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Characterization of the Generalized Life Model Based on the Generalized Order Statistics

M. Maswadah and M. Seham *

Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt

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Abstract: In this paper, a characterization of the generalized life model based on some recurrence relations for single and product moments based on the generalized order statistics are obtained. The results presented here are a generalization of the recurrence relations for single and product moments of many lifetime distributions in the literature, which are special cases such as the generalized Pareto model and the generalized Weibull model based on the ordinary order statistics.

Keywords: Characterizations; Generalized order statistics; Record values; Single and product moments; Generalized Weibull distributions; Burr type-XII distribution; Cauchy distribution; Power function distribution.

1 Introduction

Recently, in reliability theory, the generalized life model (GLM) has become more applicable in lifetime distribution that has continuous distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$, which are given respectively as:

$$F(x) = 1 - [ah(x) + b]^c, x \in (\alpha, \beta), \tag{1}$$

$$f(x) = -ach'(x)[ah(x) + b]^{c-1}, x \in (\alpha, \beta), \tag{2}$$

where $h(x)$ is a monotonic, continuous and differentiable function on (α, β), $a \neq 0$, $F(\alpha) = 0$ and $F(\beta) = 1$. Thus from (1) and (2) we have the relation

$$(ah(x) + b)f(x) = -ach'(x)[1 - F(x)]. \tag{3}$$

The class of the GLM includes among others the generalized Pareto model, the generalized Weibull model, the Burr type-XII model, and the Compound Weibull model. It a worthwhile to mention that the results presented here are a generalization of the recurrence relations for single and product moments of many distributions based on the ordinary order statistics (OS) and upper record values in the literature. The concept of the generalized order statistics (GOS) is introduced by [1]as a unified approach to the ordinary OS, record values and k-record values, which can be outlined as: The random variables $X(r,n,m,k),\ r = 1,2,...,n$ be GOS from an absolutely cdf $F(x)$ and pdf $f(x)$. Then their joint pdf can be written as:

$$f(x_1, x_2, ..., x_n) = C \prod_{i=1}^{n-1} f(x_i) [1 - F(x_i)]^m [1 - F(x_n)]^{k-1} f(x_n), \tag{4}$$

on the cone $F^{-1}(0) < x_1 < .... < x_n < F^{-1}(1)$ of R^n, where $C = \prod_{i=1}^{n} \gamma_i$, $\gamma_i = k + (n-i) + M_i > 0, M_i = \sum_{j=i}^{n-1} m_j$, $i = 1, ..., n-1$, $\gamma_n = k > 0$, and $\widetilde{m} = (m_1, m_2, ..., m_{n-1}) \in R^{n-1}$.

- If $m = 0$ and $k = 1$, then (4) is the joint pdf of the ordinary OS..
- If $m = -1$ and $k = 1$, then (4) is the joint pdf of the first n upper record values $Y_{U(1)} < Y_{U(2)} < .... < Y_{U(n)} < .....$
- If $m = -1$ and $k \neq 1$, then (4) is the joint (pdf) of the k-record values.

* Corresponding author e-mail: seham_mohamed@sci.aswu.edu.eg

Here we assume two cases:

Case I: $m_1 = m_2 = ... = m_{n-1} = m$.

Case II: $\gamma_i \neq \gamma_j,\ i,j = 1,2,...,n-1$.

For case I, we can derive from (4) the pdf of the r-th GOS $X(r,n,m,k)$ as

$$f_r(x) = \frac{c_{r-1}}{(r-1)!}\,(1-F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{5}$$

and the joint pdf of X(r,n,m,k) and X(s,n,m,k), for $x < y$ and $r < s$ as

$$f_{r,s}(x,y) = C_{r,s}(1-F(x))^m f(x) g_m^{r-1}(F(x)\ \ [h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s - 1} f(y), \tag{6}$$

where for $0 < x < 1$, we have

$$h_m(x) = \{ \begin{matrix} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -log(1-x), & m = -1 \end{matrix},$$

$$g_m(x) = \{ \begin{matrix} \frac{1}{m+1}[1-(1-x)^{m+1}], & m \neq -1 \\ -log(1-x), & m = -1 \end{matrix}$$

$C_{r-1} = \prod_{j=1}^{r} \gamma_j,\ \ r = 1,2,...,n$ and $C_{r,s} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$.

For case II, we can derive from (4) the pdf of the r-th GOS $X(r,n,m,k)$ as

$$f_r(x) = C_{r-1} \sum_{i=1}^{r} a_i(r)\,[1-F(x)]^{\gamma_i - 1} f(x), \tag{7}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, for $x < y$ and $r < s$ as

$$f_{r,s}(x,y) = C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s)\,[1-F(x)]^{\gamma_j} \left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i} \frac{f(x)f(y)}{[1-F(x)][1-F(y)]}, \tag{8}$$

where $a_i = a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i},\ \gamma_i \neq \gamma_j,\ 1 \leq i < r \leq n$
and $a_i^r(s) = \prod_{j=r+1, j\neq i}^{s} \frac{1}{\gamma_j - \gamma_i},\ \gamma_i \neq \gamma_j,\ r+1 \leq i < s \leq n$
Therefore

$$\left.\begin{aligned} a_{i(r-1)} &= (\gamma_r - \gamma_i) a_{i(r)} \\ a_i^r(r-1) &= (\gamma_r - \gamma_i) a_i^r(r) \\ a_i^r(s) &= (\gamma_r - \gamma_i) a_i^{r-1}(s) \end{aligned}\right\}. \tag{9}$$

2 Characterization of Distribution When

$\mathbf{m_i = m_j = m;\ i,\ j = 0,1,2,...n-1}$

For simplicity, let us denote the i-th moment $E(h^i(X(r,n,m,k))$ by $\mu^i_{r:n}$ and the product moments $E(h^i(X(r,n,m,k))g^j(X(s,n,m,k)))$ by $\mu^{i,j}_{r,s:n}$.

Relation1 :

For $2 \leq r \leq n$ and $\ i = 0,1,2,...$

$$\mu^{i+1}_{r:n} = \frac{c\gamma_r}{c\gamma_r + (i+1)} \mu^{i+1}_{r-1:n} - \frac{b(i+1)}{a(c\gamma_r + (i+1))} \mu^i_{r:n}, \tag{10}$$

if and only if F(x) satisfies (1).

Proof:

First, we will prove (1) implies(10). From (5), for $2 \leq r \leq n$ and $i = 0, 1, 2, \ldots$

$$\mu_{r:n}^{i+1} = \frac{C_{r-1}}{(r-1)!} \int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx.$$

Integrating by parts, treating $[1-F(x)]^{\gamma_r-1}$ as the part for integration and the rest of

the integrand for differentiation, we obtain

$$\mu_{r:n}^{i+1} = \frac{(i+1)C_{r-1}}{\gamma_r (r-1)!} \int_{\alpha}^{\beta} h^i(x) h'(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r} dx$$

$$+ \frac{(r-1)C_{r-1}}{\gamma_r (r-1)!} \int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-2}(F(x)) [1-F(x)]^{\gamma_r + m} f(x) dx.$$

Upon using $\gamma_{r-i} = \gamma_r + i(m+1), C_{r-1} = \gamma_r C_{r-2}$ and (3) we obtain

$$\mu_{r:n}^{i+1} = \mu_{r-1:n}^{i+1} - \frac{(i+1)C_{r-1}}{c\gamma_r (r-1)!} \int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx$$

$$- \frac{b(i+1)C_{r-1}}{ca\gamma_r (r-1)!} \int_{\alpha}^{\beta} h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx$$

$$= \mu_{r-1:n}^{i+1} - \frac{(i+1)}{c\gamma_r} \mu_{r:n}^{i+1} - \frac{b(i+1)}{ca\gamma_r} \mu_{r:n}^{i}.$$

The recurrence relation (10) is derived simply by rewriting the above equation, hence the ‘ if ’ part.

To prove (10) implies (1), we have from (10) that

$$[c\gamma_r + (i+1)] \frac{C_{r-1}}{(r-1)!} \int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx$$

$$= ac\gamma_r \frac{c_{r-2}}{(r-2)!} \int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-2}(F(x)) [1-F(x)]^{\gamma_r + m} f(x) dx$$

$$- b(i+1) \frac{c_{r-1}}{(r-1)!} \int_{\alpha}^{\beta} h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx.$$

Integrating the first integral on the right-hand side by parts with treating $g_m^{r-2}(F(x)) [1-F(x)]^m f(x)$

as the part for integration and the rest of the integrand for differentiation, we get after simplification that

$$\int_{\alpha}^{\beta} h^{i+1}(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx$$

$$= -ac \int_{\alpha}^{\beta} h'(x) h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r} dx$$

$$- b \int_{\alpha}^{\beta} h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} f(x) dx.$$

Thus, this equation can be rewritten as

$$\int_{\alpha}^{\beta} h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1} . [ah(x) f(x) + ach'(x)[1-F(x)] + bf(x)] dx = 0.$$

It follows from Lin [2] that $\eta(x) = h^i(x) g_m^{r-1}(F(x)) [1-F(x)]^{\gamma_r - 1}$ is complete.Thus, from the completeness

property, we have $[ah(x) + b] f(x) = -ach'(x)[1-F(x)]$. From the last equation by using separation of variables we get (1). □

Corollary 1:

For $2 \leq r \leq n,\ i = 0$ and $a = b = 1$ in (10) we get

$$\mu_{r:n} = \frac{c\gamma_r}{c\gamma_r + 1} \mu_{r-1:n} - \frac{1}{c\gamma_r + 1}.$$

Repeating this relation, we get

$$\mu_{r:n} = \prod_{i=1}^{r} \frac{c\gamma_i}{c\gamma_i+1}\mu_{1:n} - \sum_{i=1}^{r} \frac{1}{1+c\gamma_i} \prod_{j=i+1}^{r} \frac{c\gamma_j}{c\gamma_j+1},$$

with noting that $\prod_{j=r+1}^{r}(.) = 1$.

Corollary 2:

For record values $\gamma_i = k$ for all i, thus the relations (10) become

$$\mu_{r:n}^{i+1} = \frac{ck}{ck+(i+1)}\mu_{r-1:n}^{i+1} - \frac{b(i+1)}{a(ck+(i+1))}\mu_{r:n}^{i}.$$

Relation 2:

for $1 \le r < s-2 \le n$ and $i, j = 0, 1, 2, \ldots$

$$\mu_{r,s:n}^{i,j+1} = \frac{c\gamma_s}{c\gamma_s+(j+1)}\mu_{r,s-1}^{i,j+1} - \frac{b(j+1)}{a[c\gamma_s+(j+1)]}\mu_{r,s}^{i,j}, \tag{11}$$

if and only if F(x) satisfies(1).

Proof:

First, we will prove (1) implies (11).
From (6), for $1 \le r < s-2 \le n$ and $i, j = 0, 1, 2, \ldots$

$$\mu_{r,s:n}^{i,j+1} = C_{r,s}\int_{\alpha}^{\beta} g^i(x) g_m^{r-1}(F(x))\,[1-F(x)]^m f(x) I(x)dx,$$

where

$$I(x) = \int_x^{\beta} h^{j+1}(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)dy.$$

Integrating $I(x)$ by parts with treating $[1-F(y)]^{\gamma_s-1}$ as the part of integration and the rest of the integration as the part of differentiation, we obtain

$$I(x) = \frac{(s-r-1)}{\gamma_s}\int_x^{\beta} h^{j+1}(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-2}\,[1-F(y)]^{\gamma_s+m} f(y)dy$$
$$+\frac{(j+1)}{\gamma_s}\int_x^{\beta} h^{j}(y)h'(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s}dy.$$

Upon using (3) in $I(x)$ we get

$$I(x) = \frac{(s-r-1)}{\gamma_s}\int_x^{\beta} h^{j+1}(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-2}\,[1-F(y)]^{\gamma_s+m} f(y)dy$$
$$-\frac{(j+1)}{c\gamma_s}\int_x^{\beta} h^{j+1}(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)dy$$
$$-\frac{b(j+1)}{ca\gamma_s}\int_x^{\beta} h^{j}(y)\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)dy.$$

Substituting I(x) into $\mu_{r,s:n}^{i,j+1}$, using $\gamma_{r-i} = \gamma_r + i(m+1)$ and $C_{r-1} = \gamma_r C_{r-2}$ we obtain

$$\mu_{r,s:n}^{i,j+1} = \mu_{r,s-1:n}^{i,j+1} - \frac{(j+1)C_{r,s}}{c\gamma_s}\iint_{\alpha<x<y<\beta} g^i(x)h^{j+1}(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$
$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy$$
$$-\frac{b(j+1)C_{r,s}}{ca\gamma_s}\iint_{\alpha<x<y<\beta} g^i(x)h^{j}(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$
$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy.$$

Thus,

$$\mu_{r,s:n}^{i,j+1} = \mu_{r,s-1:n}^{i,j+1} - \frac{(j+1)}{c\gamma_s}\mu_{r,s:n}^{i,j+1} - \frac{b(j+1)}{ac\gamma_s}\mu_{r,s:n}^{i,j}.$$

The relation (11) can be derived simply by rewriting the above equation, hence the 'if' part.To prove (11) implies (1), we have from (11) that

$$a[c\gamma_s + (j+1)]c_{r,s}\iint_{\alpha<x<y<\beta} g^i(x)h^{j+1}(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy$$

$$= ac(s-r-1)c_{r,s}\iint_{\alpha<x<y<\beta} g^i(x)h^{j+1}(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-2}\,[1-F(y)]^{\gamma_s+m} f(y)f(x)dxdy$$

$$-b(j+1)c_{r,s}\iint_{\alpha<x<y<\beta} g^i(x)h^j(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy.$$

Integrating the first integral on the right-hand side by parts with treating $[h_m(F(y)) - h_m(F(x))]^{s-r-2}[1-F(y)]^m f(y)$ as the part for integration and the rest of the integrand for differentiation, we get after simplification that

$$a\iint_{\alpha<x<y<\beta} g^i(x)h^{j+1}(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy$$

$$= ac\iint_{\alpha<x<y<\beta} g^i(x)h^j(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-2}\,[1-F(y)]^{\gamma_s} f(y)f(x)dxdy$$

$$-b\iint_{\alpha<x<y<\beta} g^i(x)h^j(y)g_m^{r-1}(F(x))\,[1-F(x)]^m$$

$$.\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[1-F(y)]^{\gamma_s-1} f(y)f(x)dxdy.$$

Thus, this equation can be rewritten as

$$\iint_{\alpha<x<y<\beta} g^i(x)h^j(y)g_m^{r-1}(F(x))[1-F(x)]^{\gamma_s-1}\,[h_m(F(y)) - h_m(F(x))]^{s-r-1} f(x)$$

$$.[ah(y)f(y) + ach'(y)[1-F(y)] + bf(y)]dydx = 0.$$

It follows from [2] that $\eta(x) = g^i(x)h^i(x)g_m^{r-1}(F(x))[1-F(x)]^{\gamma_r-1}\,[h_m(F(y)) - h_m(F(x))]^{s-r-1} f(x)$ is complete.
Thus, from the completeness property, we have $[ah(y)+b]f(y) = -ach'(y)[1-F(y)]$. From the last equation by using separation of variables we get (1). □

Corollary 3:

The recurrence relation (11)can be used in a simple recursive manner to compute all the single and product moments of all the order statistics. By setting i=j=0 and $a = b = 1$ in (11) we get the relation

$$\mu_{s:n} = \frac{\gamma_s}{\gamma_s+1}\mu_{s-1:n} + \frac{1}{\gamma_s+1}$$

and letting $i = 1, j = 0$ and $a = b = 1$ in (11) we get the relation

$$\mu_{r,s:n} = \frac{\gamma_s}{\gamma_s+1}\mu_{r,s-1:n} + \frac{1}{\gamma_s+1}\mu_r$$

which together immediately yields for $1 \le r < s-2 \le n$,

$$Cov(X_r, X_s) = \frac{\gamma_s}{\gamma_s+1}Cov(X_r, X_{s-1}),$$

and for $r \ge 1$,

$$Cov(X_r, X_{r+1}) = \frac{\gamma_{r+1}}{\gamma_{r+1}+1}Cov(X_r).$$

3 Characterization of Distribution When

$\gamma_i \neq \gamma_j, i, j = 1, 2, ...n-1$

Proof of Relation 1:

First, we will prove (1)implies (10). From (7), for $2 \le r \le n$ and $i, j = 0, 1, 2, ...$

$$\mu_{r:n}^{i+1} = C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$= C_{r-1} \sum_{j=1}^{r} a_j(r) \frac{[\gamma_j+(\gamma_r-\gamma_j)]}{\gamma_r} \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$= \frac{C_{r-1}}{\gamma_r} \sum_{j=1}^{r} a_j(r)\gamma_j \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$+\frac{C_{r-1}}{\gamma_r} \sum_{j=1}^{r} a_j(r))(\gamma_r-\gamma_j) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx.$$

Integrating the first integral by parts with treating $[1-F(x)]^{\gamma_j-1} f(x)$ as the part of integration and using (9) in the second integral we get

$$\mu_{r:n}^{i+1} = \frac{C_{r-1}(i+1)}{\gamma_r} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i}(x)h'(x)[1-F(x)]^{\gamma_j} dx$$

$$+C_{r-2} \sum_{j=1}^{r-1} a_j(r-1) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx.$$

Using (3) in the first integral

$$\mu_{r:n}^{i+1} = \mu_{r-1:n}^{i+1} - \frac{(i+1)}{c\gamma_r} C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$-\frac{b(i+1)}{ca\gamma_r} C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i}(x)[1-F(x)]^{\gamma_j-1} f(x)dx.$$

Then

$$\mu_{r:n}^{i+1} = \mu_{r-1:n}^{i+1} - \frac{(i+1)}{c\gamma_r} \mu_{r:n}^{i+1} - \frac{b(i+1)}{ca\gamma_r} \mu_{r:n}^{i}.$$

The recurrence relation (11) is derived simply by rewriting the above equation, hence the 'if ' part. To prove (11) implies (1), we have from (11) that

$$a[c\gamma_r + (i+1)] C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$= ac\gamma_r C_{r-2} \sum_{j=1}^{r} a_j(r-1) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$-b(i+1) C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i}(x)[1-F(x)]^{\gamma_j-1} f(x)dx.$$

Using (9) in the second integral, the last equation can be written as

$$a(i+1) C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$= -ac\gamma_r C_{r-1} \sum_{j=1}^{r} \gamma_j a_j(r) \int_{\alpha}^{\beta} h^{i+1}(x)[1-F(x)]^{\gamma_j-1} f(x)dx$$

$$-b(i+1) C_{r-1} \sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i}(x)[1-F(x)]^{\gamma_j-1} f(x)dx.$$

Integrating the first integral on the right-hand side by parts with treating $[1-F(x)]^{\gamma_j-1} f(x)$ as the part for integration and the rest of the integrand for differentiation, we get after simplification that

$$\sum_{j=1}^{r} a_j(r) \int_{\alpha}^{\beta} h^{i}(x)[1-F(x)]^{\gamma_j-1} [ah(x)f(x) + ach'(x)[1-F(x)] + bf(x)]dx = 0.$$

It follows from Lin [2] that $\eta(x) = h^i(x)[1-F(x)]^{\gamma_r-1}$ is complete, thus from the completeness property, we have $[ah(x)+b]f(x) = -ach'(x)[1-F(x)]$. From the last equation by using separation of variables we get (1). □

Proof of Relation 2:

First, we will prove (1) implies (11). From (8), for $1 \le r < s-2 \le n$ and $i, j = 0, 1, 2, ...$, we have

$$\mu_{r:s}^{t,v+1} = C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \iint_{\alpha<x<y<\beta} g^t(x) h^{v+1}(y) [1-F(x)]^{\gamma_j}$$

$$.[\tfrac{1-F(y)}{1-F(x)}]^{\gamma_i} \tfrac{f(x)f(y)}{[1-f(x)][1-f(y)]} dxdy$$

$$= \tfrac{C_{s-1}}{\gamma_s} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) [\gamma_i + (\gamma_s - \gamma_i)] \int_{\alpha}^{y} g^t(x) [1-F(x)]^{\gamma_j} I(x) \tfrac{f(x)}{[1-F(x)]^{\gamma_i+1}} dx,$$

where

$$I(x) = \int_x^{\beta} h^{v+1}(y) [1-F(y)]^{\gamma_i} \tfrac{f(y)}{[1-F(y)]} dy.$$

Using (9) we get

$$\mu_{r:s}^{t,v+1} = \mu_{r:s-1}^{t,v+1} + \tfrac{C_{s-1}}{\gamma_s} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \gamma_i$$

$$. \int_{\alpha}^{y} g^t(x) [1-F(x)]^{\gamma_j} I(x) \tfrac{f(x)}{[1-F(x)]^{\gamma_i+1}} dx.$$

Integrating $I(x)$ by parts with treating $h^{v+1}(y)$ as the part of differentiation and using (3) we get

$$I(x) = -\tfrac{v+1}{c\gamma_i} \int_x^{\beta} h^{v+1}(y) [1-F(y)]^{\gamma_i-1} f(y) dy - b\tfrac{v+1}{ca\gamma_i} \int_x^{\beta} h^{v}(y) [1-F(y)]^{\gamma_i-1} f(y) dy.$$

Substituting I(x) into $\mu_{r,s}^{t,v+1}$ we get

$$\mu_{r,s:n}^{t,v+1} = \mu_{r,s-1:n}^{t,v+1} - \tfrac{(v+1)}{c\gamma_s} \mu_{r,s:n}^{t,v+1} - \tfrac{b(v+1)}{ca\gamma_s} \mu_{r,s:n}^{t,v}.$$

Rearranging the last relation we get (11), hence the ‘ if ’ part □
To prove (11) implies (1), we have from (11) that

$$a[c\gamma_s + (v+1)] C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s). \int_x^{\beta} h^{v+1}(y) [1-F(x)]^{\gamma_i-1} I(y) f(y) dy$$

$$= ac\gamma_s C_{s-2} \sum_{j=1}^{r} \sum_{i=r+1}^{s-1} a_j(r) a_i^r(s-1) \int_x^{\beta} h^{v+1}(y) [1-F(x)]^{\gamma_i-1} I(y) f(y) dy$$

$$-b(v+1) C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \int_x^{\beta} h^{v}(y) [1-F(x)]^{\gamma_i} I(y) f(y) dy,$$

where

$$I(y) = \int_{\alpha}^{y} g^t(x) [1-F(x)]^{\gamma_j} \tfrac{f(x)}{[1-F(x)]^{\gamma_i+1}} dx.$$

Using (9) in the first integral of the right-hand side we get after simplification that

$$a(v+1) C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \int_x^{\beta} h^{v+1}(y) [1-F(y)]^{\gamma_i-1} I(y) f(y) dy$$

$$= -acC_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \gamma_i \int_x^{\beta} h^{v+1}(y) [1-F(y)]^{\gamma_i-1} I(y) f(y) dy$$

$$-b(v+1) C_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \int_x^{\beta} h^{v}(y) [1-F(y)]^{\gamma_i} I(y) f(y) dy.$$

Integrating the first integral on the right-hand side by parts with treating $[1-F(y)]^{\gamma_i-1} f(y)$ as the part for integration and the rest of the integrand for differentiation and using (3) we get after simplification that

$$\sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^r(s) \int_x^{\beta} h^{v}(y) [1-F(y)]^{\gamma_i} [ah(y)f(y) + ach'(y)[1-F(y)] + bf(y)] dy = 0.$$

It follows from Lin [2] that $\eta(x) = h^v(y)[1-F(y)]^{\gamma_i-1} I(y)$ is complete, thus from the completeness property, we have $[ah(y)+b]f(y) = -ach'(y)[1-F(y)]$. From the last equation by using separation of variables we get (1). □

4 Special Cases and Remarks

1.The relation (10) can be deduced from the relation (11) by putting I equal zero.

2.Setting $k = 1$ and $m = 0$ in the deduced recurrence relations (10) and (11), we get the corresponding relations for the ordinary order statistics.

3.Setting $m = -1$ in the deduced recurrence relations (10) and (11),we get the corresponding relations for the k-th record values.

4.Setting $k = 1$ and $m = -1$ in the deduced recurrence relations (10)and (11),we get the corresponding relations for the upper record values.

5.Setting $c = \frac{1}{\lambda}$, $a = -\lambda\beta$, b=1 and $h(x) = x^{\alpha}$in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Generalized Weibull distribution, see [3].

6.Setting $c \to \infty, a = \frac{-\theta}{c}, b = 1$ and $h(x) = x^2$ in the recurrence relations (10) and (11) we deduce the corresponding recurrence relations characterize the Rayleigh distribution, see [4].

7.Setting $c \to \infty, a = \frac{-\theta}{c}$, b=1 and $h(x) = x$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Exponential distribution, see [4,5,6].

8.Setting $c \to \infty, a = \frac{-\lambda}{c}$, b=1 and $h(x) = x^p$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the ordinary Weibull distribution, see [7].

9.Setting $c = 1, a = -1, b = 1$ and $h(x) = exp(-\theta x^{-p})$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Inverse Weibull distribution, see [7].

10.Setting $c = 1, a = -a^b, b = 1$ and $h(x) = x^p$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Pareto distribution, see [8].

11.Setting $c = -m, a = \theta, b = 1$ and $h(x) = x^p$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Burr Type-XII distribution, see [8].

12.Setting $c = \frac{-1}{\theta}, a = \theta, b = 1$ and $h(x) = x$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Generalized Pareto distribution see [8,9].

13.Setting $c = 1, a = -a^{-p}, b = 1$ and $h(x) = x^p$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the power function distribution, see [10].

14.Setting $c = 1, a = -1, b = 1$ and $h(x) = x$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the uniform distribution, see [10,11].

15.Setting $c = -m, a = \frac{1}{\theta}, b = 1$ and $h(x) = x$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Lomax distribution, see [12].

16.Setting $c = -m, a = \frac{1}{\theta}, b = 1$ and $h(x) = x^p$ in the recurrence relations (10) and (11), we deduce the corresponding recurrence relations characterize the Compound Weibull distribution.

5 Conclusion

The general lifetime distribution contains some of the lifetime distributions most used in reliability and survival analyses. These distributions have the flexibility in describing the lifetime variables of constant and non-constant hazard rate and they are useful for modeling and analyzing lifetime data in medical, biological, and engineering sciences. Thus, the recurrence relations for single and product moments in this work will be useful for estimating the characteristics of these distributions such as the means, standard deviations, skewness, and kurtoses with reliable and easy way to apply, especially for researchers in social sciences and psychology.

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