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Analysis of Burger Equation Using HPM with General Fractional Derivative

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Abstract: To solve the generalised Burgers equation, the most recent operator in fractional calculus, is introduced in this research work. A more manageable form of problem can be obtained by reducing general fractional derivative into three well-known operators. We use the effective analytical method known as the homotopy perturbation method (HPM) to get generalised Burgers equation's solution. A real-world example is used to demonstrate the findings, and we also analyzed all three reduced operators. A graphical analysis is also supplied to demonstrate how the solution functions. By demonstrating how to solve generalised Burgers problem using this approach and general fractional derivative, this study makes a contribution in field of nonlinear differential equations.

Keywords: Burger equation, generalized derivative, fractional operator, fixed point condition, Homotopy, HPM.

1 Origination

Fractional calculus is an obvious next step after classical calculus. It has grown in recognition and importance during the last few decades in a number of scientific disciplines. Applications of fractional calculus are increasing, which shows that it provides better mathematical representations of everyday items. Any natural or physical phenomena whose outline is useful in the understanding of the issue is sketched using a simulation model. The fractional calculus literature is fast growing as a result of global research activities. Fractional calculus has had an influence on a wide range of fields in literature, including as biology, engineering, fluid, heat conduction, control theory, image processing, visco-elasticity, astronomy, and electricity. ([1]-[13]). As a result, the calculus of fractional order has an impact on every area of technology and study.
Fractional differential equations, which may be linear or nonlinear, are used to solve many scientific concepts. There are several differential equations of fractional order that lack definite solutions. In order to solve such problems, several unique numerical and analytical techniques are defined. HPM approach is a useful way to solve nonlinear equations because of its easy methodology and speedier convergence. This technique was developed by the renowned mathematician He. (see [14]-[16]). The key benefit of this approach is how rapidly the answer discovered in series form. Least repetitions are often necessary to give the most accurate results.
Burgers citebur later reviewed Bateman citebat's first 1915 presentation of the Burgers equation ([17]-[23]). Burger equation is present in its conventional form

$$\frac{\partial v}{\partial t} + af(v)\frac{\partial v}{\partial p} = c\frac{\partial^2 v}{\partial p^2}. \tag{1}$$

In this case, a may be any constant at all. Fractional calculus is a fascinating branch of mathematics, it handles integrals and derivatives of arbitrary order ([24]-[31]). Fractional integrals and derivatives of order $\xi > 0$ have a range of interpretations contrary to conventional definitions for derivatives and integrals. The theory of singular kernels in fractional calculus was greatly influenced by many people, including Samko, Riemann, Caputo, Kilbas, and others.

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Researchers Miller-Ross, Atangana-Baleanu, Wiman, Yang and others investigated integrals and derivatives ([32]-[36]) with kernels without singularity. General fractional derivative is believed to be the most effective way to explain the models of complicated processes. The development of fractional calculus has led to several improvements in a wide range of disciplines, including chemical science, physical science, medical research, and many others.
In fractional calculus, the behaviour of non-smooth functions is described by Caputo and generic fractional derivative, two distinct forms of fractional derivatives, were found in study. In Caputo derivative, we consider a function's initial conditions and is used to represent processes where initial conditions are unknown/unimportant. On other side, a more modern operator, general derivative, offers more freedom in modelling non-smooth systems. Since the function is supposed to be of order ξ which can be broken further into different operators. The beginning conditions of a function are not taken into consideration by the generic fractional derivative, in contrast to the Caputo derivative. We shall examine a generalised Burgers equation by applying generalised fractional derivative:

$$ {}_0^C\mathrm{D}_t^{\xi} v + af(v)\frac{\partial v}{\partial p} = c\frac{\partial^2 v}{\partial p^2} \tag{2} $$

with

$$ v(p,0) = g(p),\ p \in \Omega \tag{3} $$

where ${}_0^C\mathrm{D}_t^{\xi}$ represents generalized derivative of order $\xi \in (0,1]$ with respect to t. $p \in \Omega$ is arbitrary. Generalised derivative is a type of differential operator used in fractional calculus. A more manageable form of the problem can be obtained by reducing general fractional derivative into different operators, further. The generalised Burgers equation is shown in Equation eq (2). We solve equation (2) using the familiar homotopy perturbation technique to get a roughly solution with condition (3).

2 Preliminaries

This section provides some background on recently established generic fractional operator. Its Caputo and Riemann-Liouville derivatives of fractional exponent are provided by researchers, per [37], denoted by

$$ {}_0^C\mathrm{D}_t^{\xi} f(t) = \int_0^t \dot{f}(s)\nabla_\iota(t-s)ds, \tag{4} $$

$$ {}_0\mathrm{D}_t^{\xi} f(t) = \frac{d}{dt}\int_0^t f(s)\nabla_\iota(t-s)ds, \tag{5} $$

here $\xi \in (0,1)$ is exponent of derivative, $f : [0,+\infty) \to \mathbb{R}$ is a continuous function with $\dot{f} \in L^1_{\mathrm{loc}}(0,+\infty), 0 \leqslant t \leqslant T < +\infty$, ∇_ι is known as kernel. The operator is made to adhere to the linear condition,

$$ {}_0^C\mathrm{D}_t^{\xi}(jf(t)+kg(t)) = j{}_0^C\mathrm{D}_t^{\xi} f(t) + k{}_0^C\mathrm{D}_t^{\xi} g(t), \tag{6} $$

$$ {}_0\mathrm{D}_t^{\xi}(jf(t)+kg(t)) = j{}_0\mathrm{D}_t^{\xi} f(t) + k{}_0\mathrm{D}_t^{\xi} g(t). \tag{7} $$

It is evident that for any $t > 0$ as long as certain requirements of $\nabla_\iota(t)$ are met, a completely function of monotone type $\Im_\iota(t)$ occurs. [38],

$$ \nabla_\iota(t) * \Im_\iota(t) = \int_0^\infty \nabla_\iota(s)\Im_\iota(t-s)ds = 1, \tag{8} $$

further, for $f \in L^1_{\mathrm{loc}}(0,+\infty)$, we can rewrite above like

$$ {}_0\mathrm{D}_t^{-\xi}\left[{}_0^C\mathrm{D}_t^{\xi} f(t)\right] = f(t) - f(0), \tag{9} $$

here ${}_0\mathrm{D}_t^{-\xi}$ denotes general Riemann-Liouville integral of fractional order, given as

$$ {}_0\mathrm{D}_t^{-\xi} f(t) = \int_0^t f(s)\Im_\iota(t-s)ds. \tag{10} $$

Right side Caputo and Riemann-Liouville fractional derivatives are

$$ {}_{t}^{C}\mathrm{D}_{T}^{\xi} f(t) = \int_{t}^{T} \dot{f}(s)\nabla_r(s-t)ds, \tag{11}$$

$$ {}_{t}\mathrm{D}_{T}^{\xi} f(t) = \frac{d}{dt}\int_{t}^{T} f(s)\nabla_r(s-t)ds, \tag{12}$$

and

$$ {}_{t}\mathrm{D}_{T}^{-\xi} f(t) = \int_{t}^{T} f(s)\Im_r(s-t)ds. \tag{13}$$

Based on the findings in [39], the integration by component formula is therefore satisfied by the above-mentioned fractional order operators such

$$\int_0^T f(s)_0 D_s^{\xi} g(s)ds = \int_0^T g(s)_s {}^{C}D_s^{\xi} f(s)ds, \tag{14}$$

$$\int_0^T f(s)_0 {}^{C}D_s^{\xi} g(s)ds = \int_0^T g(s)_s D_s^{\xi} f(s)ds. \tag{15}$$

By incorporating the various kernels into the various general operator definitions, we may derive 3 specific cases of general operator. In first case when kernel is $\nabla_\iota(t) = \frac{t^{-\xi}}{\Gamma(1-\xi)}$; so we have power function $\Im_\iota(t) = \frac{t^{\xi-1}}{\Gamma(\xi)}$ reforms integral operator's associated kernel (10).
In next condition, take kernel $\nabla_\iota(t) = \frac{\Pi(\xi)}{1-\xi}\mathrm{E}_\xi\left(\frac{-\xi}{1-\xi}t^{\xi}\right)$ where E_ξ and $\mathrm{M}(\xi)$ are Mittag-Leffler and normalization functions. We also have

$$\Im_\iota(t) = \frac{1-\xi}{\Pi(\xi)}\delta(t) + \frac{\xi}{\mathrm{M}(\xi)\Gamma(\xi)}t^{\xi-1}. \tag{16}$$

So, Equations (4) and (5) may be used to get the derivatives of AB-Caputo and AB-Riemann-Liouville. The AB type integral is [27],

$$ {}_{0}\mathrm{D}_{t}^{-\xi} f(t) = \frac{1-\xi}{\mathrm{M}(p)}f(t) + \frac{\xi}{\mathrm{M}(\xi)\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}f(s)ds \tag{17}$$

Now, at the last situation, CF derivative ([38,40]) is found by taking kernel $\nabla_\iota(t) = \frac{\Pi(\xi)}{1-\xi}\exp\left(\frac{-\xi}{1-\xi}t\right)$.
The seven sections that make up this article's structure are as follows: The pre-requisites are defined in Section 2. We go into the symmetry solution's existence and uniqueness in Section 3 of this article. We go through the HPM's stages and how to apply them to the generalised Burgers problem in segment 4. The convergence analysis is covered in portion 5. In part 6, the HPM was illustrated using a simple example. In addition, we examine how this article ends in Section 7.

3 Existence and Uniqueness of Result

Here, consider T, a constant as $0 < T < \infty$, also $(p,t) \in \Xi \times (0,T]$. Now we will verify existence and uniqueness of result by Banach fixed point. For this, consider Banach space of continuous functions defined on $\Omega \times [0,T]$ ($C(\Omega \times [0,T])$) with

$$\|v\| = \max_{(p,t)\in\Omega\times[0,T]} |v(p,t)|.$$

Below are the defined notations for sake of convenience

$$\frac{\partial v}{\partial p} = v',$$

$$\frac{\partial^2 v}{\partial p^2} = v'',$$

$$f\left(v(p,t), \frac{\partial v(p,t)}{\partial p}, \frac{\partial^2 v(p,t)}{\partial p^2}\right) = -av^m(p,t)v'(p,t) + cv''(p,t), \tag{18}$$
$$:= f(p,t;v(p,t)),v'(p,t),v''(p,t)).$$

Here $f\left(v(p,t), \frac{\partial v(p,t)}{\partial p}, \frac{\partial^2 v(p,t)}{\partial p^2}\right)$ be the functional, $p \in \Omega$ and a are any arbitrary constants.
Leema Consider that ξ lies between 0 and 1, hence following equation fulfill integral equation

$$ {}_0^C\mathrm{D}_t^{\xi} v(p,t) = f(p,t,v(p,t)), v'(p,t), v''(p,t)), $$

with

$$ v(p,0) = g(p). $$

here $p \in \Omega$ and a be any constants.
Proof Above lemma can be demonstrated easily by using integral operator definition.

Theorem If $(\theta_1 + \theta_2\chi_1 + \theta_3\chi_2)\frac{t^{\xi}}{\Gamma(\xi+1)} < 1$, then function f explained in (18) fulfil Lipschitz condition.
Proof We employ the generalised integral operator to determine the existence and uniqueness of the problem. Additionally, these operators decrease in three specific situations, and for each of these situations, we may analyse the problem's existence and unity independently. In such scenario, we first take into account the first case of kernel then

$$ v(p,t) = g(p) + \frac{1}{\Gamma\xi}\int_0^t (t-P)^{\xi-1} f(p,P,v(p,P);v'(p,P);v''(p,P))dP. $$

Suppose $Hu(p,t) = v(p,t)$, and demonstrate the contraction of H. Now, by Banach fixed point theorem, we show that a fixed point belongs to H . It denotes the existence of a one fixed point.

$$ \begin{aligned} & |Hv(p,t) - Hu(p,t)| \\ & \leq \frac{1}{\Gamma(\xi)}\int_0^t (t-P)^{\xi-1} \\ & \quad |f(p,P,v(p,p);v'(p,P);v''(p,P)) - f(p,P,u(p,p);u'(p,P);u''(p,P))|\, dP, \\ & \leq \frac{1}{\Gamma(\xi)}\int_0^t (t-P)^{\xi-1} \\ & \quad (\theta_1|v(p,t)-u(p,t)| + \theta_2|v'(p,t)-u'(p,t)| + \theta_3|v''(p,t)-u''(p,t)|)\, dP, \\ & \leq \frac{(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\Gamma(\xi)}\|u-v\|\int_0^t (t-P)^{\xi-1}\, dP, \\ & \leq \frac{t^{\xi}(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\Gamma(\xi+1)}\|u-v\|. \end{aligned} $$

Therefore, using the assumption $(\theta_1 + \theta_2\chi_1 + \theta_3\chi_2)\frac{t^{\xi}}{\Gamma(\xi+1)} < 1$, we find that H is a contraction.
Further, when kernel is $\nabla_{\iota}(t) = \frac{\Pi(\xi)}{1-\xi}\mathrm{E}_{\xi}\left(\frac{-\xi}{1-\xi}t^{\xi}\right)$ then generalized operator reduces to Atangana Baleanu operator, so integral equation becomes

$$ \begin{aligned} v(p,t) = {} & \frac{1-\xi}{\mathrm{M}(p)}f(p,t) + \frac{\xi}{\mathrm{M}(\xi)\Gamma(\xi)} \\ & \int_0^t (t-P)^{\xi-1} f(p,P,v(p,P);v'(p,P);v''(p,P))dP. \end{aligned} $$

Again, suppose $Hu(p,t) = v(p,t)$, and demonstrate the contraction of H. In the same previous way, we have

$$ \begin{aligned} & |Hv(p,t) - Hu(p,t)| \\ & \leq \frac{\xi}{\mathrm{M}(\xi)\Gamma(\xi)}\int_0^t (t-P)^{\xi-1} \\ & \quad |f(p,P,v(p,P);v'(p,P);v''(p,P)) - f(p,P,u(p,P);u'(p,P);u''(p,P))|\, dP, \\ & \leq \frac{\xi}{\mathrm{M}(\xi)\Gamma(\xi)}\int_0^{\tau} (t-P)^{\xi-1} \\ & \quad (\theta_1|v(p,t)-u(p,t)| + \theta_2|v'(p,t)-u'(p,t)| + \theta_3|v''(p,t)-u''(p,t)|)\, dP, \\ & \leq \frac{\xi}{\mathrm{M}(\xi)}\frac{(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\Gamma(\xi)}\|u-v\|\int_0^{\tau} (t-p)^{\xi-1}\, dP, \\ & \leq \frac{t^{\xi}\xi(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\mathrm{M}(\xi)\Gamma(\xi+1)}\|u-v\| \\ & \quad \frac{t^{\xi}\xi(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\mathrm{M}(\xi)\Gamma(\xi+1)} < 1. \end{aligned} $$

Again, using the assumption $\frac{t^{\xi}\xi(\theta_1+\theta_2\chi_1+\theta_3\chi_2)}{\mathrm{M}(\xi)\Gamma(\xi+1)} < 1$, we again get that H is a contraction. Similarly, we can do for Caputo-Fabrizio operator.

4 Homotopy Perturbation Method

We see that there are several strategies that are occasionally offered by researchers for tackling linear or non-linear issues. Among these, the iteration technique HPM ([41]-[44]) is a potent tool for solving linear or non-linear issues. Here, we provide examples of the HPM. We wish to use the HPM to solve the above equations (eqs. 2 and 3). As a result, we define v like

$$v(p,t;q) : \Xi \times [0,T] \times [0,1] \to \mathbb{R}$$

s.t.

$$\begin{aligned} H\left(v\left(p,t;q\right),q\right) = (1-q)\left[{}_0^C D_t^\xi v(p,t;q) - {}_0^C D_t^\xi u_0(p,t)\right] + \\ q\left[{}_0^C D_t^\xi v(p,t;q) + af(v)(p,t;q)\tfrac{\partial v}{\partial p}(p,t;q) - c\tfrac{\partial^2 v}{\partial p^2}(p,t;q)\right] = 0 \end{aligned} \tag{19}$$

here q is embedding parameter and $u_0(p,t)$ is starting approximation. Now, we get

$${}_0^C D_t^\xi v(p,t;q) = {}_0^C D_t^\xi u_0(p,t) - q\left[{}_0^C D_t^\xi u_0(p,t) + af(v)(p,t;q)\frac{\partial v}{\partial p}(p,t;q) - c\frac{\partial^2 v}{\partial p^2}(p,t;q)\right] \tag{20}$$

Now expanding the $f(v)$ by using the Taylor series, we have

$$\begin{aligned} {}_0^C D_t^\xi v(p,t;q) = {}_0^C D_t^\xi u_0(p,t) - q\left[{}_0^C D_t^\xi u_0(p,t) + a\left\{f(0) + vf'(0) + \tfrac{v^2}{2!}f''(0) + \ldots\right\}\right. \\ \left.\times(p,t;q)\tfrac{\partial v}{\partial p}(p,t;q) - c\tfrac{\partial^2 v}{\partial p^2}(p,t;q)\right] \end{aligned} \tag{21}$$

or,

$$\begin{aligned} {}_0^C D_t^\xi v(p,t;q) = {}_0^C D_t^\xi u_0(p,t) - q\left[{}_0^C D_t^\xi u_0(p,t) + a\sum_{m=0}^{\infty} f^{(m)}(0)\tfrac{v^m}{m!}\right. \\ \left.\times(p,t;q)\tfrac{\partial v}{\partial p}(p,t;q) - c\tfrac{\partial^2 v}{\partial p^2}(p,t;q)\right] \end{aligned} \tag{22}$$

Now, putting $v(p,t;q) = \sum_{k=0}^{\infty} q^k v_k(p,t)$ in above equation, we get

$$\begin{aligned} {}_0^C D_t^\xi \sum_{k=0}^{\infty} q^k v_k = {}_0^C D_t^\xi u_0(p,t) - q\left[{}_0^C D_t^\xi u_0(p,t) + a\sum_{m=0}^{\infty}\tfrac{f^{(m)}(0)}{m!}\left(\sum_{k=0}^{\infty} q^k v_k\right)^m\right. \\ \left.\times\tfrac{\partial}{\partial p}\left(\sum_{k=0}^{\infty} q^k v_k\right) - c\tfrac{\partial^2}{\partial p^2}\left(\sum_{k=0}^{\infty} q^k v_k\right)\right] \end{aligned} \tag{23}$$

Now, we compare coefficients of similar powers of q in Equation 23, we get derivatives of various terms so we get the v_0, v_1, v_2 and so on. Further using those values we get v as

$$v = v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \ldots \tag{24}$$

5 Convergence of Solution

We will examine the convergence of solution given by HPM for generalised Burgers problem in this section.

Theorem Consider $v_n(p,t)$ and $u(p,t)$ are the functions in Banach space $C(\Omega \times [0,T])$ defined in (24). Further suppose that there exists a p, lies in (0,1) as $v_n(p,t) \le pv_{n-1}(p,t)\ \forall\, n \in \mathbb{N}$. Hence we get series $\sum_{l=0}^{\infty} v_l(p,t)$ converges to $u(p,t)$.
Proof Suppose that partial sum of series $\sum_{l=0}^{\infty} v_l(p,t)$ is U_n. Now atfirst we will prove that U_n is Cauchy sequence in Banach space $C(\Omega \times [0,T])$. Hence

$$\begin{aligned} &|U_s(p,t) - U_r(p,t)| \ = \\ &|(U_s(p,t) - U_{s-1}(p,t)) + (U_{s-1}(p,t) - U_{s-2}(p,t)) + \ldots + (U_{r+1}(p,t) - U_r(p,t))| \\ &\le \left|p^s v_0(p,t) + p^{s-1} v_0(p,t) + \ldots + p^{r+1} v_0(p,t)\right| \\ &= p^{r+1}(1 + p + p^2 + \ldots + p^{s-r-1})\,|v_0(p,t)| \\ &= p^{r+1}\frac{(1-p^{(s-r)})}{(1-p)}\,|v_0(p,t)|, \end{aligned} \tag{25}$$

Since p lies between 0 and 1 so

$$\|U_s(p,t) - U_r(p,t)\| \leq \frac{p^{r+1}}{(1-p)} \max_{(p,t)\in \Xi\times[0,T]} |v_0(p,t)|. \tag{26}$$

Again, since v_0 is bounded,

$$\lim_{s,r\to\infty} \|U_s(p,t) - U_r(p,t)\| = 0.$$

Hence by using the concept of Cauchy sequence, we get our expected result.

6 Examples

HPM, which is previously discussed, will be applied as a mathematical technique in this section to solve particular case of generalised Burgers equation. Equations (2) and (3), which reflect this particular case of the generalised Burgers equation, may be simply solved roughly by using the HPM.
Consider $(p,t) \in (0,1)\times(0,T]$, $\xi \in (0,1]$ and following Burger equation

$${}_0^C D_t^\xi v(p,t) = -af(v)(p,t)\frac{\partial v(p,t)}{\partial p} + c\frac{\partial^2 v(p,t)}{\partial p^2}, \tag{27}$$

with the initial condition

$$v_0 = u_0 = p^2. \tag{28}$$

if we take $f(v) = 1$ then the Burger equation becomes

$${}_0^C D_t^\xi + a\frac{\partial v}{\partial p} = c\frac{\partial^2 v}{\partial p^2} \tag{29}$$

Now applying the same methodology as above, we get

$${}_0^C D_t^\xi \sum_{k=0}^{\infty} q^k v_k = {}_0^C D_t^\xi u_0(p,t) - q\left[{}_0^C D_t^\xi u_0(p,t) + a\frac{\partial}{\partial p}\left(\sum_{k=0}^{\infty} q^k v_k\right) - c\frac{\partial^2}{\partial p^2}\left(\sum_{k=0}^{\infty} q^k v_k\right)\right] \tag{30}$$

Further, comparing the like powers of q both sides in above equation, we have

$$q^0 : {}_0^C D_t^\xi v_0 = {}_0^C D_t^\xi u_0(p,t)$$

$$q^1 : {}_0^C D_t^\xi v_1 = -\left[{}_0^C D_t^\xi u_0(p,t) + a\frac{\partial v_0}{\partial p}\right] + c\frac{\partial^2 v_0}{\partial p^2}$$

$$q^2 : {}_0^C D_t^\xi v_2 = -a\frac{\partial v_1}{\partial p} + c\frac{\partial^2 v_1}{\partial p^2}$$

and so on.
Now using generalized integral operator $\mathscr{I}_t^\xi$, we obtain

$$v_0 = I_t^\xi \left({}_0^C D_t^\xi u_0(p,t)\right)$$

$$v_1 = I_t^\xi \left[-\left({}_0^C D_t^\xi u_0(p,t) + a\frac{\partial v_0}{\partial p}\right) + c\frac{\partial^2 v_0}{\partial p^2}\right]$$

$$v_2 = I_t^\xi \left[-a\frac{\partial v_1}{\partial p} + c\frac{\partial^2 v_1}{\partial p^2}\right]$$

and so on.
Once these values have been established, they can be substituted into the power series (24) to produce the resultant value v, yielding the desired result. We will now outline each of the three possible solutions to the aforementioned issue.

6.1 Case I

We may construct three unique examples of the general operator by using different kernels to accepted definitions of operator. In the first of these three scenarios, we take general kernel as $\nabla_t(t) = \frac{t^{-\xi}}{\Gamma(1-\xi)}$; so, associated power function is $\Im_t(t) = \frac{t^{\xi-1}}{\Gamma(\xi)}$. It is capable of producing the type I integral operator with ease. Equation (2) is reduced to type I derivative operator. Consequently, we can simply solve the problem being investigated by applying the above described technique; the stages are shown below

$$v_0 = u_0 = p^2,$$

On using the above relation defined above, we get

$$\frac{\partial}{\partial p} v_0(p,t) = 2p$$

also

$$\frac{\partial^2}{\partial p^2} v_0(p,t) = 2,$$

using aforementioned values, we can write

$$v_1 = (2-\xi)(c-ap)\left(1-\xi+\frac{t^{\xi}}{\Gamma\xi}\right) \tag{31}$$

similarly, we found the value of v_2, we have

$$v_2 = \frac{a^2}{2}(2-\xi)^2\left(1-\xi+\frac{t^{\xi}}{\Gamma\xi}\right)^2 \tag{32}$$

$$\vdots$$

Consequently, it is simple to derive the remaining terms, and utilising relation (24), we can quickly determine the nearby answer.

$$v = p^2 + (2-\xi)(c-ap)\left(1-\xi+\frac{t^{\xi}}{\Gamma\xi}\right) + \frac{a^2}{2}(2-\xi)^2\left(1-\xi+\frac{t^{\xi}}{\Gamma\xi}\right)^2 + \ldots \tag{33}$$

We now pick a = 0.5 and c = 1 to plot numerical results for outcomes. The graphs for ξ= 0.5, 0.8, and 1 have been displayed.

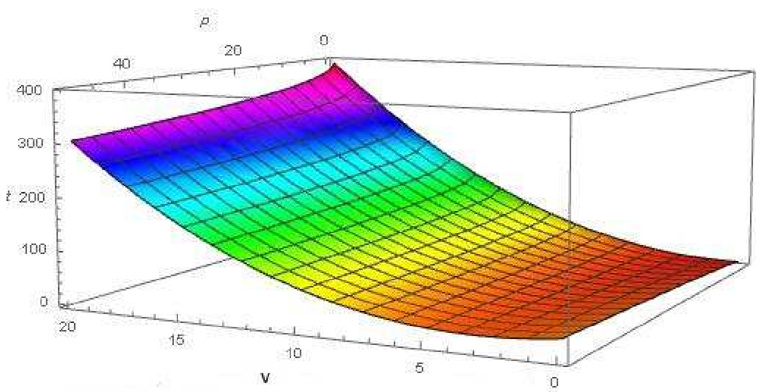

Fig. 1: Demonstration of v in first case for $\xi = 0.5, a = 0.5, c = 1$.

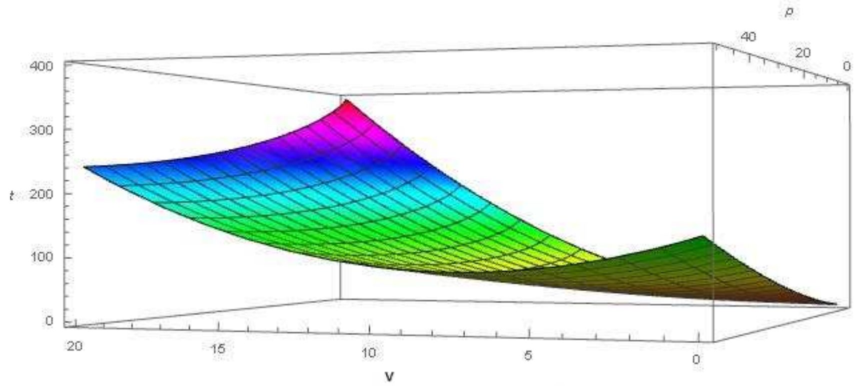

Fig. 2: Representation of v in first case for $\xi = 0.8, a = 0.5, c = 1$.

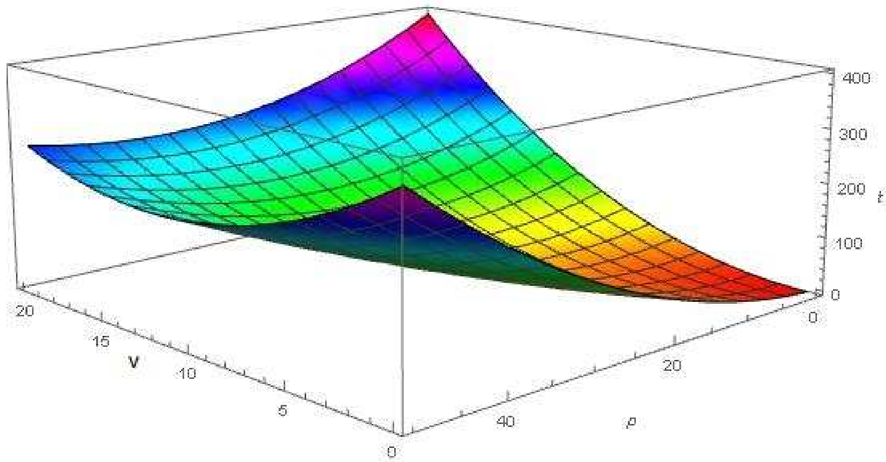

Fig. 3: Figure of v in first case for $\xi = 1, a = 0.5, c = 1$.

6.2 Case II

Now take kernel for general operator like $\nabla_\iota(t) = \frac{M(\xi)}{1-\xi}\exp\left(\frac{-\xi}{1-\xi}t\right)$, where $M(\xi)$ is normalization function. Further, proceed according to steps explained above, we obtain

$$v_0 = u_0 = p^2.$$

Now use homotopy method and relation expressed in (28), we obtain

$$\frac{\partial}{\partial p}v_0(p,t) = 2p$$

and

$$\frac{\partial^2}{\partial p^2}v_0(p,t) = 2,$$

we get

$$v_1 = 2(c - ap)(1 - \xi + \xi t) \tag{34}$$

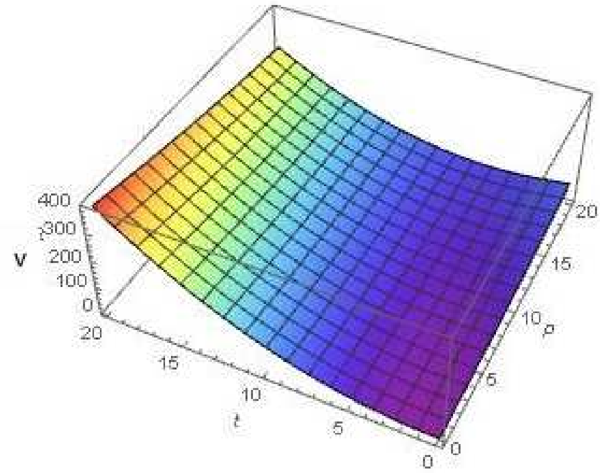

Fig. 4: Graph of v in second case for $\xi = 0.5, a = 0.5, c = 1$.

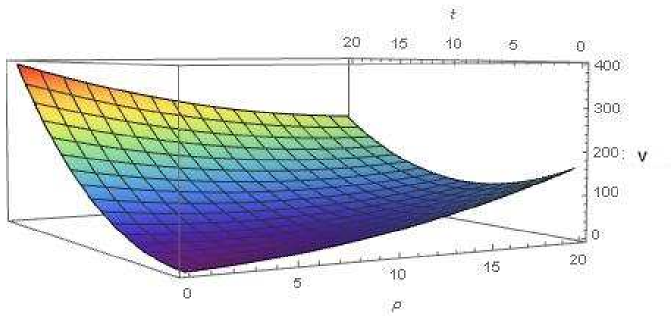

Fig. 5: Graph of v in second case for $\xi = 0.8, a = 0.5, c = 1$.

Next, v_2, we have

$$v_2 = 2a^2(1-\xi+\xi t)^2 \tag{35}$$

$$\vdots$$

Hence, in second case, the approximate solution is

$$v = p^2 + 2(c-ap)(1-\xi+\xi t) + 2a^2(1-\xi+\xi t)^2 + \ldots \tag{36}$$

Again, We take a = 0.5 and c = 1 to plot numerical results for outcomes. The graphs for ξ= 0.5, 0.8, and 1 have been displayed.

6.3 Case III

In this scenario, we take kernel's value for general operator $\nabla_\iota(t) = \frac{\Pi(\xi)}{1-\xi}\mathrm{E}_\xi\left(\frac{-\xi}{1-\xi}t^\xi\right)$, where E_ξ denote Mittag-Leffler function and M(ξ) represents normalization function. On solving in the same way, we have

$$v_0 = u_0 = p^2.$$

Again using homotopy method and relation expressed in (28), we have

$$\frac{\partial}{\partial p}v_0(p,t) = 2p$$

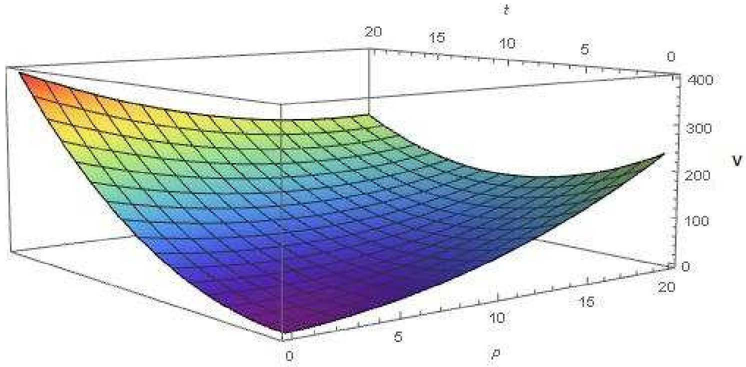

Fig. 6: Graph of v in second case for $\xi = 1, a = 0.5, c = 1$.

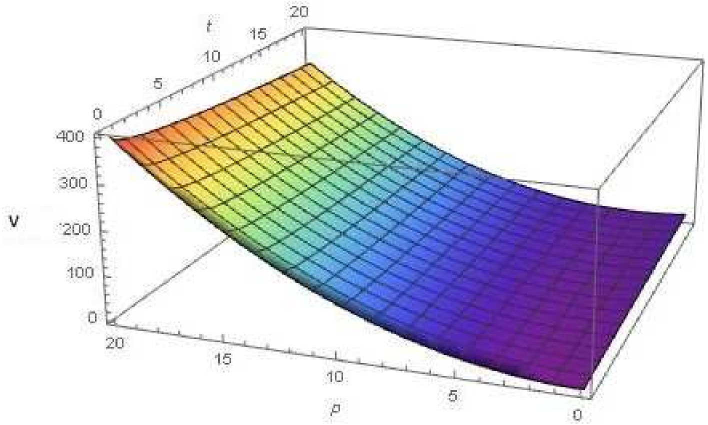

Fig. 7: Graph of v in third case for $\xi = 0.5, a = 0.5, c = 1$.

and

$$\frac{\partial^2}{\partial p^2} v_0(p,t) = 2,$$

we get

$$v_1 = \frac{(2c - 2ap)}{\Gamma \xi} \frac{t^{\xi}}{\xi} \tag{37}$$

Similarly, we can easily find v_2, as

$$v_2 = \frac{2a^2 t^{2\xi}}{(\Gamma(\xi+1))^2} \tag{38}$$

$$\vdots$$

Hence, estimated result is

$$v = p^2 + 2(c - ap) \frac{t^{\xi}}{\Gamma(\xi+1)} + 2a^2 \frac{t^{2\xi}}{(\Gamma(\xi+1))^2} + \ldots \tag{39}$$

One last time, We take a = 0.5 and c = 1 to plot numerical results for outcomes. The graphs of v for ξ= 0.5, 0.8, and 1 have been displayed.

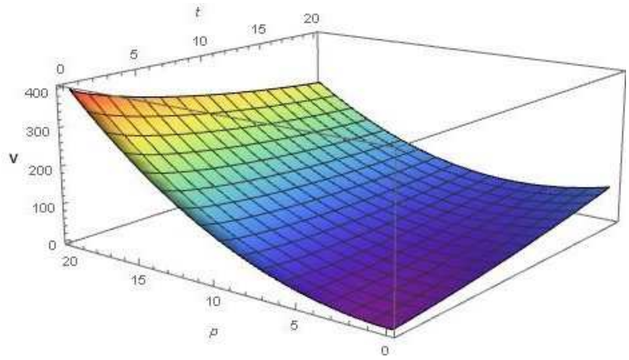

Fig. 8: Graph of v in third case for $\xi = 0.8, a = 0.5, c = 1$.

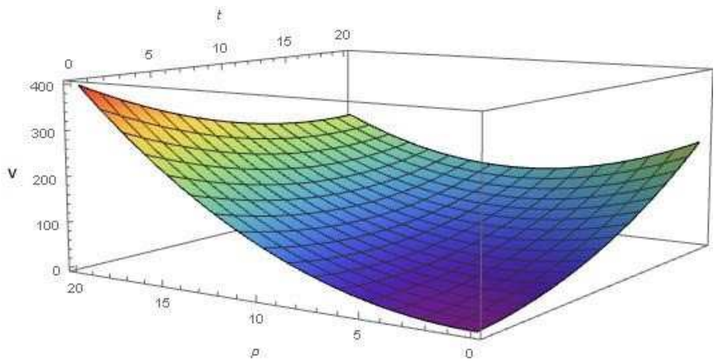

Fig. 9: Graph of v in third case for $\xi = 1, a = 0.5, c = 1$.

7 Conclusion

In this work, we used a generalised fractional operator to analyse the generalised Burgers equation. We solve several well-known instances using the HPM. We determine the defined problem's outcomes and talk about its three distinct outcomes as well. Finally, we exhibit the graph of those cases (figure 1–9) to demonstrate the effectiveness of the generalised operator. Since we already have fractional calculus, dealing with mathematical modelling is made easier. It gives us more accurate outcomes to characterise the physical models. We employ a generalised operator that gives us three specific instances of the renowned fractional operator. We have found the numerical outcomes of the conclusions reached in example. For this, we took $a = 0.5$ and $c = 1$ and made graphs of v for ξ= 0.5, 0.8 and 0.9.

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