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Strong Fuzzy Chromatic Polynomial of Intuitionistic Fuzzy Graphs (IFGs) Based on (α,β)-Levels

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Abstract: This research introduces a new idea of (α,β)-level based strong fuzzy chromatic polynomial of intuitionistic fuzzy graph(IFGs). In addition, some characteristics of (α,β)-level based strong fuzzy chromatic polynomials of IFGs are specified and proven. Besides, the strong (α,β)-fundamental set and the strong (α,β)-level graphs are defined with clear examples. Moreover, some algebraic characteristics of the strong (α,β)-level graph of IFGs and their chromatic polynomials are also projected and shown.

Keywords: Chromatic polynomial, Intuitionistic fuzzy graph, Strong (α,β)-level

1 Introduction

Zadeh [1] introduced a fuzzy set that was intended to assign a member of a given universal set X containing a set C with the degree of its membership value in C [1]. Based on this set Kaufmann [2] developed a fuzzy graph. Later, Rosenfeld [3] advanced and developed many of the structures of fuzzy graph. Atanassov [4] introduced Intuitionistic fuzzy set that is an extension of the fuzzy set. After a decade, Atanassov [5] defined IFG. Later, more properties of IFGs were presented [6][7]. M. Akram et. al., provides method for calculating the distance matrix's sum, eccentricity, diameter, and radius of IFG [8]. We apply IFGs in Election process [9], cell grouping [10] and water supply [11] and also it is used in medical diagnostics, pipeline and decision-making [12]. Mamo and Srinivasa Rao Repalle presented the fuzzy graphs' chromatic polynomial based on α-levels [13]. Rifayathali et.al., developed the coloring of IFGs [14]. Prasanna et al., introduced the strong coloring of IFGs [15]. Mohideen et. al., presented IFGs' coloring based on (α,β)-levels [16]. M. Akram developed various characteristics of level graph of the IFGs using (α,β)- [17].
Although many works were reported on the IFG and on their coloring, a study on its strong fuzzy chromatic polynomial has not been reported yet. To fill this gap, the authors have dealt with a strong fuzzy chromatic polynomial based on the strong (α,β)-levels. This article is structured as follows: Section two consists of the preliminaries that are necessary for understanding the article. The third part introduces the concept of strong (α,β)-level graphs of the IFG and their properties. And also, it introduces the idea of the strong (α,β)-fundamental set of the IFG. The fourth section defines notion of strong fuzzy chromatic polynomial of IFGs based on (α,β)-levels and presents certain related properties. Lastly, section five summarizes the article.

2 Preliminaries

This section provides essential definitions and propositions.

Definition 1. *[19] The chromatic polynomial of a simple graph W with k given colors is represented by $P(W,k)$ and it counts all ways to reach a correct vertex coloring.*

Proposition 1. *[19] Let F be a simple graph. Then, $P(F,k) = P(F-g,k) - P(F/g,k)$ where $F-g$ is obtained by deleting edge g from F and F/g is obtained by contacting edge g from F.*

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Proposition 2. *[19]. If a simple graph $F = \cup_{i=1}^{n} F_i$ and $\cap_{i=1}^{n} F_i = \emptyset$, then*

$$P(F,k) = P(F_1,k) \times P(F_2,k) \times \cdots \times P(F_n,k)$$

Definition 2. *[4] An IFS Q in W is defined as $Q = \{(w, \mu_Q(w), \gamma_Q(w)), w \in W\}$. Where $\forall w \in W$ the functions, $\mu_Q : W \to [0,1]$ and $\gamma_Q : W \to [0,1]$ shows the membership degree and non-membership degree of the element $w \in W$ respectively.*

Definition 3. *[18] An IFG $G = (V, E)$ is a graph that satisfy;*

1. $V = \{w_1, w_2, \cdots, w_n\}$ *such that $\mu_1 : V \to [0,1]$ and $\gamma_1 : V \to [0,1]$ represent the membership degree and the non-membership degree of element w_i in V respectively where $0 \le \mu_1(w_i) + \gamma_1(w_i) \le 1$ for every $w_i \in V, (i = 1,2,...,n)$.*
2. $\mu_2 : E \to [0,1]$ *and $\gamma_2 : E \to [0,1]$ are in such a way that $\mu_2(w_i, w_j) \le \min\{\mu_1(w_i), \mu_1(w_j)\}$ and $\gamma_2(w_i, w_j) \le \max\{\gamma_1(w_i), \gamma_1(w_j)\}$ satisfy the constraint $0 \le \mu_2(w_i, w_j) + \gamma_2(w_i, w_j) \le 1$ for every $(w_i, w_j) \in E \subseteq V \times V, (i, j = 1,2,...,n)$ where $\mu_2(w_i, w_j)$ and $\gamma_2(w_i, w_j)$ are the degree of membership and the degree of non-membership of the element $(w_i, w_j) \in E$ respectively.*

Definition 4. *[16], [17]. An (α,β)-level graph of an IFG, H is $H_{\alpha,\beta} = (V_{\alpha,\beta}, E_{\alpha,\beta})$ where,*

$$V_{\alpha,\beta} = \{v \in V | \mu_1(v) \ge \alpha \text{ and } \gamma_1(v) \le \beta\} \text{ and}$$

$$E_{\alpha,\beta} = \{v, w \in V | \mu_2(v,w) \ge \alpha \text{ and } \gamma_2(v,w) \le \beta\}.$$

Definition 5. *[16] Let $H = (V,E)$ be an IFG. $\chi(H) = \{(w, m(w), n(w)) | w \in W\}$ such that:*

1. $W = \{1, 2, \cdots, |V|\}$
2. $m(w) = \sup\{\alpha \in [0,1] | w \in A_{\alpha,\beta}\}$ *and* $n(w) = \inf\{\beta \in [0,1] | w \in A_{\alpha,\beta}\}$ *where* $A_{\alpha,\beta} = \{1,2,\cdots,\chi_{\alpha,\beta}\}$.

3 The strong (α,β)-levels of an IFG

This section defines a notion of $(\alpha,\beta)_s$ level graphs and strong fundamental set of an IFG. Moreover, some characterizations of the strong (α,β)-level graphs are provided with justifications.

Definition 6. *Let H be an IFG and suppose that $I = \{\alpha_m, \beta_m\}_{m=1}^{q-1}$ satisfy $\alpha_{m+1} > \alpha_m$ and $\beta_{m+1} < \beta_m$ $\forall m = 1,2,\ldots,q-1$. Then $F_s = \{(0,1)\} \cup I$, is said to be $(\alpha,\beta)_s$-fundamental set of H and read as the strong (α,β)-fundamental set of H.*

Definition 7. *Let F_s be the strong fundamental set of an IFG, $H = (V,E)$. Then the strong (α,β)-level graph of H denoted by $H_{(\alpha,\beta)_s}$ is $H_{(\alpha,\beta)_s} = \left(V_{(\alpha,\beta)_s}, E_{(\alpha,\beta)_s}\right)$ such that $V_{(\alpha,\beta)_s} = \{w \in V \mid \mu_1(w) > \alpha \text{ and } \gamma_1(w) < \beta\}$ and $E_{(\alpha,\beta)_s} = \{(w,x) \in V \times V \mid \mu_2(w,x) > \alpha \text{ and } \gamma_2(w,x) < \beta\}$ where $(\alpha,\beta)_s \in [0,1]$.*

Example 1. Take the IFG provided in figure 1 to illustrate various $(\alpha,\beta)_s$ -level graphs of G.

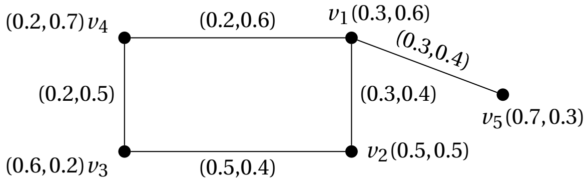

Fig. 1: IFG

The $(\alpha,\beta)_s$-fundamental set of G is: $F_s = \{(0,1)_s, (0.2,0.7)_s, (0.3,0.6)_s, (0.5,0.5)_s, (0.6,0.2)_s, (0.7,0.3)_s\}$. Based on this set, the corresponding strong level graphs are discussed in figure 2 below:

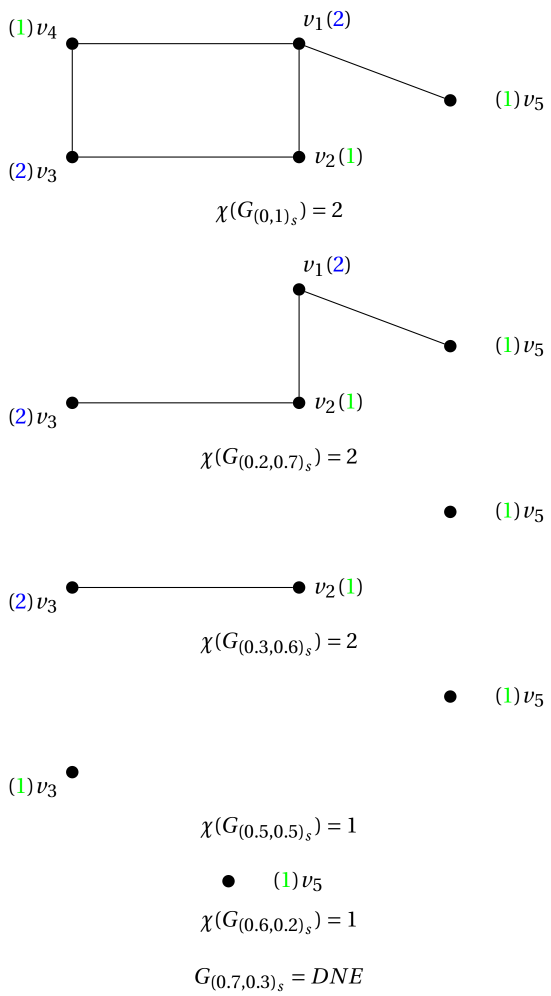

Fig. 2: The strong level graphs of the graph in figure 1

Remark.

1. In figure 2, the number assigned to each vertex is the proper color given for the vertex.
2. Since $V_{(0.7,0.3)_s}$ is an empty set, $G_{(0.7,0.3)_s}$ does not exist.
3. $\chi(G) = \{(2,(0.3,0.6)_s),(1,(0.6,0.2)_s)\}$.

3.1 Characteristics of the strong (α,β)-level graphs of an IFG

Theorem 1. *If $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ are the strong level graphs of an IFG, H with $0 \le \alpha_l < \alpha_m \le 1$ and $1 \ge \beta_l > \beta_m \ge 0$, then $V_{(\alpha_l,\beta_l)_s} \supseteq V_{(\alpha_m,\beta_m)_s}$ and $E_{(\alpha_l,\beta_l)_s} \supseteq E_{(\alpha_m,\beta_m)_s}$.*

Proof. Let $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ be two strong level graphs of an IFG, $H = (V,E)$ with $0 \le \alpha_l < \alpha_m \le 1$ and $1 \ge \beta_l > \beta_m \ge 0$. Then the strong (α,β)-level graphs of H;$H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ have the vertex sets that are given by $V_{(\alpha_1,\beta_l)_s} = \{w \in V \mid \mu_1(w) > \alpha_l \text{ and } \gamma_l(w) < \beta_l\}$ and $V_{(\alpha_m,\beta_m)_s} = \{w \in V \mid \mu_1(w) > \alpha_m \text{ and } \gamma_1(w) < \beta_m\}$ respectively. Now suppose $w \in V_{(\alpha_m,\beta_m)_s}$. Then $w \in V_{(\alpha_m,\beta_m)_s} \Rightarrow \mu_1(w) > \alpha_m$ and $\gamma_1(w) < \beta_m$. Since $\alpha_m > \alpha_l$ and $\beta_m < \beta_l$. Then it implies, $\mu_1(w) > \alpha_m > \alpha_l$ and $\gamma_1(w) < \beta_m < \beta_l$. Further, this indicates $\mu_1(w) > \alpha_l$ and $\gamma_1(w) < \beta_l$. This shows, $w \in V_{(\alpha_l,\beta_l)_s}$. Hence, $V_{(\alpha_l,\beta_l)_s} \supseteq V_{(\alpha_m,\beta_m)_s}$.
Similarly, edge sets of $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ are: $E_{(\alpha_l,\beta_l)_s} = \{(w,y) \in E \mid \mu_2(w,y) > \alpha_l \text{ and } \gamma_2(w,y) < \beta_l\}$ and

$E_{(\alpha_m,\beta_m)_s} = \{(w,y) \in E \mid \mu_2(w,y) > \alpha_m$ and $\gamma_2(w,y) < \beta_m\}$ respectively. Suppose that $(w,y) \in E_{(\alpha_m,\beta_m)_s}$. Now $(w,y) \in E_{(\alpha_m,\beta_m)_s} \Rightarrow \mu_2(w,y) > \alpha_m$ and $\gamma_2(w,y) < \beta_m$. Since $\alpha_m > \alpha_l$ and $\beta_m < \beta_l$. Then this implies, $\mu_2(w,y) > \alpha_m > \alpha_l$ and $\gamma_2(w,y) < \beta_m < \beta_l$. Further, this indicates $\mu_2(w,y) > \alpha_l$ and $\gamma_2(w,y) < \beta_l$ which shows $(w,y) \in E_{(\alpha_l,\beta_l)_s}$. Thus, $E_{(\alpha_l,\beta_l)_s} \supseteq E_{(\alpha_m,\beta_m)_s}$.

Corollary 1. *If $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ are the strong level graphs of an IFG, H such that $\alpha_l < \alpha_m$ and $\beta_l > \beta_m$, then $|V_{(\alpha_m,\beta_m)_s}| \le |V_{(\alpha_l,\beta_l)_s}|$ and $|E_{(\alpha_m,\beta_m)_s}| \le |E_{(\alpha_l,\beta_l)_s}|$.*

Proof. Take an IFG, $H = (V,E)$. Let $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ be the strong level graphs of H with $\alpha_l < \alpha_m$ and $\beta_l > \beta_m$. Then by theorem 1, $V_{(\alpha_l,\beta_l)_s} \supseteq V_{(\alpha_m,\beta_m)_s}$ and $E_{(\alpha_l,\beta_l)_s} \supseteq E_{(\alpha_m,\beta_m)_s}$. This shows $|V_{(\alpha_m,\beta_m)_s}| \le |V_{(\alpha_l,\beta_l)_s}|$ and $|E_{(\alpha_m,\beta_m)_s}| \le |E_{(\alpha_l,\beta_l)_s}|$. Hence, the corollary holds.

Theorem 2. *If $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ are the strong level graphs of an IFG, H such that $\alpha_l < \alpha_m$ and $\beta_l > \beta_m$, then $H_{(\alpha_m,\beta_m)_s}$ is a sub graph of $H_{(\alpha_l,\beta_l)_s}$.*

Proof. Take an IFG, $H = (V,E)$. Let $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ be the strong level graphs of H and let $(\alpha_l,\beta_l)_s$ and $(\alpha_m,\beta_m)_s$ be given. Since $\alpha_l < \alpha_m$ and $\beta_l > \beta_m$, $V_{(\alpha_m,\beta_m)_s} \subseteq V_{(\alpha_l,\beta_l)_s}$ and $E_{(\alpha_m,\beta_m)_s} \subseteq E_{(\alpha_l,\beta_l)_s}$. Hence, $H_{(\alpha_m,\beta_m)_s}$ is a sub graph of $H_{(\alpha_l,\beta_l)_s}$.

Corollary 2. *If $(\alpha_k,\beta_k)_s, (\alpha_l,\beta_l)_s$ and $(\alpha_m,\beta_m)_s$ are the strong levels of an IFG, $G = (V,E)$ such that $\alpha_k < \alpha_l, \alpha_l < \alpha_m, \beta_k > \beta_l$ and $\beta_l > \beta_m$, then $G_{(\alpha_m,\beta_m)_s}$ is a sub graph of $G_{(\alpha_k,\beta_k)_s}$.*

Proof. The statement holds automatically from the theorem 2 and applying the transitivity property.

Theorem 3. *If $H_{(\alpha_k,\beta_k)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ are the strong level graphs of an IFG, H such that $\alpha_k < \alpha_m$ and $\beta_m < \beta_k$, then $\left(H_{(\alpha_k,\beta_k)_s} \cup H_{(\alpha_m,\beta_m)_s}\right) = H_{(\alpha_k,\beta_k)_s}$.*

Proof. Let $H_{(\alpha_k,\beta_k)_s}$ and $H_{(\alpha_m,\beta_m)_s}$ be the level graphs of an IFG, H such that $\alpha_k < \alpha_m$ and $\beta_k > \beta_m$. Then $\alpha_k < \alpha_m$ and $\beta_k > \beta_m$ implies that $V_{(\alpha_m,\beta_m)_s} \subseteq V_{(\alpha_k,\beta_k)_s}$ and $E_{(\alpha_m,\beta_m)_s} \subseteq E_{(\alpha_k,\beta_k)_s}$. Now by the definition of union of sets, $V_{(\alpha_k,\beta_k)_s} \cup V_{(\alpha_m,\beta_m)_s} = V_{(\alpha_k,\beta_k)_s}$ and $E_{(\alpha_k,\beta_k)_s} \cup E_{(\alpha_m,\beta_m)_s} = E_{(\alpha_k,\beta_k)_s}$. Hence, $\left(H_{(\alpha_k,\beta_k)_s} \cup H_{(\alpha_m,\beta_m)_s}\right) = H_{(\alpha_k,\beta_k)_s}$.

Theorem 4. *If $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_k,\beta_k)_s}$ are the strong level graphs of an IFG, H such that $\alpha_l < \alpha_k$ and $\beta_l > \beta_k$, then $\left(H_{(\alpha_l,\beta_l)_s} \cap H_{(\alpha_k,\beta_k)_s}\right) = H_{(\alpha_k,\beta_k)_s}$.*

Proof. Let $H_{(\alpha_l,\beta_l)_s}$ and $H_{(\alpha_k,\beta_k)_s}$ be the level graphs of an IFG, H such that $\alpha_l < \alpha_k$ and $\beta_l > \beta_k$. Then $\alpha_l < \alpha_k$ and $\beta_l > \beta_k$ implies that $V_{(\alpha_k,\beta_k)_s} \subseteq V_{(\alpha_l,\beta_l)_s}$ and $E_{(\alpha_k,\beta_k)_s} \subseteq E_{(\alpha_l,\beta_l)_s}$. By the definition of intersection of two sets, $V_{(\alpha_l,\beta_l)_s} \cap V_{(\alpha_k,\beta_k)_s} = V_{(\alpha_k,\beta_k)_s}$ and $E_{(\alpha_l,\beta_l)_s} \cap E_{(\alpha_k,\beta_k)_s} = E_{(\alpha_k,\beta_k)_s}$. Hence, $\left(H_{(\alpha_k,\beta_k)_s} \cap H_{(\alpha_l,\beta_l)_s}\right) = H_{(\alpha_k,\beta_k)_s}$.

4 Strong fuzzy chromatic polynomial of the IFGs based on (α,β) -levels

This section defines (α,β)-level based strong fuzzy chromatic polynomial of an IFG and derive the chromatic polynomials for illustrative example. Also it states and shows some related concepts.

Definition 8. *Let k colors be given and let F_s be the strong fundamental set of an IFG, H. Then strong fuzzy chromatic polynomial of H based on (α,β)-levels which is denoted by $P^I_{(\alpha,\beta)_s}(H,k)$ is defined as: $P^I_{(\alpha,\beta)_s}(H,k) = P(H_{(\alpha,\beta)_s},k)$, $\forall(\alpha,\beta)_s \in F_s$.*

Example 2. Take k distinct colors to properly color an IFG, G in figure 1. Then $P^I_{(\alpha,\beta)_s}(G,k)$ is $P(G_{(\alpha,\beta)_s},k)$ $\forall(\alpha,\beta)_s \in F_s$ in figure 2 and computed as follows:

1. For $\alpha = 0, \beta = 1$, $P^I_{(0,1)_s}(G,k) = P(G_{(0,1)_s},k)$ is computed using proposition 1 as in figure 3 below: $P^I_{(0,1)_s}(G,k) = P(G_{(0,1)_s},k) = P(P_4,k).P(N_1,k) - P(K_3,k).P(N_1) - P(P_4,k) + P(K_3,k) = k \times [k(k-1)^3] - k \times [k(k-1)(k-2)] - k(k-1)^3 + k(k-1)(k-2) = k(k-1)^2(k^2-3k+3)$
2. For $\alpha = 0.2, \beta = 0.7$, the strong level graph corresponding to the level is a Path graph P_4. Therefore, $P^I_{(0.2,0.7)_s}(G,k) = P(G_{(0.2,0.7)_s},k) = P(P_4,k) = k(k-1)^3$.
3. For $\alpha = 0.3, \beta = 0.6$, the strong level graph corresponding to the level is a forest graph containing Path graphs P_2 and P_1. Therefore, $P^I_{(0.3,0.6)_s}(G,k) = P(G_{(0.3,0.6)_s},k) = P(P_1,k) \times (P_2,k) = k^3 - k^2$
4. For $\alpha = 0.5, \beta = 0.5$, we have the strong level graph N_2. Thus, $P^I_{(0.5,0.5)_s}(G,k) = P(G_{(0.5,0.5)_s},k) = P(N_2,k) = k^2$.
5. Lastly, for $\alpha = 0.6, \beta = 0.2$, we have the strong level graph N_1. Thus, $P^I_{(0.6,0.2)_s}(G,k) = P(G_{(0.6,0.2)_s},k) = P(N_1,k) = k$.

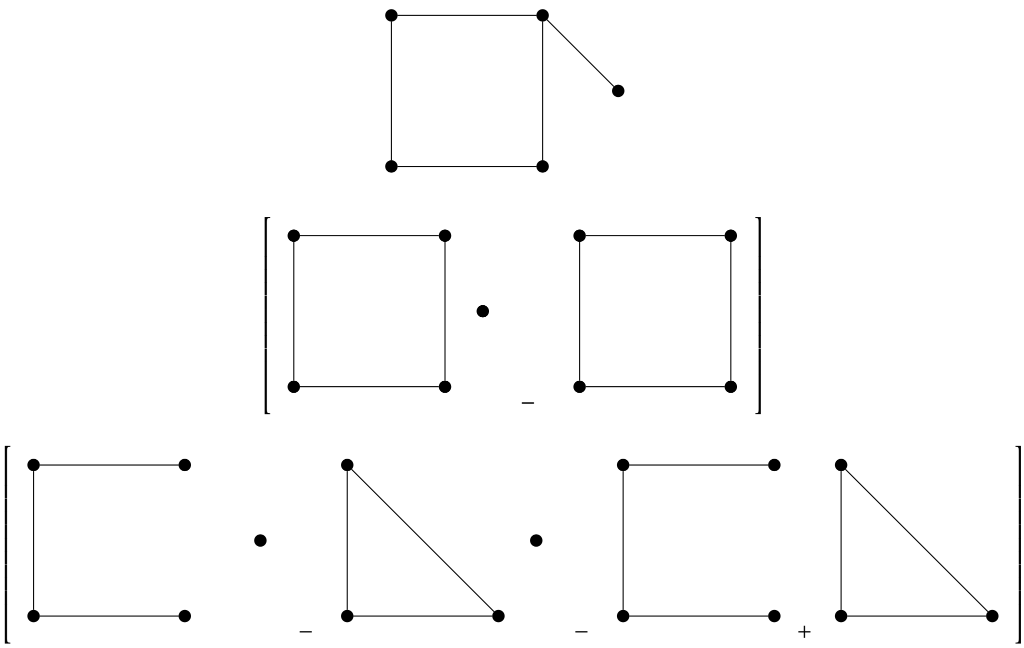

Fig. 3: The computation of $P(G_{(0,1)_s}, k)$

Therefore, $P^I_{(\alpha,\beta)_s}(G,k)$ of a graph in figure 1 is:

$$P^I_{(\alpha,\beta)_s}(G,k) = \begin{cases} (k(k-1)^2)(k^2-3k+3) & \text{if } \alpha = 0, \beta = 1 \\ k(k-1)^3 & \text{if } \alpha = 0.2, \beta = 0.7 \\ k^3 - k^2 & \text{if } \alpha = 0.3, \beta = 0.6 \\ k^2 & \text{if } \alpha = 0.5, \beta = 0.5 \\ k & \text{if}\alpha = 0.6, \beta = 0.2 \end{cases}$$

Generally, the comparison of the vertex set, the edge set, the chromatic numbers, and the chromatic polynomials of various strong levels of a graph in figure 1 are put in table 1.

Table 1: The comparison of $|V_{(\alpha,\beta)_s}|$, $|E_{(\alpha,\beta)_s}|$, $\chi(G_{\alpha,\beta)_s})$ and $P^I_{(\alpha,\beta)_s}(G,k)$.

Strong level	$\lvert V_{(\alpha,\beta)_s}\rvert$	$\lvert E_{(\alpha,\beta)_s}\rvert$	$\chi(G_{\alpha,\beta)_s})$	$P^I_{(\alpha,\beta)_s}(G,k)$
$\alpha = 0$ $\beta = 1$	5	5	2	$k(k-1)^2(k^2-3k+3)$
$\alpha = 0.2$ $\beta = 0.7$	4	3	2	$k(k-1)^3$
$\alpha = 0.3$ $\beta = 0.6$	3	1	2	$k^3 - k^2$
$\alpha = 0.5$ $\beta = 0.5$	2	0	1	k^2
$\alpha = 0.6$ $\beta = 0.2$	1	0	1	k

*Remark.*As α increases or as β decreases:

1. Both the cardinal of vertices and the cardinal of edges of an IFG based on the strong (α,β)-level decrease.
2. Both the chromatic number and the degree of strong fuzzy chromatic polynomial of an IFG based on (α,β)-level decrease.

4.1 The Characteristics of strong fuzzy chromatic polynomial of IFGs based on (α,β)-levels

Theorem 5.*If $H_{(\alpha_1,\beta_1)_s}$ and $H_{(\alpha_2,\beta_2)_s}$ are the strong level graphs of an IFG, $H=(V,E)$ with $\alpha_1<\alpha_2$ and $\beta_1>\beta_2$, then $deg\left(P(H_{(\alpha_2,\beta_2)_s},k)\right)\le deg\left(P(H_{(\alpha_1,\beta_1)_s},k)\right)$.*

*Proof.*Assume that $H=(V,E)$ is an IFG and also assume that $H_{(\alpha_1,\beta_1)_s}$ and $H_{(\alpha_2,\beta_2)_s}$ are the strong level graphs of H such that $\alpha_1<\alpha_2$ and $\beta_1>\beta_2$. Since for any graph F, $deg(P(F,k))=|V(F)|$, we have $deg\left(P(H_{(\alpha_1,\beta_1)_s},k)\right)=|V_{(\alpha_1,\beta_1)_s}|$ and $deg\left(P(H_{(\alpha_2,\beta_2)_s},k)\right)=|V_{(\alpha_2,\beta_2)_s}|$. Since $\alpha_1<\alpha_2$ and $\beta_1>\beta_2$, by applying corollary 1, We obtain $|V_{(\alpha_1,\beta_1)_s}|\ge|V_{(\alpha_2,\beta_2)_s}|$. Thus, $deg\left(P(H_{(\alpha_2,\beta_2)_s},k)\right)\le deg\left(P(H_{(\alpha_1,\beta_1)_s},k)\right)$.

Corollary 3.*If $(\alpha_l,\beta_l)_s$, $(\alpha_m,\beta_m)_s$ and $(\alpha_n,\beta_n)_s$ are the strong intuitionistic fuzzy levels of an IFG, $H=(V,E)$ such that $\alpha_m>\alpha_l$, $\alpha_n>\alpha_m$, $\beta_m<\beta_l$ and $\beta_n<\beta_m$, then $deg\left(P(H_{(\alpha_n,\beta_n)_s},k)\right)\le deg\left(P(H_{(\alpha_l,\beta_l)_s},k)\right)$.*

*Proof.*Assume that $H=(V,E)$ is an IFG and $H_{(\alpha_l,\beta_l)_s}, H_{(\alpha_m,\beta_m)_s}$ and $H_{(\alpha_n,\beta_n)_s}$ are the strong level graphs of H with $\alpha_m>\alpha_l$, $\alpha_n>\alpha_m$, $\beta_m<\beta_l$ and $\beta_n<\beta_m$. Now the transitivity of three real numbers implies that $\alpha_1<\alpha_n$ and $\beta_1>\beta_n$. Further, by applying theorem 3, $deg\left(P(H_{(\alpha_n,\beta_n)_s},k)\right)\le deg\left(P(H_{(\alpha_l,\beta_l)_s},k)\right)$.

Theorem 6.*If H_c is an underlying crisp graph of an IFG, H and $F_s=\{(0,1)_s\}\cup\{(\alpha_i,\beta_i)_s\}_{i=1}^{l}$ is the strong fundamental set of H such that $0<\alpha_i$ and $\beta_i<1\ \forall i=1,2,\cdots,l$, then $P^I_{(0,1)_s}(H,k)=P(H_c,k)$.*

*Proof.*Let H be an IFG and $F_s=\{(0,1)_s\}\cup\{(\alpha_i,\beta_i)_s\}_{i=1}^{l}$ be a strong fundamental set of H. Suppose that H_c is the underlying graph and assume $0<\alpha_i$ and $\beta_i<1\ \forall i=1,2,\cdots,l$ of (α_i,β_i) in F_s. Then by applying theorem 2, $H_{(\alpha_i,\beta_i)_s}$ is a sub graph of $H_{(0,1)_s}$ for each $i=1,2,\cdots,l$. Since every strong level graph of H is sub graph of $H_{(0,1)_s}$ and $H_{(0,1)_s}$ is the underlying graph of H. $H_{(0,1)_s}$ contains every vertex and every edge of H_c. Thus, $H_{(0,1)_s}$ and H_c are similar graphs and contain equal chromatic polynomial. Mathematically, $P(H_{(0,1)_s},k)=P(H_c,k)$. Hence, $P^I_{(0,1)_s}(H,k)=P(H_c,k)$.

Theorem 7.*If F and H are the IFG components of an IFG G, then $P^I_{(\alpha,\beta)_s}(G,k)=P^I_{(\alpha,\beta)_s}(F,k)\times P^I_{(\alpha,\beta)_s}(H,k)$.*

*Proof.*Let F and H be IFG components of an IFG, G. Now suppose that $G_{(\alpha,\beta)_s}, F_{(\alpha,\beta)_s}$ and $H_{(\alpha,\beta)_s}$ are the strong (α,β)-level graphs of G,F, and H in a respective manner. Since F and H are IFG components of G, $F_{(\alpha,\beta)_s}$ and $H_{(\alpha,\beta)_s}$ are underlying crisp components of $G_{(\alpha,\beta)_s}$. Now, by applying proposition 2: $P^I_{(\alpha,\beta)_s}(G,k)=P^I_{(\alpha,\beta)_s}(F,k)\times P^I_{(\alpha,\beta)_s}(H,k)$.

Theorem 8.*If $W_{(\alpha_l,\beta_l)_s}$ and $W_{(\alpha_m,\beta_m)_s}$ are the level graphs of an IFG, W with $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$, then $P^I\left(W_{(\alpha_l,\beta_l)_s}\cup W_{(\alpha_m,\beta_m)_s},k\right)=P^I\left(W_{(\alpha_l,\beta_l)_s},k\right)$.*

*Proof.*Let $W_{(\alpha_l,\beta_l)_s}$ and $W_{(\alpha_m,\beta_m)_s}$ are the level graphs of an IFG, W with $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$. Then $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$ implies that $V_{(\alpha_m,\beta_m)_s}\subseteq V_{(\alpha_l,\beta_l)_s}$ and $E_{(\alpha_m,\beta_m)_s}\subseteq E_{(\alpha_l,\beta_l)_s}$. Now by the definition of union of sets, $V_{(\alpha_l,\beta_l)_s}\cup V_{(\alpha_m,\beta_m)_s}=V_{(\alpha_l,\beta_l)_s}$ and $E_{(\alpha_l,\beta_l)_s}\cup E_{(\alpha_m,\beta_m)_s}=E_{(\alpha_l,\beta_l)_s}$. This implies, $W_{(\alpha_l,\beta_l)_s}\cup W_{(\alpha_m,\beta_m)_s}=W_{(\alpha_l,\beta_l)_s}$. Hence, $P^I\left(W_{(\alpha_l,\beta_l)_s}\cup W_{(\alpha_m,\beta_m)_s},k\right)=P^I\left(W_{(\alpha_l,\beta_l)_s},k\right)$.

Theorem 9.*If $W_{(\alpha_l,\beta_l)_s}$ and $W_{(\alpha_m,\beta_m)_s}$ are the level graphs of an IFG, W with $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$, then $P^I\left(W_{(\alpha_l,\beta_l)_s}\cap W_{(\alpha_m,\beta_m)_s},k\right)=P^I\left(W_{(\alpha_m,\beta_m)_s},k\right)$.*

*Proof.*Let $W_{(\alpha_l,\beta_l)_s}$ and $W_{(\alpha_m,\beta_m)_s}$ be the level graphs of an IFG, W with $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$. Then $\alpha_l<\alpha_m$ and $\beta_l>\beta_m$ implies that $V_{(\alpha_m,\beta_m)_s}\subseteq V_{(\alpha_l,\beta_l)_s}$ and $E_{(\alpha_m,\beta_m)_s}\subseteq E_{(\alpha_l,\beta_l)_s}$. Now by the definition of intersection of two sets, $V_{(\alpha_l,\beta_l)_s}\cap V_{(\alpha_m,\beta_m)_s}=V_{(\alpha_m,\beta_m)_s}$ and $E_{(\alpha_l,\beta_l)_s}\cap E_{(\alpha_m,\beta_m)_s}=E_{(\alpha_m,\beta_m)_s}$. This indicates $W_{(\alpha_l,\beta_l)_s}\cap W_{(\alpha_m,\beta_m)_s}=W_{(\alpha_m,\beta_m)_s}$. Hence, $P^I\left(W_{(\alpha_l,\beta_l)_s}\cap W_{(\alpha_m,\beta_m)_s},k\right)=P^I\left(W_{(\alpha_m,\beta_m)_s},k\right)$.

5 Conclusions

In this research, the concept of the $(\alpha,\beta)_s$ -fundamental set of an IFG and the idea of (α,β) -level based strong fuzzy chromatic polynomial of an IFG has been introduced. In addition, some characterizations of $(\alpha,\beta)_s$-level graphs and (α,β) -level based strong fuzzy chromatic polynomial of an IFG have been discussed and illustrated. Moreover, the strong fuzzy chromatic polynomials of an IFG for the various $(\alpha,\beta)_s$ -levels have been computed and associated.

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