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Duality in a class of vector Köthe-Orlicz spaces

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Abstract: We deal with a complete normed space E, a scalar sequence space λ, and an Orlicz mapping M to introduce and study some properties of the spaces $\lambda_M\{E\}$ of all $E-$valued sequences that are absolutely (λ,M)-summable. Denote by $\lambda_M\{E\}_r$ the subspace of $\lambda_M\{E\}$ whose elements are AK-sequences. We describe the continuous linear forms on this space in term of E^*-valued sequences that are absolutely (λ^*,N)-summable, where N is the Orlicz mapping complement of M.

Keywords: Duality, vector and scalar sequence spaces, normed spaces, Orlicz space

1 Introduction

The notion of absolutely and weakly λ-summable sequences in a locally convex space, for λ a perfect Köthe scalar sequence space, was first introduced by A. Pietsch [1] to characterize the nuclearity of locally convex spaces.

Since then many authors have been interested to the study of these spaces defined by a combination of a Köthe scalar sequence spaces and a linear vector space. They consider on λ, not only its normal topology, but general polar topologies. The space of absolutely λ summable sequence has been intensively studied by many authors as in [2,3]. Later, an extension to the modular function has been introduced in [4,5]. The authors in [6,7,8,9,10,11,12] were mainly interested in the weakly λ-summability. In [12], the author involved the Orlicz mapping to define a new class of these spaces

In this note, we deal with an Orlicz mapping M and a a scalar sequence space λ, supposed to be perfect, to generalize the notion of the absolute $\lambda-$summability by defining $\lambda_M\{E\}$, the space of all absolute (λ,M)-summable ones in a complete normed space E.
Notice that, for $M(t)=t$, the space $\lambda_M\{E\}$ is nothing else but $\lambda\{E\}$ of all $E-$ valued sequences that are $\lambda-$ summable studied in [2,3].

In this paper, w study some of properties of $\lambda_M\{E\}$, such as the description of the topological dual.

2 Preliminaries

Throughout this paper, if F is a normed space then we denote by F^*, B_{F^*} and $\|\cdot\|_{F^*}$, respectively, the topological dual, the closed unit ball and the norm of F.

Let the symbol ω stand for the linear space of all complex sequences with respect to the standard component operations. For all $n\in\mathbb{N}$, by, e_n we mean the standard unit vector of order n in ω. A linear subspace λ of ω is said to be normal, whenever α and β are in ω, and $\alpha\le\beta$ and $\beta\in\lambda$ then $\beta\in\lambda$.
If λ is a sequence space, its α-dual will be denoted λ^* and defined as

$$\lambda^*=\left\{(\beta_n)\in\omega:\sum_{n=1}^{\infty}|\alpha_n\beta_n|<\infty,\ \forall(\alpha_n)_n\in\lambda\right\}.$$

It is easy to check the inclusion $\lambda\subset\lambda^{**}=(\lambda^*)^*$. We say that λ is perfect whenever the equality $\lambda=\lambda^{**}$ hols. Everywhere it occurs in this note, λ means a complete and perfect normed sequence space such that

(a) $\|\cdot\|_\lambda$ is solid, that is, whatever γ and δ in λ, if $\gamma\le\delta$ then $\|\gamma\|_\lambda\le\|\delta\|_\lambda$.
(b) Every $(\beta_n)_n$ in λ is the limit of the sequence $(\beta_1,\ldots,\beta_n,0,\ldots)$, $n\in\mathbb{N}$, the finite sections of β with respect to the norm in $\|\cdot\|_\lambda$. In other words, the space λ satisfies the AK-property.

These two conditions make the continuous dual of λ coincide with λ^*. By using Hahn-Banach Theorem, the

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standard norm $\|\cdot\|_{\lambda^*}$ of λ^* can then be given as

$$\|\gamma\|_{\lambda^*} = \sup\left\{\sum_{n=1}^{\infty}|\delta_n\gamma_n|,\ \delta=(\delta_n)_n\in\lambda,\ \|\delta\|_\lambda\leq 1\right\}.$$

Moreover, it will be needed to assumed that the dual space $(\lambda^*,\|\cdot\|_{\lambda^*})$ of λ^* satisfies also the is also AK-property. In that case, λ will be a reflexive complete normed space.

An Orlicz mapping is a non-decreasing, non-negative, convex and continuous, function M defined for every $t\geq 0$, with the properties that $M(0)=0, M(x)>0$ for $x>0$ and $\lim_{x\to\infty} M(x)=\infty$.

It is possible to represent an Orlicz mapping M in the integral form

$$M(x)=\int_0^x m(t)dt,$$

where m is positive, continuous at the right for every $t>0$, and $m(0)=0$. Let n be defined by for $t\geq 0$,

$$n(t)=\sup\{u: m(u)\leq t.\ \forall f \text{ for} \geq 0\}.$$

So, n satisfies the same conditions as m. Let N be defined by

$$N(u)=\int_0^u n(x)dx$$

Then N is also an Orlicz mapping. We say that N complements M and M complements N. They satisfy

$$ts\leq M(t)+N(s), \text{ for } t,s\geq 0. \tag{1}$$

For an Orlicz mapping M, define the space ℓ_M by

$$\ell_M=\left\{(\alpha_n)_n\in\omega:\exists\sigma>0, \sum_{n=1}^{\infty}M\left(\frac{|\alpha_n|}{\sigma}\right)<\infty,\right\}.$$

Since M is non-decreasing, it is easy to verify that the space ℓ_M is normal. Moreover, the quantity

$$\|(\beta_n)_n\|_M=\inf\left\{\sigma>0, \sum_{n=1}^{\infty}M\left(\frac{|\beta_n|}{\sigma}\right)\leq 1\right\},$$

is a solid norm on ℓ_M for which ℓ_M is a complete normed space.

For $M(t)=t^p$, and $1\leq p<\infty$, the space ℓ_M coincides with the classical complete normed spaces $\ell_{p},$.

Because of its convexity, M satisfies always the inequality $M(tx)\leq tM(x)$, for every $0\leq t\leq 1$.
We will assume that there is $L>0$, verifying $M(2x)\leq LM(x)$, for all $x\in[0,\infty)$. This condition on M is known as the condition Δ_2.

Particularly, from this condition, one derives that ℓ_M and ℓ_N are $\alpha-$dual each other (Corollary 4.2 of [5]) and are then perfect reflexive normed spaces.

3 The vector sequence space $\lambda_M\{E\}$

For a complete normed space E, $\omega(E)$ will denote the vector space of all $E-$valued sequences, and by $\lambda_M\{E\}$ we mean the subset of $\omega(E)$ constituted by all sequences in E that are absolutely $(\lambda,M)-$ summable. By this we mean

$$\lambda_M\{E\}=\{(x_n)_n\in\omega(E):\forall(\alpha_n)_n\in\lambda^*,(\|\alpha_nx_n\|_E)_n\in\ell_M\}.$$

We have

Theorem 1. *With respect to the standard component operations, $\lambda_M\{E\}$, is a linear space, and the quantity*

$$\begin{aligned}\|x\|_{\lambda_M\{E\}} &= \left\|\left(\alpha_n\|x_n\|\right)_n\right\|_M\\ &= \sup_{\alpha\in B_{\lambda^*}}\inf\left\{\sigma>0:\sum_{n=1}^{\infty}M(\|\alpha_nx_n\|/\sigma)\leq 1\right\}\end{aligned}$$

is a norm on $\lambda_M\{E\}$.

Proof. It follows quickly from the subadditivity of the norm of E and the fact that ℓ_M is normal that

$$\ell_M(E)=\{x=(x_n)_n\in\omega(E):(\|x_n\|_E)_n\in\ell_M\}$$

is a linear subspace of $\omega(E)$. Now, for all $(\alpha_n)_n\in\lambda^*$, define $\varphi_\alpha:\omega(E)\to\omega(E)$ by $\varphi_\alpha(x)=(\alpha_na(x_n))$. Clearly, φ_α is a linear mapping, and

$$\lambda_M\{E\}=\bigcap_{\alpha\in\lambda^*}\varphi_\alpha^{-1}(\ell_M(E)).$$

Then, $\lambda_M\{E\}$ is a linear space.

Next, we shall show that the quantity in (2) is finite. To this purpose, let $x=(x_n)_n$ in $\lambda_M\{E\}$ fixed, and consider the operator $f_x:\lambda^*\to\ell_M$ such that $f_x(\gamma)=(\gamma_n\|x_n\|)_n$. An application of the closed graph theorem yields the continuity of f_x. It follows that

$$\begin{aligned}\|x\|_{\lambda_M\{E\}} &= \sup_{\gamma\in B_{\lambda^*}}\inf\left\{\sigma>0:\sum_{n=1}^{\infty}M(\|\gamma_nx_n\|/\sigma)\leq 1\right\}\\ &= \sup_{\gamma\in B_{\lambda^*}}\|(\gamma_n\|x_n\|)\|_M=\sup_{\gamma\in B_{\lambda^*}}\|f_x(\gamma)\|_M,\end{aligned}$$

which gives the required property. The other conditions of the norm are easily checked. ■

Now, we prove that the projections are continuous.

Lemma 1. *If i, is a natural number, let P_i be the the projection from $\lambda_M\{E\}$ to E, given as*

$$\text{if } x=(x_n)\in\lambda_M\{E\}, \text{ then } P_i(x)=x_i.$$

Then, P_i is a linear and continuous mapping.

Proof. Let $i \in \mathbb{N}$, and $(\gamma_n)_n \in B_{\lambda^*}$ such that $\gamma_i > 0$. Let $K = 1/(\gamma_i \|e_i\|_M)$. For all $x = (x_n) \in \lambda_M\{E\}$, since the norm $\|\cdot\|_M$ is solid and $\gamma_i \|x_i\| e_i \leq (\|\gamma_n x_n\|)_n$, we have

$$\gamma_i \|x_i\| \|e_i\|_M = \|\gamma_i x_i\| \|e_i\|_M \leq \|(\|\gamma_n x_n\|)_n\|_M = \|(x_n)_n\|_{\lambda_M\{E\}}.$$

Thus,

$$\forall x = (x_n) \in \lambda_M\{E\}, \quad \|x_i\|_E \leq K \|x\|_{\lambda_M\{E\}},$$

from which one derives the continuity of P_i. ■

Theorem 2. *The space $\lambda_M\{E\}$ so normed is a complete normed space for which E and λ are closed linear subspaces.*

Proof. We will show first that, if $\alpha = (\alpha_k)_k \in \lambda$ and $t \in E$, then $(\alpha_k t)_k \in \lambda_M\{E\}$,

Consider $0 \neq \alpha = (\alpha_k)_k \in \lambda$, $\beta = (\beta_k)_k \in \lambda^*$ and $0 \neq t \in E$. Let $\sigma = \sum_k \|\alpha_k \beta_k t\|$ and $\eta_k = \|\alpha_k \beta_k t\| / \sigma$, for every k. Then,

$$\sum_k M(\|\alpha_k \beta_k t\| / \sigma) = \sum_k M(\eta_k) \leq \sum_k \eta_k M(1) = M(1) < \infty.$$

So, $(\alpha_k t)_k \in \lambda_M\{E\}$. Now, Let us show that for all $\alpha = (\alpha_k)_k \in \lambda$ and $t \in E$,

$$\|(\alpha_k t)_k\|_{\lambda_M\{E\}} \leq (1 + M(1)) \|\alpha\|_\lambda \|t\|_E. \quad (2)$$

The assertion (2) is trivial when $t = 0$. Assume that $t \neq 0$. Let $\sigma_0 = (1 + M(1)) \|t\|_E \|\alpha\|_\lambda$. If $\beta = (\beta_n)_n \in \lambda^*$ with $\|\beta\|_{\lambda^*} \leq 1$, since M is convex,

$$\begin{aligned}\sum_{n=1}^{\infty} M\left(\frac{|\alpha_n \beta_n| \|t\||}{\sigma_0}\right) &\leq \sum_{n=1}^{\infty} \frac{|\alpha_n \beta_n| |\|t\||}{\sigma_0} M(1) \\ &\leq \frac{\|\alpha\|_\lambda \|t\|_E}{\sigma} M(1) = \frac{M(1)}{M(1)+1} \leq 1.\end{aligned}$$

Thus,

$$\|(\alpha_n t)_n\|_{\lambda_M\{E\}} = \|(\alpha_n \|t\|)_n\|_M \leq \sigma_0 = (M(1)+1) \|\alpha\|_\lambda \|t\|_E.$$

For a nonzero $\tau = (\tau_n)_n$ fixed in λ, the well defined mapping f_τ from E to $\lambda_M\{E\}$ with $f_\tau(t) = (\tau_n t)_n$ is linear and one to one ; its continuity holds by (2).

Suppose that $(t_k)_k$ is any sequence of members of E that satisfies the convergence of $(\tau t_k)_k$ in $\lambda_M\{E\}$ to $y = (x_n)_n$. For any natural number m such that $\tau_m \neq 0$, we conclude from Lemma 1 the convergence of the sequence $(t_k)_k$ to $\frac{1}{\tau_m} x_m$. Suppose then that $(t_k)_k$ tends to t as $k \to \infty$.. Thus, when $\tau_n \neq 0$, $x_n = t$ and $x_n = 0$ for $\tau_n = 0$. Then, $y = \tau t$, which means that E can be assimilated as a closed subspace of $\lambda_M\{E\}$. The same argument applies to prove that λ can also be assimilated with a closed subspace of $\lambda_M\{E\}$.

Consider a Cauchy sequence $x^k = (x_n^k), k = 1, 2, \ldots,$ in $\lambda_M\{E\}$. Let i be a natural number. Thanks to the continuity of the mapping P_i stated and proved in the lemma 1, the projected sequence $x_i^k, k = 1, 2, \ldots,$ is , in fact, in E ; a Cauchy sequence, denote its limit by $x_i \in E$.

We clam that $x = (x_i)_i \in \lambda_M\{E\}$ and that $(x^k)_k$ tends to x as $k \to \infty$. For a fixed $\alpha = (\alpha_n) \in \lambda^*$, We will verify that the mapping $\varphi_\alpha : y = (y_n) \in \lambda_M\{E\} \to (\alpha_n \|y_n\|) \in \ell_M$ is uniformly continuous. Since the norm $\|\cdot\|_M$ of ℓ_M is solid, for all $y = (y_n)$ and $z = (z_n) \in \lambda_M\{E\}$, we can write

$$\begin{aligned}\|\varphi_\alpha(y) - \varphi_\alpha(z)\|_M &= \|(\alpha_n \|y_n\|)_n - (\alpha_n \|z_n\|)_n\|_M \\ &= \|(\alpha_n(\|y_n\| - \|z_n\|))_n\|_M \\ &\leq \|(\alpha_n |\|y_n\| - \|z_n\||)_n\|_M \\ &\leq \|(\alpha_n \|y_n - z_n\|)_n\|_M = \|\varphi_\alpha(y - z)\|_M.\end{aligned}$$

So, $\varphi_\alpha(x^k) = \{\alpha_n \|x_n^k\|\}_{k=1}^\infty$. Since ℓ_M is a complete normed space, this sequence converges to a limit that we denote by $\beta = (\beta_n)$ in ℓ_M. Let k be a natural number. Then

$$\begin{aligned}\alpha_k \|x_k\|_E &= \alpha_k \| \lim_{p\to\infty} x_k^p \|_E \\ &= \lim_{p\to\infty} \alpha_k \|x_k^p\| = \beta_k.\end{aligned}$$

So, $(\alpha_n \|x_n\|)_n = \beta \in \ell_M$. By what we proved that $x \in \lambda_M\{E\}$.

A more difficult task is to prove the convergence of $\{x^k\}_{k=1}^\infty$ to x. Consider a positive real number ε.

We can select a natural number N for which, if $\alpha = (\alpha_n)$ is laying in B_{λ^*} and p and q are natural numbers greater than N, there exists $0 < \sigma < \varepsilon$ that satisfies

$$\sup_{K \in \mathbb{N}} \sum_{n=1}^{K} M\left(\frac{\|\alpha_n (x_n^p - x_n^q)\|}{\sigma}\right) = \sum_{n=1}^{\infty} M\left(\frac{\|\alpha_n (x_n^q - x_n^p)\|}{\sigma}\right) \leq 1.$$

Thanks to the is continuity of M, letting $p \to \infty$, we find $\sum_{n=1}^{K} M\left(\frac{\|\alpha_n (x_n^q - x_n)\|}{\varepsilon}\right) \leq 1$ for every natural number K greater than N. One can then conclude that

$$\begin{aligned}\|x^p - x\|_{\lambda_M\{E\}} = &\sup_{\alpha \in B_\lambda^*} \inf \\ &\left\{\sigma > 0 : \sum_{n=1}^{\infty} M\left(\frac{|\alpha_n (x_n^p - x_n)|}{\sigma}\right) \leq 1\right\} \leq \varepsilon,\end{aligned}$$

whenever p is greater than N. The proof is over. ■

4 On the continuous dual of $\lambda_M\{E\}$

For $x = (x_n) \in \omega(E)$, let $\{x^{(k)}\}_{k=1}^\infty$ denote the sequence of the finite sections of x. That is

$$x^{(k)} = (x_1, x_2, \ldots, x_k, 0 \ldots) = \sum_{n=1}^{k} x_n e_n.$$

It is immediately seen that $\lambda_M\{E\}$ contains the finite sections of all its elements. In other words, if $y=(y_n)\in\lambda_M\{E\}$, then $\{y^{(k)}\}_{k=1}^{\infty}\subset\lambda_M\{E\}$. Using the Σ notation for $y^{(k)}$, we see that if y is an AK-sequence, that is $\{y^{(k)}\}_{k=1}^{\infty}$ converges to y, in $\lambda_M\{E\}$, then

$$y=\lim_{k\to\infty}y^{(k)}=\sum_{n=1}^{\infty}y_n e_n. \qquad (3)$$

Let $\lambda_M\{E\}_r$ denote the subspace of elements of $\lambda_M\{E\}$ satisfying the equation (3). The vector sequence space $\lambda_M\{E\}$ is said to have the AK-property, if it coincides with $\lambda_M\{E\}_r$.

The following result relates topologically these two spaces.

Theorem 3. *$\lambda_M\{E\}_r$ is a closed subspace of $\lambda_M\{E\}$.*

Proof. Since the norm $\|\cdot\|_M$ of ℓ_M is solid, the definition of the norm $\|\cdot\|_{\lambda_M\{E\}}$ of $\lambda_M\{E\}$ reveals that it is monotonic ; in particular, if $y=(y_n)\in\lambda_M\{E\}$ then $\|y^{(k)}\|_{\lambda_M\{E\}}\le\|y\|_{\lambda_M\{E\}}$. Consider an element $y\in\lambda_M\{E\}$ which is laying in the closure $\overline{\lambda_M\{E\}_r}$ of $\lambda_M\{E\}_r$ and a positive number δ. One has $z\in\lambda_M\{E\}_r$ and $K\in\mathbb{N}$ for which $\|y-z\|_{\lambda_M\{E\}}<\delta/3$ and $\|z^{(k)}-z\|_{\lambda_M\{E\}}<\delta/3$ if $k\ge K$. So, since $\|\cdot\|_{\lambda_M\{E\}}$ is monotonic,

$$\begin{aligned}\|y^{(k)}-y\|_{\lambda_M\{E\}}&\le\|y^{(k)}-z^{(k)}\|_{\lambda_M\{E\}}+\|z-z^{(k)}\|_{\lambda_M\{E\}}\\&\quad+\|y-z\|_{\lambda_M\{E\}}\\&<2\|y-z\|_{\lambda_M\{E\}}\\&\quad+\delta/3<\delta,\end{aligned}$$

if $k\ge K$. This means that $y\in\lambda_M\{E\}_r$ and $\lambda_M\{E\}_r$ is indeed closed in $\lambda_M\{E\}$. ■

Theorem 4. *Suppose that φ is a mapping which is linear and continuous on $\lambda_M\{E\}$. Define, for every natural number n, the mapping x_n^* on E by setting $x_n^*(t)=\varphi(te_n)$. Then, $(x_n^*)_n\in\lambda_N^*\{E^*\}$. In other words, $(x_n^*)_n$ is absolutely (λ^*,N)-summable in the dual space E^* of E.*

Proof. The continuity of φ provides a positive constant K with the property that

$$|\varphi(y)|\le K\|y\|_{\lambda_M\{E\}},\ \text{whenever}\ y=(y_n)_n\in\lambda_M\{E\}.$$

Now, for a natural number n and a vector z in E, the inequality (2) yields

$$|x_n^*(z)|=|\varphi(ze_n)|\le K\|ze_n\|_{\lambda_M\{E\}}\le K(M(1)+1)\|e_n\|_\lambda\|z\|_E. \qquad (4)$$

By the inequality (4), one has $(x_n^*)_n\in\omega(E^*)$.
The proof that $(x_n^*)_n\in\lambda^*(E^*,N)$ is the only what is remaining. To do so, consider $\alpha=(\alpha_n)\in\lambda$. We will prove that $(\alpha_n\|x_n^*\|)_n\in\ell_N=\ell_M^*$. Let $\gamma=(\gamma_n)\in\ell_M$.

Since $\|\gamma_n\alpha_n x_n^*\|=\sup\{|x_n^*(\gamma_n\alpha_n t)|:t\in B_E\}$, if $\delta>0$, one can find, for every natural number n, a vector t_n in the closed unit ball B_E of E satisfying

$$\|\gamma_n\alpha_n x_n^*\|\le|x_n^*(\gamma_n\alpha_n t_n)|+\frac{\delta}{2^n}.$$

Let $(\varepsilon_n)_n\in\omega$ be such that $|\varphi(\gamma_n\alpha_n t_n e_n)|=\varepsilon_n\varphi(\gamma_n\alpha_n t_n e_n)$. For every $k\in\mathbb{N}$, we have,

$$\begin{aligned}\sum_n^k\|\gamma_n\alpha_n x_n^*\|&\le\sum_n^k|x_n^*(\gamma_n\alpha_n t_n)|+\sum_n^k\frac{\varepsilon}{2^n}=\sum_n^k|\varphi(\gamma_n\alpha_n t_n e_n)|\\&+\delta=\Big|\sum_n^k\varphi(\varepsilon_n\gamma_n\alpha_n t_n e_n)\Big|+\delta\\&=\left|\varphi\left(\sum_n^k\varepsilon_n\gamma_n\alpha_n t_n e_n\right)\right|+\delta\\&\le K\Big\|\sum_n^k\varepsilon_n\gamma_n\alpha_n t_n e_n\Big\|_{\lambda_M\{E\}}+\delta.\end{aligned}$$

Let $(\beta_n)_n\in B_{\lambda^*}$.Then,

$$\Big\|\sum_n^k\varepsilon_n\gamma_n\alpha_n\beta_n\|t_n\|e_n\Big\|_M\le\Big\|\sum_n^k\alpha_n\beta_n\gamma_n e_n\Big\|_M\le\|\alpha\|\|\gamma\|.$$

Thus, $\|\sum_n^k\varepsilon_n\gamma_n\alpha_n t_n e_n\|_{\lambda_M\{E\}}\le\|\alpha\|\|\gamma\|$, which proves that the series $\sum_n^\infty|\gamma_n|\|\alpha_n x_n^*\|$ converges. So, $(x_n^*)_n\in\lambda_N^*\{E^*\}$.

For the $\alpha-$duality, we prove what follows.

Lemma 2. *Denote by $(\lambda_M\{E\})^\times$ the $\alpha-$ dual of $\lambda_M\{E\}$:*

$$(\lambda_M\{E\})^\times=\{(a_n)_n\subset E^*:\sum_{n=1}^\infty|a_n(x_n)|<\infty,\forall(x_n)_n\in\lambda_M\{E\}\}.$$

Then one has the double inclusion $(\lambda_M\{E\}_r)^*\subset(\lambda_M\{E\})^\times\subset(\lambda_N^*\{E^*\})$.

Proof. We first show the inclusion $(\lambda_M\{E\})^*\subset(\lambda_M\{E\})^\times$.
As in the proof of the theorem 4, let φ be in $(\lambda_M\{E\})^*$. For every natural number n and vector $z\in E$, define $b_n(z)=\varphi(ze_n)$. By the continuity of φ one has $\rho>0$ such that

$$|\varphi(y)|\le\rho\|y\|_{\lambda_M\{E\}},\ \text{whenever}\ y=(y_n)_n\ \text{belongs to}\ \lambda_M\{E\}.$$

In particular, we get, by (2),

$$|b_n(z)|=|\varphi(ze_n)|\le\rho\|te_n\|_{\lambda_M\{E\}}\le\rho(M(1)+1)\|e_n\|_\lambda\|z\|_E,$$

for all $n\in\mathbb{N}$ and $z\in E$.
This means that $(b_n)_n\in\omega(E^*)$.
Now, we are ready to prove that $(b_n)_n\in(\lambda_M\{E\}_r)^\times$. Let $x=(x_n)_n\in\lambda_M\{E\}_r$. By the equation (3),

$x = \lim_{k\to\infty} x^{(k)} = \sum_{n=1}^{\infty} x_n e_n$. Since φ is continuous on $\lambda_M\{E\}_r$, we can write

$$\varphi(x) = \varphi(\lim_{k\to\infty} x^{(k)}) = \lim_{k\to\infty} \varphi(x^{(k)}) \quad (5)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} \varphi(x_n e_n) = \sum_{n=1}^{\infty} \varphi(x_n e_n)$$

$$= \sum_{n=1}^{\infty} b_n(x_n). \quad (6)$$

Then, the series $\sum_{n=1}^{\infty} a_n(x_n)$ converges. Actually, it converges absolutely. In fact, let $(\varepsilon_n)_n$ a sequence of real numbers such that

$$|a_n(x_n)| = \varepsilon_n a_n(x_n), \text{for every } n \in \mathbb{N}.$$

It is not hard to verify that $y = (\varepsilon_n x_n)_n$ belongs to $\lambda_M\{E\}_r$ and that

$$\sum_{n=1}^{\infty} |a_n(x_n)| = \varphi(y).$$

Now, let $a = (a_n)_n \in (\lambda_M\{E\})^{\times}$. We have to prove that for all $\alpha = (\alpha_n)_n \in \lambda$, $(\alpha_n \|a_n\|_{E^*}) \in \ell_N$. As in the second part of the proof of (4), since $\ell_N = \ell_M^*$, it is enough to prove that the series $\sum_{n=1}^{\infty} |\gamma_n \alpha_n| \|a_n\|$ converges, for all $(\gamma_n)_n \in \ell_M$. For every $n \in \mathbb{N}$, since

$$\|\gamma_n \alpha_n a_n\| = \sup\{|a_n(\gamma_n \alpha_n t)| : t \in B_E\},$$

there exists $t_n \in B_E$ such that

$$\|\gamma_n \alpha_n a_n\| \le |a_n(\gamma_n \alpha_n t_n)| + \frac{1}{2^n}.$$

But, $(\gamma_n \alpha_n t_n)_n \in (\lambda_M\{E\}$. Indeed, if $\beta = (\beta_n)_n \in \lambda^*$ then $(\gamma_n \alpha_n \|t_n\|)_n \le \|\alpha\| \|\beta\| (\gamma_n)_n$ and ℓ_M is normal.

Now, $\sum_{n=1}^{\infty} |a_n(\gamma_n \alpha_n t_n)| < \infty$ since $a \in (\lambda_M\{E\})^{\times}$ and $(\gamma_n \alpha_n t_n)_n \in \lambda_M\{E\}$, and then $\sum_{n=1}^{\infty} |\gamma_n| \|\alpha_n a_n\| \le \sum_{n=1}^{\infty} |a_n(\gamma_n \alpha_n t_n)| + 1$ is finite too. This completes the proof.

Theorem 5. *Let ψ be the correspondence from $(\lambda_M\{E\}_r)^*$ to $(\lambda^*\{E^*, N\})$ which assigns to every continuous linear form on $\lambda_M\{E\}_r$ the element of $(\lambda^*\{E^*, N\})$ defined by the sequence $b = (b_n)_n$ given in the theorem 4. Then, ψ defines a one- to- one continuous mapping, when these two spaces are endowed with their standard respective norms.*

Proof. If $\varphi \in (\lambda_M\{E\}_r)^*$ the sequence $a = (a_n)_n$ represents φ as seen in (5). So, ψ is well defined. Moreover, using (5), one can see that ψ is linear and one to one. Now, let us prove that ψ is continuous.

Let $a = (a_n)_n$ be an element of $\in \lambda_N^*\{E^*\}$, and $\alpha = (\alpha_n)_n \in B_\lambda$. Since $\ell_N = \ell_M^*$, the norm of $\|(\alpha_n \|a_n\|)_n\|_N$ is defined by,

$$\|(\alpha_n \|a_n\|)_n\|_N = \sup\left\{\sum_{n=1}^{\infty} |\gamma_n| \alpha_n \|a_n\|)_n : \gamma = (\gamma_n) \in B_{\ell_M}\right\}.$$

For $\varepsilon > 0$, a sequence $(t_n)_n \subset B_E$ can be found such that, for every $n \in \mathbb{N}$,

$$\gamma_n \alpha_n \|a_n\| \le a_n(\gamma_n \alpha_n t_n) + \frac{\varepsilon}{2^n}.$$

But, as can be easily seen, $y = (\gamma_n \alpha_n t_n)_n \in \lambda_M\{E\}_r$, and then, if φ is represented by the sequence $(a_n)_n$, we have

$$\sum_{n=1}^{\infty} |\gamma_n| \alpha_n \|a_n\| \le |\varphi(y)| \le \varepsilon + \|\varphi\|_{(\lambda_M\{E\}_r)^*} \|y\|$$

$$\le \varepsilon + \|\alpha\| \|\gamma\|.$$

■

Conflict of Interest

The authors declare that there is no conflict regarding the publication of this paper.

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