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Spectrum of a Perturbed Harmonic Oscillator

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Abstract: In this article the spectrum of a perturbed harmonic oscillator is calculated. New correctly solvable boundary value problems are used for the corresponding harmonic oscillator on a two-dimensional punctured sphere. The eigenfunctions of the harmonic oscillator are studied in detail and a set of elements from the obtained functionals is introduced.

Keywords: Harmonic oscillator, two-dimensional punctured sphere, well-posed problems, Green's functions.

1 Introduction

The spectrum of a one-dimensional harmonic oscillator, which represents an unbounded self-adjoint operator in the function space $L_2(\mathbb{R})$, consists only of normal eigenvalues. In any textbook on quantum mechanics, one can find an explicit form of the eigenvalues of a one-dimensional harmonic oscillator

$$\lambda_n = 2n+1, n = 0,1,\ldots$$

Moreover, the resolvent of a one-dimensional harmonic oscillator is a compact operator. It is also known that the system of eigenfunctions of a one-dimensional harmonic oscillator has the form

$$f_n(x) = (\pi^{\frac{1}{2}} 2^n n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), n = 0,1,\ldots$$

where $H_n(x) = e^{\frac{x^2}{2}} (e^{-\frac{x^2}{2}})^{(n)}$ are the Hermite polynomials. Spectral analysis of perturbations of a one-dimensional harmonic oscillator

$$L = -\frac{d^2}{dx^2} + x^2 + W(x)$$

in the special case when

$$W(x) = \sum_{j=1}^{N} c_j \delta(x-b_j), N < \infty$$

was studied in the works [1,2,3]. Here we present the result of [4], where the case when $W(x) = s[\delta(x-b) + \delta(x+b)], b \neq 0, b \in \mathbb{R}, s \in \mathbb{C}$ is studied. In this case, the operator has a discrete spectrum consisting of simple eigenvalues. The asymptotic formula for the eigenvalues is proved in the work [4]

$$\lambda_n = (2n+1) + s^2 \frac{k(n)}{n} + \rho(n),$$

where $k(n) = \frac{1}{2\pi}[(-1)^{n+1} \sin(2b\sqrt{2n}) - \frac{1}{2}\sin(4b\sqrt{2n})]$, $|\rho(n)| \leq C \frac{\log n}{n^{\frac{3}{2}}}$.

The two-dimensional harmonic oscillator is also well studied. In particular, the spectrum of the operator H_0, where

$$H_0 = -\Delta + x^2, x^2 = x_1^2 + x_2^2, x \in \mathbb{R}^2,$$

consists of eigenvalues $\lambda_n = 2n+2, n \geq 0$. The multiplicity of the eigenvalue λ_n is equal to $(n+1)$. The corresponding eigenfunctions have the form

$$\varphi_l^{(n)}(x) = f_l(x_1) f_{n-l}(x_2), l = 0,\ldots,n.$$

The perturbation spectrum of a two-dimensional harmonic oscillator is studied in the work [5]

$$H = H_0 + W,$$

where W is the operator of multiplication by a bounded measurable finite real function. In the work [5], the

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following theorem on the localization of the perturbation spectrum of a two-dimensional harmonic oscillator is proved.

Theorem [5]. Let n be a sufficiently large natural number. Then the eigenvalues of the operator H lying in a neighborhood of the point $2(n+1)$ satisfy the inequality

$$|2(n+1)-z| \leq M^2 ||W||_\infty \left(\frac{1}{\sqrt{n}} + O\left(\frac{\ln n}{n}\right)\right),$$

where $||W||_\infty$ is the norm in the space $L_\infty(\mathbb{R}^2)$.

Thus, regular perturbations of a two-dimensional harmonic oscillator are studied in [5]. In this paper, we study the spectra of some singular perturbations of a two-dimensional harmonic oscillator. The beginning of such research is laid in the paper [6].

2 Known facts about the harmonic oscillator

Let $\mathbb{R}^2 = \{x = (x_1, x_2)\}$ be a Euclidean space. We choose [6] a fixed reversible self-adjoint harmonic oscillator $B_0 = B_0^* = -\Delta + x_1^2 + x_2^2$ acting in the functional space $L_2(\mathbb{R}^2)$. The domain of this operator is denoted as $D(B_0) = \{u \in W_2^2(\mathbb{R}^2) : -\Delta u + x^2 u \in L_2(\mathbb{R}^2)\}$.

It is known [7] that the eigenvalues of the operator B_0 are calculated by the formulas $\lambda_n = 2n+2, n \geq 0$. Since the condition $\lambda_n \geq 1$ is satisfied for all λ_n, then there exists an inverse operator B_0^{-1}, which is an integral operator $B_0^{-1} f(x_1, x_2; t_1, t_2) = \int\int_{\mathbb{R}^2} \varepsilon(x_1, x_2; t_1, t_2) \cdot \cdot f(t_1, t_2) dt_1 dt_2$. The kernel of the integral operator $\varepsilon(x_1, x_2; t_1, t_2)$ defines Green's function of the operator B_0.

Green's function $\varepsilon(x_1, x_2; t_1, t_2)$ satisfies the equality $(-\Delta + x_1^2 + x_2^2)\varepsilon(x_1, x_2; t_1, t_2) = 0$ for $(x_1, x_2) \neq (t_1, t_2)$, which follows from the definition of the inverse operator B_0^{-1}. Derivatives $\frac{\partial \varepsilon}{\partial t_1}(x_1, x_2; t_1, t_2)$, $\frac{\partial \varepsilon}{\partial t_2}(x_1, x_2; t_1, t_2)$ satisfy the equalities

$$(-\Delta + x_1^2 + x_2^2)\frac{\partial \varepsilon}{\partial t_1}(x_1, x_2; t_1, t_2) = 0,$$

$$(-\Delta + x_1^2 + x_2^2)\frac{\partial \varepsilon}{\partial t_2}(x_1, x_2; t_1, t_2) = 0$$

for $(x_1, x_2) \neq (t_1, t_2)$. Thus, the functions $\varepsilon(x_1, x_2; t_1, t_2)$, $\frac{\partial \varepsilon}{\partial t_1}(x_1, x_2; t_1, t_2)$, $\frac{\partial \varepsilon}{\partial t_2}(x_1, x_2; t_1, t_2)$ are solutions of the homogeneous equation

$$(-\Delta + x_1^2 + x_2^2)u(x_1, x_2) = 0$$

at $(x_1, x_2) \neq (t_1, t_2)$.

The eigenfunctions [6] of the operator B_0 are given by the formula

$$f_l(x_1) f_{n-l}(x_2) = (2^l l! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x_1^2}{2}} H_l(x_1) \cdot$$
$$\cdot (2^{n-l}(n-l)! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x_2^2}{2}} H_{n-l}(x_2),$$

where $H_l(x)$ represents the Hermite polynomials. We get the following representations

$$f_l(t) = \alpha_l \{\cos[t\sqrt{2l+1} - \frac{l\pi}{2}][u_0(t) - \frac{u_2(t)}{4(2l+1)} +$$
$$+ O(\frac{1}{l^2})]\} + \frac{\alpha_l}{2\sqrt{2l+1}}\{\sin[t\sqrt{2l+1} -$$
$$-\frac{l\pi}{2}][u_1(t) - \frac{u_3(t)}{4(2l+1)} + O(\frac{1}{l^2})]\},$$

where $u_0(t) \equiv 1, \quad u_l(t) = \int_0^t L u_{l-1}(t) dt, \quad L = -\frac{d^2}{dt^2} + t^2,$

$$\alpha_l = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt[4]{2l+1}} \{1 - \frac{1}{32(2l+1)^2} + O(\frac{1}{l^3})\}.$$

Note [8] that the eigenfunctions of the operator B_0 on any compact set $K \subset \mathbb{R}^2$ have a global estimate

$$|f_l(t)| \leq \frac{C_0}{\sqrt[4]{2l+1}}. \tag{1}$$

The system of eigenfunctions of the operator B_0 is an orthogonal basis in the functional space $L_2(\mathbb{R}^2)$.

It is known [9] that Green's function $\varepsilon(x_1, x_2; t_1, t_2)$ expands in terms of the eigenfunctions of the operator B_0 and has the following form

$$\varepsilon(x_1, x_2; t_1, t_2) = \sum_{n=0}^{\infty} \frac{1}{2n+2} \sum_{l=0}^{n} f_l(x_1) f_{n-l}(x_2) f_l(t_1) f_{n-l}(t_2). \tag{2}$$

We present a theorem proved in [6].

Theorem Let Green's function $\varepsilon(x_1, x_2; t_1, t_2)$ of a two-dimensional harmonic oscillator be defined for all $x \neq \pm t$ and be continuous function of $(x_1, x_2; t_1, t_2)$ in the domain. Then Green's function $\varepsilon(x_1, x_2; t_1, t_2)$ has the representation

$$\varepsilon(x_1, x_2; t_1, t_2) = \sqrt{\frac{2}{|x-t|\pi}} \varepsilon_-(x_1, x_2; t_1, t_2) +$$
$$+ \sqrt{\frac{2}{|x+t|\pi}} \varepsilon_+(x_1, x_2; t_1, t_2) + k(x_1, x_2; t_1, t_2),$$

Note that $\varepsilon_-(x_1, x_2; t_1, t_2)$, $\varepsilon_+(x_1, x_2; t_1, t_2)$, $k(x_1, x_2; t_1, t_2)$ are continuous functions of two pairs of variables $(x_1, x_2; t_1, t_2)$.

3 Delta-shaped perturbations of a harmonic oscillator

In this section, we formulate new correctly solvable problems for the operator $(-\Delta + x_1^2 + x_2^2)$ on the punctured plane $\mathbb{R}_0^2 = \mathbb{R}^2 \backslash \{x_1^0, x_2^0\}$. Here $x_1^0 \neq 0, x_2^0 \neq 0$ are fixed points in $\mathbb{R}^2$. Through $Q_+^0(\delta), Q_-^0(\delta)$ we denote two open circles of radius δ centered at the points x_1^0, x_2^0, respectively, i.e. $Q_+^0(\delta) = \{x \in \mathbb{R}^2 : |x - x_1^0| < \delta\}, Q_-^0(\delta) = \{x \in \mathbb{R}^2 : |x - x_2^0| < \delta\}$.

Let $h(x)$ be a function such that there are finite values of the functionals $\gamma_1, \ldots, \gamma_6$. The linear functionals $\gamma_1(h), \ldots, \gamma_6(h)$ are given by the formulas:

$$\gamma_1(h) = -\lim_{\delta\to 0}\int_{\partial Q_+^0(\delta)} \frac{\partial}{\partial v_t} h(t) ds_t,$$

$$\gamma_2(h) = -\lim_{\delta\to 0}\int_{\partial Q_-^0(\delta)} \frac{\partial}{\partial v_t} h(t) ds_t,$$

$$\gamma_3(h) = \lim_{\delta\to 0}\int_{\partial Q_+^0(\delta)} (h(t)\frac{t_1 + x_1^0}{\delta}) ds_t,$$

$$\gamma_4(h) = \lim_{\delta\to 0}\int_{\partial Q_+^0(\delta)} (h(t)\frac{t_2 - x_2^0}{\delta}) ds_t,$$

$$\gamma_5(h) = \lim_{\delta\to 0}\int_{\partial Q_-^0(\delta)} (h(t)\frac{t_1 + x_1^0}{\delta}) ds_t,$$

$$\gamma_6(h) = \lim_{\delta\to 0}\int_{\partial Q_-^0(\delta)} (h(t)\frac{t_2 - x_2^0}{\delta}) ds_t,$$

where v_t is the outward normal to $\partial Q_\pm^0(\delta)$.

In particular, the following functions can be chosen as $h(x)$:

$$\psi_1(x) = \varepsilon(x_1, x_2; x_1^0, x_2^0),$$
$$\psi_2(x) = \varepsilon(x_1, x_2; -x_1^0, -x_2^0),$$
$$\psi_3(x) = \frac{\partial}{\partial t_1}\varepsilon(x_1, x_2; x_1^0, x_2^0),$$
$$\psi_4(x) = \frac{\partial}{\partial t_2}\varepsilon(x_1, x_2; x_1^0, x_2^0),$$
$$\psi_5(x) = \frac{\partial}{\partial t_1}\varepsilon(x_1, x_2; -x_1^0, -x_2^0),$$
$$\psi_6(x) = \frac{\partial}{\partial t_2}\varepsilon(x_1, x_2; -x_1^0, -x_2^0).$$

Lemma 3.1 For $j, k = 1,2,3,4,5,6$ the biorthogonal relations are valid

$$\gamma_j(\psi_k) = \delta_{jk}, \tag{3}$$

where δ_{jk} is the Kronecker symbol.

Using the chosen potentials, we introduce the class of functions:

$$\widehat{W}_2^2(\mathbb{R}_0^2) = \{h(x) = B_0^{-1} f(x) + \sum_{j=1}^{6} \alpha_j \psi_j(x),$$

$$\forall f \in L_2(\mathbb{R}^2), \forall \alpha_j \in \mathbb{C}, j = \overline{1,6}\}.$$

Therefore, the following assertion plays an important role.

Lemma 3.2 Let h be an arbitrary element of the class $\widehat{W}_2^2(\mathbb{R}_0^2)$. Then there exists a smooth function $g(x)$ such that h can be represented as

$$h(x) = g(x) - \sum_{k=1}^{6} \gamma_k(h)\psi_k(x), \tag{4}$$

where $g(x)$ is some smooth function of the space $W_2^2(\mathbb{R}^2)$, $\gamma_j(h), j = 1,...,6$ are the linear functionals defined above. The specified representation 4 is unique.

Note that Lemma 3.1 and Lemma 3.2 are proved in [6].

According to Lemma 3.2, we introduce the operator J. The operator J maps an arbitrary element $h \in \widehat{W}_2^2(\mathbb{R}_0^2)$ into an element $g \in W_2^2(\mathbb{R}^2)$, where the element g is from the 4 representation.

The following theorem is proved in the work [10].

Theorem 3.1. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ be an arbitrary set of linear functionals on the space $L_2(\mathbb{R}^2)$. Then for any $f \in L_2(\mathbb{R}^2)$ the following problem

$$B_0 Jh(x) = f(x), \quad x \in \mathbb{R}_0^2, \quad \gamma_k(h) = \alpha_k(B_0 Jh),\ k = 1,...,6 \tag{5}$$

has a unique solution of class $\widehat{W}_2^2(\mathbb{R}_0^2)$. The operator corresponding to the problem 5 will be denoted as B_α.

In the next section, we study the spectrum of the operator B_α.

4 Spectrum of a perturbed harmonic oscillator

According to Theorem 3.1, the operator B_α is introduced. In this section, we study the eigenvalue problem $B_0 Ju = \lambda Ju$, $\gamma_k(u) = \lambda \cdot \alpha_k(Ju), k = 1, \ldots, 6$.

To calculate the eigenvalues, we use the equation $B_0 Ju = \lambda Ju$, which assumes that the equation $B_0 g = \lambda g$. is satisfied. The eigenvalues of the operator B_0 are known and written as $\lambda_n = 2n + 2$. The function $\varphi_l^{(n)}(x)$ is an eigenfunction $\varphi_l^{(n)}(x) = f_l(x_1) f_{n-l}(x_2),\ l = 0, ..., n$.

Denote by $g(x) = Ju(x)$. Then $B_0 g = \lambda g(x)$. Since the eigenvalues of the operator B_0 are known, then $\lambda_n = 2n + 2,\ n \geq 0$. The functions $Ju_{nl}(x) = \varphi_l^{(n)}(x),\ l = 0, 1, ..., n$. Then for $k = \overline{1,6}$ we have

$$\gamma_k(u_{nl}) = \lambda_n \alpha_k(\varphi_l^{(n)}).$$

As a result, the eigenfunction of the operator B_α takes the form

$$u_{nl}(x) = \varphi_l^{(n)}(x) - \lambda_n \sum_{k=1}^{6} \alpha_k(\varphi_l^{(n)}) \cdot \psi_k(x), \tag{6}$$

$$l = 0, ..., n;\ n \geq 0.$$

Thus, the assertion is proved.

Theorem 4.1. Let $\alpha_1, ..., \alpha_6$ be arbitrary linear continuous functionals from $L_2(\mathbb{R}^2)$. Then the spectrum of the operator B_α coincides with the spectrum of the operator B_0, i.e.

$$\lambda_n(B_\alpha) = \lambda_n(B_0).$$

The system of eigenfunctions of the operator B_α is given by the formulas 6.

Finally, let's consider an example.

Example 1. As $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ we take zero functionals, and as the functional $\alpha_1(f)$ we choose the following functional $\alpha_1(f) = \int\int_{\mathbb{R}^2} \psi_1(x) f(x) dx$. In this case, the eigenfunctions of the operator $B_{\alpha_1 00000}$ will take the form

$$u_{nl}(x) = \varphi_l^{(n)}(x) + \varphi_l^{(n)}(x^0) \cdot \psi_1(x).$$

where $l = 0, 1, \ldots$

5 Conclusion

In this article, we have studied the eigenfunctions of a perturbed harmonic oscillator. Descriptions of linear functionals for an arbitrary point are given. The relationship between eigenvalues and eigenfunctions of the operator, perturbed and unperturbed operators is shown. The property of discreteness of the spectrum of a harmonic oscillator on the straight axis is proved in the work. An example is also given where the first functional is defined through the integral, and the other five are equal to zero. The example is an explicit use of a theorem whose proof used previously obtained results for a perturbed operator.

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