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Monotonic and Asymptotic Properties of Solutions of Emden-Fowler Neutral Differential Equations and Their Applications

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Abstract: The objective of this work is to improve the relationships between the solutions of neutral differential equations and their corresponding functions in the classical case. We use these relationships to optimize the conditions that test the oscillation of solutions to Emden-Fowler neutral differential equations. We consider both cases $p < 1$ and $p > 1$. In the case $p > 1$, we test the oscillation of the solutions without imposing the conventional constraints on the delay functions. The approach adopted depends on deducing new properties for the positive solutions of the studied equation, and these properties are of an iterative nature. The iterative nature of properties helps to create relationships and conditions that can be used more than once. By applying the results to special cases of the studied equation, we can clarify the importance of the new results and compare them with the relevant previous results.

Keywords: Emden-Fowler equation; neutral differential equation; the canonical case; oscillation.

1 Introduction

Emden–Fowler equation is attributed to the astrophysicist Jacob Emden (Swiss: 1862–1940) and astronomer Sir Ralph Fowler (English: 1889–1944). Fowler [1] discussed the nature of the solutions to the Emden equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\frac{\mathrm{d}}{\mathrm{d}s}x(s)\right)+s^2x(s)=0,$$

and explained some of its applications in astrophysics. Moreover, in [2], He investigated the equation to explain many fluid mechanics phenomena. The generalization of this equation and its use to simulate various physical processes have since attracted increasing interest [3,4]. The Emden–Fowler with a forced term

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p(s)\frac{\mathrm{d}}{\mathrm{d}s}x(s)\right)+q(s)\left|x(s)\right|^{\alpha-1}\operatorname{sgn}x(s)=g(s),$$

where $s \geq s_0$, $\alpha \geq 1$, and $p, q \in \mathbf{C}([s_0,\infty))$, arises from a certain radial solution of the equation of Klein–Gordon, which is the relativistic version of the Schrödinger equation and used to describe spinless particles. The oscillatory behavior of this equation on a time scale has been studied in [5,6]. There are many physical applications (theoretical, and chemical physics) and engineering applications for Emden-Fowler differential equations with delay and neutral arguments. Therefore, it is easy to note the great interest in studying the asymptotic and oscillatory properties of solutions to these equations; see, for example, [7,8,9,12,13,14,15]. In many applications, a kind of delay differential equations appear which are called neutral differential equations (NDE), see [10].

In this study, we investigate some monotonic properties of positive solutions of the Emden–Fowler NDE

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(a(s)\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}z(s)\right]^{\alpha}\right)+q(s)x^{\alpha}(h(s))=0, \quad (1)$$

where $s \geq s_0$, $n \geq 2$ is an even integer, $\alpha > 0$ is a ratio of odd integers, and $z(s) = x(s) + p(s)x(\delta(s))$. We use these properties to deduce some new inequalities and relationships, and then explain the importance of the new

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relationships by applying them in the oscillation theory. Throughout this study, we hypothesize the following:

H1: $a \in \mathbf{C}^1([s_0,\infty),(0,\infty))$, $a'(s) \geq 0$, p, $q \in \mathbf{C}([s_0,\infty),[0,\infty))$, $p(s) \leq p_0$, and q does not vanish eventually;
H2: δ, $h \in \mathbf{C}([s_0,\infty),\mathbb{R})$, $\delta(s) \leq s$, $h(s) < s$, $h'(s) \geq 0$, and $\lim_{s\to\infty}\delta(s) = \lim_{s\to\infty}h(s) = \infty$.

For a solution of (1), we mean a function $x \in \mathbf{C}([s_x,\infty),\mathbb{R})$, $s_x \geq s_0$, which satisfies $z \in \mathbf{C}^{n-1}([s_x,\infty))$, $a\cdot\left[z^{(n-1)}\right]^{\alpha} \in \mathbf{C}^1([s_x,\infty))$ and x satisfies (1) on $[s_x,\infty)$. We consider only those solutions of equation (1) which are not vanish eventually. If a solution x of (1) is eventually positive or negative, then it is said to be non-oscillatory; otherwise, it is said to be oscillatory.

In the canonical case, that is,

$$\int_{s_0}^{\infty} a^{-1/\alpha}(\ell)\,\mathrm{d}\ell = \infty. \tag{2}$$

Baculikova and Dzurina [11] presented conditions for oscillation of the NDE

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(a(s)\left[\frac{\mathrm{d}}{\mathrm{d}s}z(s)\right]^{\alpha}\right) + q(s)x^{\beta}(h(s)) = 0, \tag{3}$$

under the following conditions

$$\tau\circ\sigma = \sigma\circ\tau,\ \tau'(s) \geq \tau_0 > 0 \text{ and } \sigma(s) \leq \tau(s). \tag{4}$$

The results in [11] stand out because they test the oscillation of (3) when $p_0 \geq 1$ as well as $p_0 < 1$, unlike most other research in the literature that only examine the oscillation when $p < 1$.

For equation (1), Grace et al. [16] and Moaaz et al. [17] used various methods, and enhanced the well-known oscillation results that were documented in the literature.

Theorem 1. *[16, Theorem 6] All solutions of equation (1) oscillate if*

$$\limsup_{s\to\infty}\int_{s_1}^{s}\left[\varphi(\ell)G(\ell)\exp\left(-\int_{h(\ell)}^{\ell}\frac{\alpha}{a^{1/\alpha}(l)\,w(l)}\mathrm{d}l\right) - \frac{a(\ell)\left(\varphi'_+(\ell)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi^{\alpha}(\ell)}\mathrm{d}\ell\right] = \infty,$$

where $\varphi \in \mathbf{C}([s_0,\infty),(0,\infty))$, $G(s) = (1-p(h(s)))^{\alpha}q(s)$, $\varphi'_+(s) := \max\{\varphi'(s),0\}$, *and*

$$w(s) := \int_{s_1}^{s} a^{-1/\alpha}(\ell)\,\mathrm{d}\ell + \frac{1}{\alpha}\int_{s_1}^{s}\left(\int_{s_1}^{\ell} a^{-1/\alpha}(l)\,\mathrm{d}l\right)\left(\int_{s_1}^{h(\ell)} a^{-1/\alpha}(\ell)\,\mathrm{d}\ell\right)^{\alpha} G(\ell)\,\mathrm{d}\ell.$$

Recently, Pátíková and Fišnarová [18] and Jadlovská [19] used improved techniques to investigate the oscillation of equation (1). The results were obtained in [18] using modified Riccati technique. In [19], Jadlovská considered both cases $\delta(s) \leq s$ and $\delta(s) \geq s$.

All of these and other results were used by the traditional relationship $x > (1-p)z$ that links the solution x and its corresponding function z, as well as the known monotonic properties. Nevertheless, Jadlovská [19] improved the monotonic properties of the solutions to the equation (1) by using an iterative approach.

It is easy to notice that the difficulties of studying oscillations for equations of the higher order are more than those of the second order. This is due to a fundamental issue in the theory of oscillation, which is the classification of positive solutions. With the increase in the order of the equation, the probabilities of the signals of the derivatives of the solutions increase, especially in the non-canonical case. Another interesting problem in the study of oscillation is obtaining criteria in the case of x = 1 without finding restrictions on the delay functions. It is worth noting that we will address this problem through the results in this paper.

In this paper, in the canonical case (2), we obtain new monotonic properties of solutions to equation (1) using an iterative approach, which is an extension of the approach used in [16]. Moreover, we employ new properties to obtain a suitable improvement for the relationship $x > (1-p)z$. Then, we find new oscillation criteria using improved characteristics. Through comparisons and examples, we explain the effect of improving characteristics on the oscillation criteria.

2 Second-order equation

In this section, we present new results that improve the monotonic properties and oscillation criteria for solutions of equation (1) when $n = 2$.

2.1 Properties of solutions

In the following, we deduce some new properties and relationships for the positive solutions of the studied equation. For convenience, we define

$$f_{[0]}(s) = s,\ f_{[j]}(s) = f\left(f_{[j-1]}(s)\right), \text{ for } j = 1,2,\ldots,$$

$$\eta_0(v,u) = \int_u^v \frac{1}{a^{1/\alpha}(\ell)}\mathrm{d}\ell,$$

$$B_i(s,m) := \begin{cases} \widetilde{p}_i(s;m) \text{ for } p_0 < 1, \\ \widehat{p}_i(s;m) \text{ for } p_0 > \frac{\eta_0(s,s_0)}{\eta_0(\delta(s),s_0)}, \end{cases}$$

$$\mu_i(s,s_1) = \eta_0(s,s_1) + \frac{1}{\alpha}\int_{s_1}^{s}\eta_0(\ell,s_1)\,\eta_0^{\alpha}(h(\ell),s_1)\,q(\ell)\,B_i^{\alpha}(h(\ell),m)\,\mathrm{d}\ell,$$

$$\eta_{i+1}(s,s_1) = \exp\left[\int_{s_1}^{s}\frac{\mathrm{d}\ell}{\mu_i(\ell,s_1)\,a^{1/\alpha}(\ell)}\right],$$

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for $i = 0, 1, \ldots$, where

$$\widetilde{p}_i(s;m) = \sum_{k=0}^{m} \prod_{i=0}^{2k} p\left(\delta_{[i]}\right) \left[\frac{1}{p\left(\delta_{[2k]}\right)} - 1\right] \frac{\eta_i\left(\delta_{[2k]}, s_0\right)}{\eta_i(s, s_0)},$$

and

$$\widehat{p}_i(s;m) = \sum_{k=1}^{m} \prod_{i=1}^{2k-1} \frac{1}{p\left(\delta_{[i]}^{-1}\right)} \left[\frac{\eta_i\left(\delta_{[2k-1]}^{-1}, s_0\right)}{\eta_i\left(\delta_{[2k]}^{-1}, s_0\right)} - \frac{1}{p\left(\delta_{[2k]}^{-1}\right)}\right].$$

Lemma 1. *[11, Lemma 3] Each corresponding function z of an eventually positive solution x to equation (1) is positive, increasing, and satisfies $\left(a(s)\left[z'(s)\right]^{\alpha}\right)' \leq 0$.*

Lemma 2. *[20, Lemma 1] For any eventually positive solution x to equation (1), if $p_0 < 1$, then*

$$x > \sum_{k=0}^{m} \left(\prod_{i=0}^{2k} p\left(\delta_{[i]}\right)\right) \left[\frac{z\left(\delta_{[2k]}\right)}{p\left(\delta_{[2k]}\right)} - z\left(\delta_{[2k+1]}\right)\right], \quad (5)$$

for any integer $m \geq 0$.

Lemma 3. *For any eventually positive solution x to equation (1), if $p_0 > 1$, then*

$$x > \sum_{k=1}^{m} \left(\prod_{i=1}^{2k-1} \frac{1}{p\left(\delta_{[i]}^{-1}\right)}\right) \left[z\left(\delta_{[2k-1]}^{-1}\right) - \frac{z\left(\delta_{[2k]}^{-1}\right)}{p\left(\delta_{[2k]}^{-1}\right)}\right],$$

for all $\varepsilon \in (0,1)$.

Proof. From the relationship between x and z, we find that

$$\begin{aligned} x(s) &= \frac{1}{p\left(\delta^{-1}\right)} \left[z\left(\delta^{-1}\right) - x\left(\delta^{-1}\right)\right] \\ &= \frac{1}{p\left(\delta^{-1}\right)} z\left(\delta^{-1}\right) - \frac{1}{p\left(\delta^{-1}\right)} \frac{\left[z\left(\delta_{[2]}^{-1}\right) - x\left(\delta_{[2]}^{-1}\right)\right]}{p\left(\delta_{[2]}^{-1}\right)} \\ &= \frac{1}{p\left(\delta^{-1}\right)} z\left(\delta^{-1}\right) - \prod_{i=1}^{2} \frac{1}{p\left(\delta_{[i]}^{-1}\right)} z\left(\delta_{[2]}^{-1}\right) \\ &\quad + \prod_{i=1}^{3} \frac{1}{p\left(\delta_{[i]}^{-1}\right)} \left[z\left(\delta_{[3]}^{-1}\right) - x\left(\delta_{[3]}^{-1}\right)\right], \end{aligned}$$

and so

$$x > \sum_{k=1}^{m} \left(\prod_{i=1}^{2k-1} \frac{1}{p\left(\delta_{[i]}^{-1}\right)}\right) \left[z\left(\delta_{[2k-1]}^{-1}\right) - \frac{z\left(\delta_{[2k]}^{-1}\right)}{p\left(\delta_{[2k]}^{-1}\right)}\right].$$

The proof is complete.

Lemma 4. *Each corresponding function z of an eventually positive solution x to equation (1) satisfies $\left(z(s)/\eta_0(s,s_0)\right)' \leq 0$.*

Proof. Since $\left(a(s)\left[z'(s)\right]^{\alpha}\right)' \leq 0$, we have

$$\begin{aligned} z(s) &\geq \int_{s_0}^{s} \frac{\left(a(\ell)\left[z'(\ell)\right]^{\alpha}\right)^{1/\alpha}}{a^{1/\alpha}(\ell)} \mathrm{d}\ell \\ &\geq \eta_0(s,s_0)\, a^{1/\alpha}(s)\, z'(s), \end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{z(s)}{\eta_0(s,s_0)}\right) \leq 0.$$

The proof is complete.

Lemma 5. *For any eventually positive solution x to equation (1), we have $x(s) > B_0(s,m)\, z(s)$, eventually.*

Proof. Using the facts that $z'(s) > 0$ and $\left(z(s)/\eta_0(s,s_0)\right)' \leq 0$, it follows from Lemma 2 that

$$\begin{aligned} x &> \sum_{k=0}^{m} \left(\prod_{i=0}^{2k} p\left(\delta_{[i]}\right)\right) \left[\frac{z\left(\delta_{[2k]}\right)}{p\left(\delta_{[2k]}\right)} - z\left(\delta_{[2k]}\right)\right] \\ &> z \sum_{k=0}^{m} \left(\prod_{i=0}^{2k} p\left(\delta_{[i]}\right)\right) \left[\frac{1}{p\left(\delta_{[2k]}\right)} - 1\right] \frac{\eta_0\left(\delta_{[2k]}, s_0\right)}{\eta_0(s,s_0)}. \end{aligned}$$

On the other hand, using the monotonic properties of z with the results of Lemma 3, we get

$$\begin{aligned} x &> \sum_{k=1}^{m} \prod_{i=1}^{2k-1} \frac{1}{p\left(\delta_{[i]}^{-1}\right)} \left[\frac{\eta_0\left(\delta_{[2k-1]}^{-1}, s_0\right)}{\eta_0\left(\delta_{[2k]}^{-1}, s_0\right)} - \frac{1}{p\left(\delta_{[2k]}^{-1}\right)}\right] z\left(\delta_{[2k]}^{-1}\right) \\ &> z \sum_{k=1}^{m} \prod_{i=1}^{2k-1} \frac{1}{p\left(\delta_{[i]}^{-1}\right)} \left[\frac{\eta_0\left(\delta_{[2k-1]}^{-1}, s_0\right)}{\eta_0\left(\delta_{[2k]}^{-1}, s_0\right)} - \frac{1}{p\left(\delta_{[2k]}^{-1}\right)}\right]. \end{aligned}$$

The proof is complete.

In the following results, we improve the monotonic properties of the solutions, and then obtain an improved relationship between x and z.

Lemma 6. *For any eventually positive solution x to equation (1), we have, eventually,*

$$z(s) \geq \mu_{i-1}(s,s_1)\, a^{1/\alpha}(s)\, z'(s), \quad (6)$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{z(s)}{\eta_i(s,s_1)}\right) \leq 0, \quad (7)$$

and

$$x(s) > B_i(s,m)\, z(s), \quad (8)$$

for $i = 1, 2, \ldots$.

Proof. For convenience, we assume that $w(s) = a^{1/\alpha}(s)\, z'(s)$. It follows from Lemma 5 that $x(s) > B_0(s,m)\, z(s)$ for $s \geq s_1$. Then, (1) becomes

$$\left[w^{\alpha}(s)\right]' \leq -q(s)\, B_0^{\alpha}(h(s),m)\, z^{\alpha}(h(s)).$$

Thus, we have

$$\begin{aligned} &\left[z(s) - \eta_0(s,s_1)\, w(s)\right]' \\ &= -\eta_0(s,s_1)\, w'(s) \\ &= -\eta_0(s,s_1) \left(\left[w^{\alpha}(s)\right]^{1/\alpha}\right)' \\ &= -\eta_0(s,s_1) \frac{1}{\alpha} w^{1-\alpha}(s) \left[w^{\alpha}(s)\right]' \\ &\geq \frac{1}{\alpha} \eta_0(s,s_1)\, w^{1-\alpha}(s)\, q(s)\, B_0^{\alpha}(h(s),m)\, z^{\alpha}(h(s)). \quad (9) \end{aligned}$$

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From Lemma 4, we note that $z(s) - \eta_0(s,s_1)w(s) \geq 0$. Integrating (9) from s_1 to s, we obtain

$$z(s) \geq \eta_0(s,s_1)w(s) + \frac{1}{\alpha}\int_{s_1}^{s} \frac{\eta_0(\ell,s_1)}{w^{\alpha-1}(\ell)} q(\ell) B_0^{\alpha}(h(\ell),m) z^{\alpha}(h(\ell))\,\mathrm{d}\ell. \quad (10)$$

Using the facts that $(z(s)/\eta_0(s,s_1))' \leq 0$ and $w'(s) \leq 0$, we arrive at

$$z(h(\ell)) \geq \eta_0(h(\ell),s_1)w(h(\ell)) \geq \eta_0(h(\ell),s_1)w(s), \text{ for } s \geq \ell,$$

which with (10) gives

$$z(s) \geq w(s)[\eta_0(s,s_1) + \frac{1}{\alpha}\int_{s_1}^{s} \frac{\eta_0(\ell,s_1)}{\eta_0^{-\alpha}(h(\ell),s_1)} q(\ell) B_0^{\alpha}(h(\ell),m)\,\mathrm{d}\ell] = \mu_0(s,s_1)w(s).$$

Multiplying this inequality by

$$\exp\left[-\int_{s_1}^{s} \frac{\mathrm{d}\ell}{\mu_0(\ell,s_1)a^{1/\alpha}(\ell)}\right],$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{z(s)}{\eta_1(s,s_1)}\right) \leq 0.$$

Now, as in the proof of Lemma 5, we get $x(s) > B_1(s,m)z(s)$.

By repeating the same approach, we get that (6), (7) and (8) hold, for $i = 1,2,\ldots$. The proof is complete.

2.2 Oscillation results

In the following, we use the results in the previous section to obtain an improved oscillation criteria for equation (1). We will deduce the conditions that guarantee that there are no positive solutions. From the fact that every negative value of a positive solution to equation (1) is also a solution, then by excluding positive solutions, we also exclude negative solutions.

Theorem 2. *All solutions of equation (1) oscillate if*

$$\liminf_{s\to\infty} \int_{h(s)}^{s} q(\ell) B_r^{\alpha}(h(\ell),m) \mu_r^{\alpha}(h(\ell),s_0)\,\mathrm{d}\ell > \frac{1}{\mathrm{e}}, \quad (11)$$

for some $r, m \in \mathbb{N}$.

Proof. Assume the contrary that x is an eventually positive solution to equation (1). From Lemma 6, we have

$$x(s) > B_r(s,m)\mu_r(s,s_0)a^{1/\alpha}(s)z'(s),$$

for $r, m \in \mathbb{N}$. Setting $H(s) := a(s)[z'(s)]^{\alpha}$, we have from (1) that

$$H'(s) + q(s)B_r^{\alpha}(h(s),m)\mu_r^{\alpha}(h(s),s_0)H(h(s)) \leq 0.$$

Using Theorem 1 in [21], we conclude that the equation

$$H'(s) + q(s)B_r^{\alpha}(h(s),m)\mu_r^{\alpha}(h(s),s_0)H(h(s)) = 0 \quad (12)$$

has also a positive solution. It follows from Theorem 2 in [22] that (12) is oscillatory under condition (11), a contradiction. The proof is complete.

Theorem 1 used the Riccati technique, where it assumed the Riccati substitution on the form

$$\omega(s) = \varphi(s)a(s)\left[\frac{z'(s)}{z(s)}\right]^{\alpha}.$$

Then, it utilized the relationships $x(s) > (1-p(s))z(s)$ and $(z(s)/\eta_0(s))' \leq 0$, to compensate for $\left(a(s)[z'(s)]^{\alpha}\right)'$ and the ratio $(z\circ\sigma)(s)/z(s)$. The next theorem is obtained directly from the use of relationships (7) and (8) instead of traditional relationships in Theorem 1, and therefore its proof is omitted.

Theorem 3. *All solutions of equation (1) oscillate if there is a $\varphi \in \mathbf{C}([s_0,\infty),(0,\infty))$ such that*

$$\limsup_{s\to\infty} \int_{s_1}^{s} \left[\varphi(\ell) q(\ell) B_r^{\alpha}(h(\ell),m)\left[\frac{\eta_{r+1}(h(\ell),s_1)}{\eta_{r+1}(\ell,s_1)}\right]^{\alpha} - \frac{a(\ell)\left(\varphi'_+(\ell)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi^{\alpha}(\ell)}\right]\mathrm{d}\ell = \infty, \quad (13)$$

for some $r, m \in \mathbb{N}$.

Remark. It is easy to note that Theorem 1 is a special case of Theorem 3, where it is obtained by setting $r = 0$ and $m = 0$.

Example 1. Consider the NDE of Euler type

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}[x(s) + p_0 x(\delta_0 s)] + \frac{q_0}{s^2}x(h_0 s) = 0, \quad (14)$$

where p_0, q_0 are positive and δ_0, $h_0 \in (0,1)$. It is easy to check that

$$B_0 = \begin{cases} [1-p_0]\sum_{k=0}^{m} p_0^{2k}\delta_0^{2k} & \text{for } p_0 < 1, \\ [\delta_0 p_0 - 1]\sum_{k=1}^{m} p_0^{-2k} & \text{for } p_0 > \frac{1}{\delta_0}, \end{cases}$$

and the sequences of functions (μ_i), (η_i), and (B_{i+1}) are defined as the following, respectively:

$$\mu_i(s,s_1) = \frac{1}{\lambda_i}s,$$

$$\eta_{i+1}(s,s_1) = s^{\lambda_i},$$

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and

$$B_{i+1} = \begin{cases} [1-p_0]\sum_{k=0}^{m} p_0^{2k}\left[\delta_0^{2k}\right]^{\lambda_i} & \text{for } p_0 < 1, \\ \left[p_0\delta_0^{\lambda_i}-1\right]\sum_{k=1}^{m} p_0^{-2k} & \text{for } p_0 > \frac{1}{\delta_0}, \end{cases}$$

for $i = 0, 1, \ldots$, where

$$\lambda_i := \frac{1}{1+h_0q_0B_i}, \text{ for } i = 1, 2, \ldots .$$

Condition (11) reduces to

$$\frac{1}{\lambda_r}h_0q_0B_r \ln\frac{1}{h_0} > \frac{1}{\mathrm{e}}. \tag{15}$$

On the other hand, by choosing $\varphi(s) = s$, condition (13) becomes

$$\limsup_{s\to\infty}\int_{s_1}^{s}\left(h_0^{\lambda_r}B_rq_0 - \frac{1}{4}\right)\frac{1}{\ell}\mathrm{d}\ell = \infty,$$

which is satisfied if

$$h_0^{\lambda_r}B_rq_0 > \frac{1}{4}. \tag{16}$$

From Theorems 2 and 3, all solutions of equation (1) oscillate if either (15) or (16) holds.

*Remark.*Consider the following special case of equation (14):

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[x(s)+\frac{4}{5}x\left(\frac{9}{10}s\right)\right]+\frac{q_0}{s^2}x\left(\frac{1}{2}s\right)=0. \tag{17}$$

We have $B_0 = 0.415282$ and

$$\lambda_0 = \frac{1}{1+(0.20764)\,q_0}.$$

Condition (15) and (16) reduce, respectively, to $q_0 \gtrapprox 1.84739$ and $q_0 \gtrapprox 1.0623$.
Corollary 1 in [18] confirms that all solutions of equation (17) oscillate if $q_0 > 2.5$. While Theorem 1 ensures that all solutions of equation (17) oscillate if $q_0 \gtrapprox 2.2057$. Accordingly, our results improve results in [16, 18].

*Remark.*Consider the following special case of equation (14):

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[x(s)+5x\left(\frac{4}{5}s\right)\right]+\frac{q_0}{s^2}x\left(\frac{1}{10}s\right)=0. \tag{18}$$

We have $B_0 = 0.125$ and

$$\lambda_0 = \frac{1}{1+(0.1)\,(0.125)\,q_0}.$$

Condition (15) and (16) reduce, respectively, to $q_0 \gtrapprox 11.21$ and $q_0 \gtrapprox 14.150$.
Theorem 2.2 in [9] confirms that all solutions of equation (18) oscillate if $q_0 \gtrapprox 18.125$. While Corollary 2 in [11] ensures that all solutions of equation (17) oscillate if $q_0 \gtrapprox 12.826$. Accordingly, our results complement results in [9, 11].

*Remark.*It should also be noted that results in [9, 11] considered all cases of p_0, but required constraints in (4) and $h(s) \leq \delta(s)$. While our results do not require these constraints.

*Remark.*To illustrate the importance of criteria of an iterative nature, we consider the special case of (14) when $\delta_0 = 0.8$, $h_0 = 0.5$, $p = 0.71555$, and $q = 1$. Table 1 shows the values of quantity $h_0^{\lambda_r}B_rq_0$ when $r = 0, 1, 2, 3$. Therefore, condition (13) is satisfied when $r = 3$, while fails when $r = 0, 1, 2$.

r	0	1	2	3
$h_0^{\lambda_r}B_rq_0$	0.238763	0.249598	0.249996	0.25001

3 Higher-order equation

In this section, we present new results that improve the monotonic properties and oscillation criteria for solutions of equation (1) when $n \geq 4$. For convenience, we define $\beta = (\alpha+1)^{\alpha+1}$,

$$\phi_0(v,u) := \eta_0(v,u)$$
$$\phi_{j+1}(v,u) := \int_u^v \phi_j(\ell,u)\,\mathrm{d}\ell,$$

for $j = 0, 1, \ldots, n-3$,

$$\mathscr{P}_k(s;m) = \begin{cases} P_1(s;k,m) & \text{for } p_0 < 1, \\ P_2(s;k,m) & \text{for } p_0 > \frac{s}{\delta(s)}, \end{cases}$$

for $i = 0, 1, \ldots$, where

$$P_1(s;\kappa,m) = \sum_{l=0}^{m}\prod_{i=0}^{2l} p\left(\delta_{[i]}\right)\left(\frac{1}{p\left(\delta_{[2l]}\right)}-1\right)\left[\frac{\delta_{[2l]}(s)}{s}\right]^{k/\varepsilon},$$

and

$$P_2(s;k,m) := \sum_{l=1}^{m}\prod_{i=1}^{2l-1}\frac{1}{p\left(\delta_{[i]}^{-1}\right)}\left[\left(\frac{\delta_{[2l-1]}^{-1}}{\delta_{[2l]}^{-1}}\right)^{k/\varepsilon}-\frac{1}{p\left(\delta_{[2l]}^{-1}\right)}\right].$$

Lemma 7.*[23] If $g \in \mathbf{C}^m([s_0,\infty),(0,\infty))$, $g^{(j)}(s) > 0$ for $j = 1, 2, \ldots, m$, and $g^{(m+1)}(s) \leq 0$, then $g(s) \geq \frac{\varepsilon}{m}sg'(s)$, eventually, for all $\varepsilon \in (0,1)$.*

3.1 Properties of solutions

Next, we say that $z(s)$ belongs to class $\mathscr{S}_k$, $0 < k < n$, if $z(s)$ satisfies the following properties:

(i)$z^{(j)}(s) > 0$ for $j = 0, 1, \ldots, k$;
(ii)$(-1)^{j-k}z^{(j)}(s) > 0$ for $j = k+1, k+2, \ldots, n-1$.

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In view of Lemma 2.2.1 in [24], we obtain the following lemma:

Lemma 8. *Each corresponding function $z(s)$ of an eventually positive solution $x(s)$ to equation (1) satisfies $\left(a(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right)' \leq 0$ and belongs to one of the classes $\mathscr{S}_k$ for $k=1,3,...,n-1$.*

Lemma 9. *For $z \in \mathscr{S}_k$, we have that $z(s)/s^{k/\varepsilon}$ is decreasing, for $k=1,3,...,n-1$.*

Proof. Assume that $z \in \mathscr{S}_k$ for $k=1,3,...,n-1$. Then, $z^{(j)}(s)>0$ for $j=0,1,...,k$, and $z^{(k+1)}(s)<0$. Using Lemma 7 with $m=k$, we obtain $z(s) \geq \frac{\varepsilon}{k} s z'(s)$. Then,

$$\left(\frac{z(s)}{s^{k/\varepsilon}}\right)' = \frac{1}{s^{k/\varepsilon+1}}\left[sz'(s)-\frac{k}{\varepsilon}z(s)\right] \leq 0.$$

The proof is complete.

Lemma 10. *For $z \in \mathscr{S}_k$, we have that $x(s) > \mathscr{P}_k(s,n)\, z(s)$, eventually, for $k=1,3,...,n-1$.*

Proof. Using Lemma 9, we obtain $z(s)/s^{k/\varepsilon}$ is decreasing. Then, we find

$$z\left(\delta_{[2l]}(s)\right) \geq \left[\frac{\delta_{[2l]}(s)}{s}\right]^{k/\varepsilon} z(s)$$

and

$$z\left(\delta^{-1}_{[2l-1]}(s)\right) \geq \left[\frac{\delta^{-1}_{[2l-1]}(s)}{\delta^{-1}_{[2l]}(s)}\right]^{k/\varepsilon} z\left(\delta^{-1}_{[2l]}(s)\right).$$

It follows from Lemma 2 that

$$x > \sum_{l=0}^{m}\left(\prod_{i=0}^{2l} p\left(\delta_{[i]}\right)\right)\left[\frac{z(\delta_{[2l]})}{p(\delta_{[2l]})} - z\left(\delta_{[2l]}\right)\right]$$
$$> z\sum_{l=0}^{m}\left(\prod_{i=0}^{2l} p\left(\delta_{[i]}\right)\right)\left[\frac{1}{p(\delta_{[2l]})}-1\right]\left[\frac{\delta_{[2l]}(s)}{s}\right]^{k/\varepsilon}.$$

Moreover, using the results of Lemma 3, we obtain

$$x > \sum_{l=1}^{m}\prod_{i=1}^{2l-1}\frac{1}{p\left(\delta^{-1}_{[i]}\right)}\left[\left(\frac{\delta^{-1}_{[2l-1]}}{\delta^{-1}_{[2l]}}\right)^{k/\varepsilon} - \frac{1}{p\left(\delta^{-1}_{[2l]}\right)}\right] z\left(\delta^{-1}_{[2l]}\right)$$
$$> z\sum_{l=1}^{m}\prod_{i=1}^{2l-1}\frac{1}{p\left(\delta^{-1}_{[i]}\right)}\left[\left(\frac{\delta^{-1}_{[2l-1]}}{\delta^{-1}_{[2l]}}\right)^{k/\varepsilon} - \frac{1}{p\left(\delta^{-1}_{[2l]}\right)}\right].$$

The proof is complete.

Lemma 11. *For $z \in \mathscr{S}_{n-1}$, we have that $z'(s) > \phi_{n-3}(s,s_1)\, a^{1/\alpha}(s)\, z^{(n-1)}(s)$, and $z(s)/\phi_{n-2}(s)$ is decreasing, eventually.*

Proof. Assume that $z \in \mathscr{S}_{n-1}$. Since $\left(a(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right)' \leq 0$, w ehave

$$z^{(n-2)}(s) > \int_{s_1}^{s}\frac{a^{1/\alpha}(\ell)\, z^{(n-1)}(\ell)}{a^{1/\alpha}(\ell)}\mathrm{d}\ell$$
$$\geq \phi_0(s,s_1)\, a^{1/\alpha}(s)\, z^{(n-1)}(s). \qquad (19)$$

Integrating (19) $n-3$ times from s_1 to s, we get

$$z'(s) > \phi_{n-3}(s,s_1)\, a^{1/\alpha}(s)\, z^{(n-1)}(s).$$

From (19), we have $z^{(n-2)}(s)/\phi_0(s)$ is decreasing. Thus,

$$z^{(n-3)}(s) > \int_{s_1}^{s}\phi_0(\ell,s_1)\frac{z^{(n-2)}(\ell)}{\phi_0(\ell,s_1)}\mathrm{d}\ell$$
$$\geq \phi_1(s,s_1)\frac{z^{(n-2)}(s)}{\phi_0(s,s_1)}.$$

Hence, $z^{(n-3)}(s)/\phi_1(s)$ is decreasing. By repeating the same approach, we arrive at $z(s)/\phi_{n-2}(s)$ is decreasing. The proof is complete.

3.2 Oscillation results

Lemma 12. *[24] Assume that $\upsilon \in \mathbf{C}^n([s_0,\infty),(0,\infty))$, $\upsilon^{(n)}$ does not vanish eventually, and $\upsilon^{(n)}$ is of one sign. If $\upsilon^{(n-1)}(s)\,\upsilon^{(n)}(s) \leq 0$ and $\lim_{s\to\infty}\upsilon(s) \neq 0$, then, eventually,*

$$\upsilon(s) \geq \frac{\varepsilon}{(n-1)!}s^{n-1}\left|\upsilon^{(n-1)}(s)\right|,$$

for $\varepsilon \in (0,1)$.

Theorem 4. *All solutions of equation (1) oscillate if*

$$\liminf_{s\to\infty}\int_{h(s)}^{s}\frac{\left[h^{n-1}(\ell)\right]^{\alpha}}{a(h(\ell))}q(\ell)\,\mathscr{P}_k^{\alpha}(h(\ell),m)\,\mathrm{d}\ell$$
$$> \frac{[(n-1)!]^{\alpha}}{\mathrm{e}}, \qquad (20)$$

for all $k=1,3,...,n-1$, and for some $m \geq 0$.

Proof. Assume the contrary that $x(s)$ is an eventually positive solution to equation (1). From Lemma 8, $z \in \mathscr{S}_k$ for $k=1,3,...,n-1$. From Lemma 10, equation (1) becomes

$$\left(a(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right)' + q(s)\,\mathscr{P}_k^{\alpha}(h(s),m)\,z^{\alpha}(h(s)) \leq 0. \qquad (21)$$

It follows from Lemma 12 that

$$0 \geq \left(a(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right)'$$
$$+q(s)\,\mathscr{P}_k^{\alpha}(h(s),m)\left[\frac{\varepsilon h^{n-1}(s)}{(n-1)!}\right]^{\alpha}\left[z^{(n-1)}(h(s))\right]^{\alpha}.$$

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Setting $u(s) := a(s)\left[z^{(n-1)}(s)\right]^{\alpha} > 0$, we have $u(s)$ is a positive solution of

$$u'(s) + \frac{q(s)\mathscr{P}_k^{\alpha}(h(s),m)}{a(h(s))}\left[\frac{\varepsilon h^{n-1}(s)}{(n-1)!}\right]^{\alpha} u(h(s)) \le 0.$$

From Theorem 1 in [21], the delay equation

$$u'(s) + \frac{q(s)\mathscr{P}_k^{\alpha}(h(s),m)}{a(h(s))}\left[\frac{\varepsilon h^{n-1}(s)}{(n-1)!}\right]^{\alpha} u(h(s)) = 0 \quad (22)$$

also has a positive solution. Nevertheless, condition (20) confirms that the solutions to equation (22) are oscillatory.

Theorem 5. *All solutions of equation (1) oscillate if (20) holds for $k = 1,3,\dots,n-3$, and*

$$\limsup_{s\to\infty}\int_{s_1}^{s}\left[\rho(\ell)q(\ell)\mathscr{P}_k^{\alpha}(h(\ell),m)\left(\frac{\phi_{n-2}(h(\ell))}{\phi_{n-2}(\ell)}\right)^{\alpha} - \frac{1}{\beta}\frac{(\rho'(\ell))^{\alpha+1}}{\rho^{\alpha}(\ell)\phi_{n-3}^{\alpha}(\ell,s_1)}\right]d\ell = \infty, \quad (23)$$

for some $m \ge 0$.

Proof. Assume the contrary that $x(s)$ is an eventually positive solution to equation (1). From Lemma 8, $z \in \mathscr{S}_k$ for $k = 1,3,\dots,n-1$. From Lemma 10, equation (1) turns into the form (21). From Theorem 4, we note that condition (20) contradicts the possibility that $z(s)$ belongs to one of the categories $\mathscr{S}_k$ for $k = 1,3,\dots,n-3$.

Assume that $z \in \mathscr{S}_{n-1}$. We define the function

$$w(s) := \rho(s)a(s)\left[\frac{z^{(n-1)}(s)}{z(s)}\right]^{\alpha} > 0.$$

Then,

$$w'(s) = \frac{\rho'(s)}{\rho(s)}w(s) + \rho(s)\left[\frac{\left(a(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right)'}{z^{\alpha}(s)} - \frac{a(s)\left[z^{(n-1)}(s)\right]^{\alpha}}{z^{\alpha+1}(s)}\alpha z'(s)\right],$$

which with (21) gives

$$w'(s) \le \frac{\rho'(s)}{\rho(s)}w(s) - \rho(s)q(s)\mathscr{P}_k^{\alpha}(h,m)\frac{z^{\alpha}(h(s))}{z^{\alpha}(s)} - \rho(s)\frac{a(s)\left[z^{(n-1)}(s)\right]^{\alpha}}{z^{\alpha+1}(s)}\alpha z'(s).$$

From Lemma 11, we have $z'(s) > \phi_{n-3}(s,s_1)a^{1/\alpha}(s)z^{(n-1)}(s)$, and thus

$$\begin{aligned} w'(s) &\le \frac{\rho'(s)}{\rho(s)}w(s) - \rho(s)q(s)\mathscr{P}_k^{\alpha}(h,m)\frac{z^{\alpha}(h(s))}{z^{\alpha}(s)} - \alpha\rho(s)a^{1+1/\alpha}(s)\phi_{n-3}(s,s_1)\left[\frac{z^{(n-1)}(s)}{z(s)}\right]^{\alpha+1} \\ &= \frac{\rho'(s)}{\rho(s)}w(s) - \rho(s)q(s)\mathscr{P}_k^{\alpha}(h,m)\frac{z^{\alpha}(h(s))}{z^{\alpha}(s)} - \alpha\frac{\phi_{n-3}(s,s_1)}{\rho^{1/\alpha}(s)}w^{1+1/\alpha}(s). \end{aligned}$$

Using the fact that $(z(s)/\phi_{n-2}(s))' < 0$, we obtain

$$w'(s) \le \frac{\rho'(s)}{\rho(s)}w(s) - \rho(s)q(s)\mathscr{P}_k^{\alpha}(h,m)\left(\frac{\phi_{n-2}(h(s))}{\phi_{n-2}(s)}\right)^{\alpha} - \alpha\frac{\phi_{n-3}(s,s_1)}{\rho^{1/\alpha}(s)}w^{1+1/\alpha}(s). \quad (24)$$

Using the inequality

$$Aw - Cw^{1+1/\alpha} \le \frac{1}{\beta}\alpha^{\alpha}A^{\alpha+1}C^{-\alpha},$$

with

$$A = \frac{\rho'(s)}{\rho(s)} \text{ and } C = \alpha\frac{\phi_{n-3}(s,s_1)}{\rho^{1/\alpha}(s)},$$

we find

$$\frac{\rho'(s)}{\rho(s)}w(s) - \alpha\frac{\phi_{n-3}(s,s_1)}{\rho^{1/\alpha}(s)}w^{1+1/\alpha}(s) \le \frac{1}{\beta}\frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)\phi_{n-3}^{\alpha}(s,s_1)}.$$

Therefore, (24) reduces to

$$w'(s) \le -\rho(s)q(s)\mathscr{P}_k^{\alpha}(h,m)\left(\frac{\phi_{n-2}(h(s))}{\phi_{n-2}(s)}\right)^{\alpha} + \frac{1}{\beta}\frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)\phi_{n-3}^{\alpha}(s,s_1)}.$$

Integrating this inequality from s_1 to s, we get

$$w(s_1) \ge \int_{s_1}^{s}\left[\rho(\ell)q(\ell)\mathscr{P}_k^{\alpha}(h(\ell),m)\left(\frac{\phi_{n-2}(h(\ell))}{\phi_{n-2}(\ell)}\right)^{\alpha} - \frac{1}{\beta}\frac{(\rho'(\ell))^{\alpha+1}}{\rho^{\alpha}(\ell)\phi_{n-3}^{\alpha}(\ell,s_1)}\right]d\ell,$$

which contradicts condition (23).

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Corollary 1. *All solutions of the equation*

$$\left(a(s)\left[z'''(s)\right]^{\alpha}\right)' + q(s)x^{\alpha}(h(s)) = 0$$

oscillate if

$$\liminf_{s\to\infty}\int_{h(s)}^{s}\frac{\left[h^{3}(\ell)\right]^{\alpha}}{a(h(\ell))}q(\ell)\mathscr{P}_{1}^{\alpha}(h(\ell),m)\,\mathrm{d}\ell > \frac{6^{\alpha}}{\mathrm{e}}$$

and

$$\limsup_{s\to\infty}\int_{s_1}^{s}\left[\rho(\ell)q(\ell)\mathscr{P}_{k}^{\alpha}(h(\ell),m)\left(\frac{\phi_2(h(\ell))}{\phi_2(\ell)}\right)^{\alpha} - \frac{1}{\beta}\frac{(\rho'(\ell))^{\alpha+1}}{\rho^{\alpha}(\ell)\phi_{1-}^{\alpha}(\ell,s_1)}\right]\mathrm{d}\ell = \infty.$$

Remark. It is easy to see that our previous results are an improvement of the related results where the function $\mathscr{P}_k(s,m)$ is used instead of $(1-p(s))$ which provides a better estimate of the relationship between the solution and its corresponding function.

4 Conclusion

In this work, we investigated the oscillatory behavior of Emden-Fowler NDEs of second order. We obtained the new oscillation criteria based on the inference of improved monotonic properties of an iterative nature. Examples and remarks compared our results with relevant results in the literature, and emphasized the importance of the new criteria.

Despite the large number of works that dealt with the issue of studying the oscillatory behavior of solutions to second-order differential equations, this issue is still very rich in interesting analytical points. It is interesting to extend our results to higher-order equations.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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