ds\\&\le M(\|\varphi(0)\|+\psi_g\big(k(h+b)^{\frac{1}{p}}\big)+\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\left[\psi_f\big(k+k(b+h)^{\frac{1}{p}}\big)\|d_2\|_p+\|e_2\|_p\right].\end{aligned}$$

Then

$$\begin{aligned}\|(\Lambda u)(t)\|&\le\|(\Lambda_1 u)(t)\|+\|(\Lambda_2 u)(t)\|\\&\le\|\varphi\|_C+M\|\varphi(0)\|+2\psi_g\big(k(h+b)^{\frac{1}{p}}\big)+\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\\&\quad\left[\psi_f\big(k+k(b+h)^{\frac{1}{p}}\big)\|d_2\|_p+\|e_2\|_p\right].\end{aligned}$$

So $\Lambda u\in L^p([-h,b],E)$.
Now, we show that there is a $k\in\mathbb{N}$ such that $\Lambda(B_k)\subseteq B_k$.
Suppose contrary that for each $k\in\mathbb{N}$ there is $u^k\in B_k$ and $t^k\in[0,b]$ such that $\|\Lambda u(t^k)\|>k$. Then

$$\begin{aligned}k&<\|(\Lambda u)(t^k)\|\\&\le\|\varphi\|_C+M\|\varphi(0)\|+2\psi_g\big(k(h+b)^{\frac{1}{p}}\big)+\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\left[\psi_f\big(k+k(b+h)^{\frac{1}{p}}\big)\|d_2\|_p+\|e_2\|_p\right].\end{aligned}$$

Therefore,

$$1 \leq \liminf_{k\to\infty} \frac{1}{k}\left[\|\varphi\|_C + M\|\varphi(0)\| + 2\psi_g\big(k(h+b)^{\frac{1}{p}}\big)\right] + \frac{1}{k}\left[\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha-\frac{1}{p}}\left(\psi_f\big(k+k(b+h)^{\frac{1}{p}}\big)\|d_2\|_p + \|e_2\|_p\right)\right].$$

Passing to the limits in the last inequality, one gets a contradiction. So, there is a $k\in\mathbb{N}$ such that $\Lambda(B_k)\subseteq B_k$.
From now on, we will restrict Λ on B_k.

Second, from the assumptions imposed on f and g, Λ is a continuous map on $L^p([-h,b],E)$.

Third, we will verify that Λ satisfies the inequality (2) in the Corollary 2.
The hypothesis $(\mathbf{H_1})$ and lemma 8 imply that $\Lambda B_k \subset C([-h,b],E)$ is bounded and equicontinuous on $[-h,b]$, so is $\operatorname{conv}(\Lambda B_k)$.
Let $B\subset \operatorname{conv}(\Lambda B_k)$. As E is separable and from lemma 1, we have

$$\begin{aligned}\chi((\Lambda_2 B)(t)) &\leq \chi\Big(T_\alpha(t)(\varphi(0)+g(B))\Big) + \int_0^t (t-s)^{\alpha-1}\chi\Big(S_\alpha(t-s)f(s,B(s),B_s)\Big)\,ds\\ &\leq Mc_1\eta_1(\chi_p(B)) + \frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha-\frac{1}{p}}\chi_p(B)\end{aligned}$$

for a.e. $t\in(0,b]$, where $B(t)=\{u(t): u\in B\}\subseteq E$, $B_t=\{u_t: u\in B\}\subseteq L^p([-h,0],E)$.

$$\chi_p(\Lambda_2 B) \leq b^{\frac{1}{p}} Mc_1\eta_1(\chi_p(B)) + \frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha}\chi_p(B). \tag{4}$$

For $z_1,z_2\in\Lambda_1(B)$, there exist $u_1,u_2\in B$ such that

$$z_1(t)=\varphi(t)-g(u_1), z_2(t)=\varphi(t)-g(u_2) \text{ if } t\in[-h,0].$$

Then

$$\|z_1(t)-z_2(t)\|_E \leq \|g(u_1)-g(u_2)\|_E \text{ if } t\in[-h,0].$$

We have

$$\|z_1-z_2\|_p \leq (h+b)^{\frac{1}{p}}\|g(u_1)-g(u_2)\|_E.$$

Thus

$$\chi_p(\Lambda_1(B)) \leq (h+b)^{\frac{1}{p}}\chi(g(B)).$$

Employing $(\mathbf{H_2})(3)$, we have

$$\chi_p(\Lambda_1(B)) \leq (h+b)^{\frac{1}{p}}c_1\eta_1(\chi_p(B)). \tag{5}$$

Combining the last inequality with (4), we arrive at

$$\chi_p(\Lambda B) \leq \Big((h+b)^{\frac{1}{p}} + b^{\frac{1}{p}}M\Big)c_1\eta_1(\chi_p(B)) + \frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha}\chi_p(B). \tag{6}$$

Note that, the inequality (6) may not remain valid in the case of $B\subset B_k$ as B_k is not equicontinuous on $[-h,b]$. So one must look for another closed convex and bounded subset of $L^p([-h,b],E)$ such that Λ is a χ_p-contraction on it.
Let $U=L^p-\operatorname{conv}(\Lambda B_k)$, where $L^p-\operatorname{conv}$ means the convex hull in $L^p([-h,b],E)$. Then $\Lambda U\subset U$ as $\Lambda B_k\subset B_k$, and B_k is closed and convex in $L^p([-h,b],E)$. For any closed subset $V\subset U$, let $B=V\cap \operatorname{conv}(\Lambda B_k)$. Then
$V=L^p-\operatorname{cl}(B)$, where $L^p-\operatorname{cl}$ means closure in $L^p([-h,b],E)$. Furthermore $\Lambda V\subset L^p-\operatorname{cl}(\Lambda B)$, as Λ is continuous on $L^p([-h,b],E)$. By (6) this implies that

$$\begin{aligned}\chi_p(\Lambda V) &\leq \chi_p(L^p-\operatorname{cl}(\Lambda B)) = \chi_p(\Lambda B)\\ &\leq \Big((h+b)^{\frac{1}{p}} + b^{\frac{1}{p}}M\Big)c_1\eta_1(\chi_p(B)) + \frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha}\chi_p(B).\end{aligned}$$

We put $\eta(t)=\Big((h+b)^{\frac{1}{p}}+b^{\frac{1}{p}}M\Big)c_1\eta_1(t)+\frac{2ML}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha}$, for all $t\in[0,b]$. Then

$$\chi_p(\Lambda V) \leq \eta\big(\chi_p(V)\big)$$

From the hypothesis $(\mathbf{H_1})(2)$, we have $\eta(t)<t$, for all $t>0$. Moreover, $\frac{\eta(t)}{t}$ is a non-decreasing mapping.

So by Corollary 2 $\Lambda: U\to U$ admits a fixed point u on B_k, which is a mild solution of Equation (1).

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