Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070422

Lag synchronization of hyperchaotic complex nonlinear systems via passive control

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Received: 22 Dec. 2012, Revised: 12 Feb. 2012, Accepted: 4 Mar. 2012
Published online: 1 Jul. 2013

Abstract: In this paper we study and investigate the Lag synchronization (LS) of hyperchaotic complex nonlinear systems by using passive control theory. LS of these hyperchaotic complex systems via passive control has not received enough attentions, as far as we know, in the literature. Based on the property of the passive system the passive controller is stated to achieve LS of two identical hyperchaotic complex Lorenz systems. These complex systems appear in many important fields of physics and engineering. The analytical results of the controllers, which have been calculated by using our scheme, are tested numerically and good agreement is obtained.

Keywords: Lag synchronization, Passive control, Minimum phase, Hyperchaotic, Complex

1 Introduction

There are several kinds of synchronization between two chaotic or hyperchaotic systems have been identified as complete (or full) synchronization [1,2], generalized synchronization [3], phase synchronization [4] and lag synchronization (LS) [5,6,7,8,9,10,11,12,13,14,15]. They represent the difference in the degree of correlation between interacting systems [6]. Among these synchronizations, complete synchronization is the strongest in the degree of correlation and describes the interaction of two identical systems, leading to their trajectories remaining identical in the course of temporal evolution, i.e., $x^d(t) = x^r(t)$, where $x^d(t)$ and $x^r(t)$ are the states of the drive and response systems, respectively. Generalized synchronization, as introduced for drive-response systems, is defined as the presence of a functional relationship between the states of the response and drive systems, i.e., $x^d(t) = f(x^r(t))$. Phase synchronization is the situation where two coupled hyperchaotic systems keep their phases in step with each other while their amplitudes remain uncorrelated.

In the typical synchronization regimes, lag synchronization has been proposed as the coincidence of the states of chaotic systems in which one of the systems is delayed by a finite time. Many experimental investigations and computer simulations of chaos synchronization in unidirectional coupled external cavity semiconductor lasers [16,17,18] have demonstrated the presence of lag time between the drive and response lasers intensities. The similar experiments for chaotic circuits [19] have also demonstrated the complete synchronization, i.e., the states of two chaotic systems remain identical in the course of temporal evolution, is practically impossible for the presence of the signal transmission time and evolution time of response system itself. Thus, knowledge of the lag synchronization is of considerable practical importance. In LS the state of the response system at time t is asymptotically synchronous with the drive system at time $t-\tau$, namely, $\lim_{t\longrightarrow\infty} \left\|x^r(t) - x^d(t-\tau)\right\| = 0$, where $x^d(t)$ and $x^r(t)$ are the states of the drive and response real systems, respectively. Complete synchronization is special case of LS when $\tau = 0$.
Recently, some control methods, such as observer-based scheme, [20] impulsive control, [21] projective approach, [22,23] and adaptive control, [24,25] have been applied to the lag synchronization for chaotic and hyperchaotic systems.

Passivity is part of a broader and general theory of dissipativity, which can be found in Refs. [26,27] and various others references therein. The main idea of passivity theory is that the passive properties of system can keep the system internally stable. So, to make the

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system stable, one can design a controller which renders the closed loop system passive using passivity theory. During the last decade the passivity theory has played an important role in designing an asymptotically stabilizing controller for nonlinear systems [28,29,30]. These references show that the main feature of the passive control approach lies in the fact that the controller design includes, at a fundamental level, the system structural properties that can be exploited to solve a given control problem. It is also shown that the passive control has many advantages: e.g. clear physical interpretation, less control effort required or ease in implementation. Therefore, passive control has been widely investigated and applied [27,28,29,30].
LS via passivity is not detected yet in the literature. So, the motivation of this paper is to investigate the LS of n-dimensional hyperchaotic complex nonlinear systems by using passive control theory and compare our results with those in [31]. In [31] LS of n-dimensional hyperchaotic complex nonlinear systems is investigated by using nonlinear control scheme.

The organization of this paper is as follows. Section 2 presents the description of n-dimensional hyperchaotic complex nonlinear systems. Basic conceptions of passivity theory are stated in Section 3. In Section 4 we achieve LS between two identical hyperchaotic complex Lorenz systems. The analytical forms of the controllers of this example are derived based on passivity theory. In Section 5, the numerical results are illustrated to emphasize the validity of the theoretical investigations. Finally, conclusions are drawn in Section 6.

2 Description of hyperchaotic complex nonlinear systems

A complex dynamical system is called hyperchaotic if it is deterministic, has long-term aperiodic behavior, and exhibits sensitive dependence on the initial conditions. A hyperchaotic complex attractor is defined as a complex chaotic attractor with at least two positive Lyapunov exponents. The sum of Lyapunov exponents must be negative to ensure that system is dissipative. It is even more complicated than chaotic complex systems and has more unstable manifolds. Due to hyperchaotic complex systems with characteristics of high capacity, high security and high efficiency, it has a broadly applied potential in nonlinear circuits, secure communications, lasers, neural networks, biological systems and so on. Therefore, research on hyperchaotic complex nonlinear systems is extremely important nowadays [31,32,33,34, 35,36,37,38,39,40,41,42].

Consider the $n-$dimensional hyperchaotic complex nonlinear system as follows:

$$\dot{\mathbf{z}} = \boldsymbol{\psi}(\mathbf{z}, \overline{\mathbf{z}}), \tag{1}$$

where the state complex vector $\mathbf{z} = (z_1, z_2, ..., z_n)^T \in \mathbf{C}^n$, $z_j = \eta_{j1} + i\eta_{j2}$ $(j = 1,2,...,n)$, $i = \sqrt{-1}$, $\boldsymbol{\psi} = (\psi_1, \psi_2, ..., \psi_n)^T$ is a smooth nonlinear vector field, Dot represents derivative with respect to time and an overbar denotes complex conjugate variables.
In this paper we study the phenomenon of the LS of hyperchaotic complex nonlinear systems of the form (1) via passive control theory which is not detected yet in the literature.

Remark 1: The $n-$dimensional hyperchaotic complex nonlinear system (1) can be rewritten in the real form as:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Psi}(\boldsymbol{\eta}), \tag{2}$$

where $\boldsymbol{\eta} = (\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}, ..., \eta_{n1}, \eta_{n2})^T \in \mathbb{R}^{2n}$ is a state real vector, $(j = 1,2,...,n)$, η_{j1}, η_{j2} are real functions and $\boldsymbol{\Psi}$ is a smooth nonlinear vector field.

Remark 2: Most of hyperchaotic complex systems can be described by (1), such as complex Lorenz, Chen and Lü systems [31,32]. For example, the hyperchaotic complex Lorenz system [33,34] is:

$$\begin{aligned}\dot{z}_1 &= \alpha(z_2 - z_1) + iz_4,\\ \dot{z}_2 &= \gamma z_1 - z_2 - z_1 z_3 + iz_4,\\ \dot{z}_3 &= 1/2\,(\overline{z}_1 z_2 + \overline{z}_2 z_1) - \beta z_3,\\ \dot{z}_4 &= 1/2\,(\overline{z}_1 z_2 + \overline{z}_2 z_1) - \sigma z_4,\end{aligned} \tag{3}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3, \psi_4)^T = (\alpha(z_2 - z_1) + iz_4,\ \gamma z_1 - z_2 - z_1 z_3 + +iz_4,\ 1/2\,(\overline{z}_1 z_2 + \overline{z}_2 z_1) - \beta z_3,\ 1/2\,(\overline{z}_1 z_2 + \overline{z}_2 z_1) - \sigma z_4)^T$, α, γ and β are positive parameters, $z_1 = \eta_{11} + i\eta_{12}$, $z_2 = \eta_{21} + i\eta_{22}$ are complex functions and η_{jl} $(j = 1,2, l = 1,2)$, $z_3 = \eta_{31}$, $z_4 = \eta_{41}$ are real functions and σ is a control parameter.
This hyperchaotic example has been introduced recently in our work [33]. For the case $\alpha = 20$, $\beta = 5$, $\gamma = 40$ and $\sigma = 13$ we calculated the Lyapunov exponents as: $\zeta_1 = 1.7745$, $\zeta_2 = 0.2043$, $\zeta_3 = 0$, $\zeta_4 = -18.7918$, $\zeta_5 = -30.8533$, $\zeta_6 = -38.8889$ and its Lyapunov dimension is $D \cong 3.10530$. Therefore system (3) has a hyperchaotic behavior since ζ_1 and ζ_2 are positive, see Fig. 1, for more dynamical properties, see Ref. [33].

3 Basic conceptions of passivity theory in hyperchaotic complex nonlinear systems

We consider the hyperchaotic complex nonlinear systems with the controller in the general form as follows:

$$\begin{cases}\dot{\mathbf{z}} = \boldsymbol{\psi}(\mathbf{z}, \overline{\mathbf{z}}) + \boldsymbol{\phi}(\mathbf{z}, \overline{\mathbf{z}})\boldsymbol{\Xi},\\ \mathbf{y} = \mathbf{h}(\mathbf{z}, \overline{\mathbf{z}}),\end{cases} \tag{4}$$

where $\boldsymbol{\Xi} = (v_1, v_2, ..., v_m)^T$ is the input (or "controller"), $v_s = v_{s1} + iv_{s2}$ $(s = 1,2,...,m)$, $n > m$, $\mathbf{y} = (y_1, y_2, ..., y_m)^T$ is the output, $\boldsymbol{\psi}$ and the m columns of $\boldsymbol{\phi}$ are smooth vector fields and $\mathbf{h}$ is a smooth mapping.
The real form of system (4) can be written as:

$$\begin{cases}\dot{\boldsymbol{\eta}} = \boldsymbol{\Psi}(\boldsymbol{\eta}) + \boldsymbol{\Phi}(\boldsymbol{\eta})\mathbf{v},\\ \boldsymbol{\lambda} = \mathbf{H}(\boldsymbol{\eta}),\end{cases} \tag{5}$$

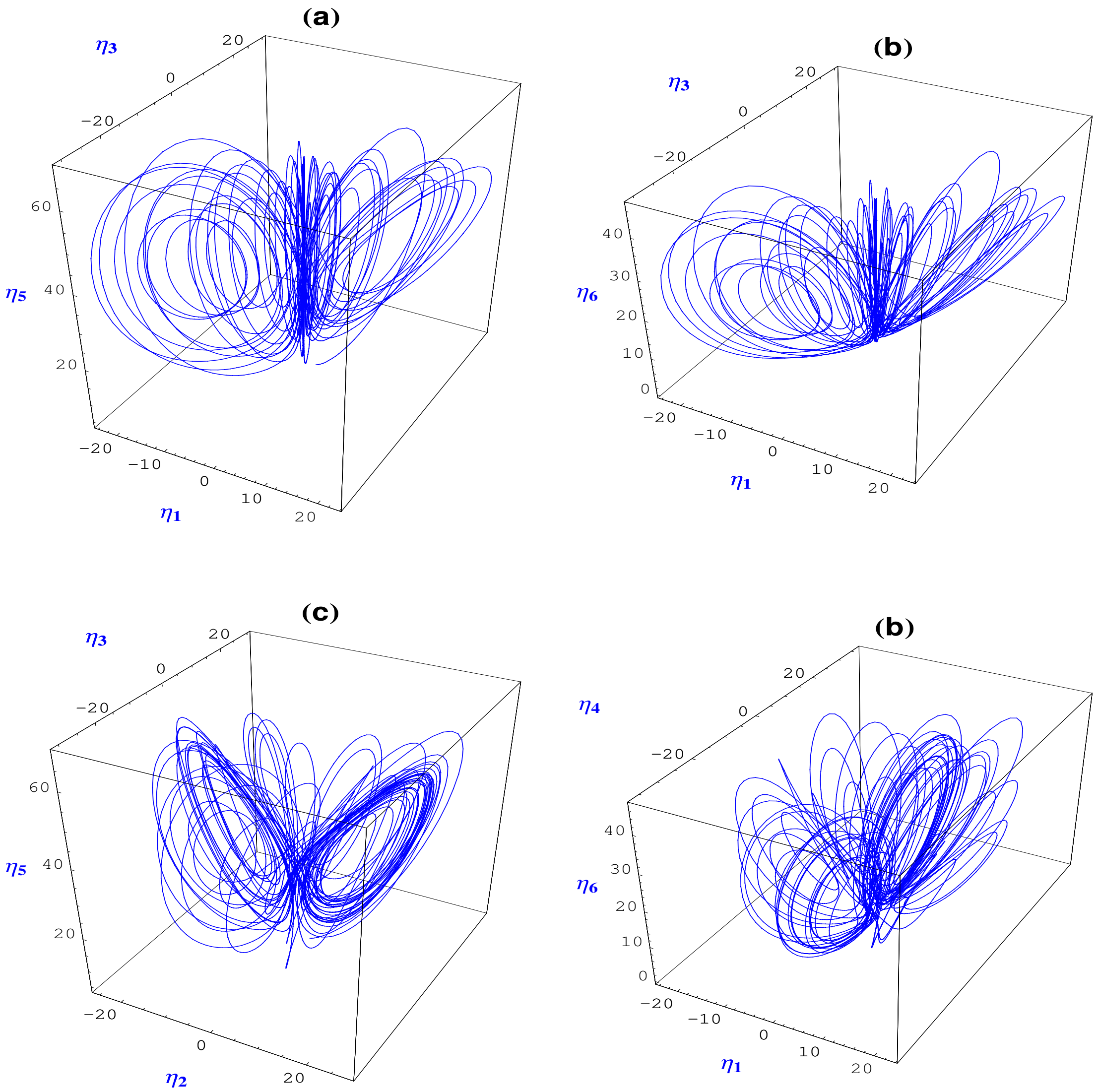

Fig. 1: Hyperhaotic attractors of system:
(a) In (η_1, η_3, η_6) space. (b) In (η_1, η_3, η_6) space. (c) In (η_2, η_3, η_5) space. (d) In (η_1, η_4, η_6) space.

where $\mathbf{v} = (v_{11}, v_{12}, v_{21}, v_{22}, ..., v_{m1}, v_{m2})^T$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, ..., \lambda_{2m})^T$. For every initial condition $\boldsymbol{\eta}(\mathbf{0})$ and every input $\mathbf{v}(\cdot)$, there is a maximally defined solution $\boldsymbol{\eta}(\cdot)$ of the system (5), and corresponding output $\boldsymbol{\lambda}(\cdot)$ [26,27]. We suppose that the vector field $\boldsymbol{\psi}$ has at least one equilibrium point. Without loss of generality, we can assume that the equilibrium point is $\boldsymbol{\eta} = \mathbf{0}$. If the equilibrium point is not at $\boldsymbol{\eta} = \mathbf{0}$, we can shift the equilibrium point to $\boldsymbol{\eta} = \mathbf{0}$ by coordinate transform [43].

Definition 1 [43]: The system (5) is a minimum phase system if $\mathbf{L}_{\boldsymbol{\Phi}}\mathbf{H}(\mathbf{0}) = \frac{\partial \mathbf{H}}{\partial \boldsymbol{\eta}}\boldsymbol{\Phi}(\boldsymbol{\eta})$ is nonsingular and $\boldsymbol{\eta} = \mathbf{0}$ is one of asymptotically stabilized equilibrium points of $\boldsymbol{\Psi}(\boldsymbol{\eta})$.

Definition 2 [44]: The system of the form (5) is said to have relative degree [1, 1, ..., 1] at $\boldsymbol{\eta} = \mathbf{0}$ if the matrix $\mathbf{L}_{\boldsymbol{\Phi}}\mathbf{H}(\mathbf{0})$ is nonsingular.

Definition 3 [45]: The system (5) is passive if the following two conditions are satisfied:
(1) $\boldsymbol{\Psi}(\boldsymbol{\eta})$ and $\boldsymbol{\Phi}(\boldsymbol{\eta})$ exist, $\boldsymbol{\Psi}$ and the $2m$ columns of $\boldsymbol{\Phi}$ are smooth vector fields and $\mathbf{H}(\boldsymbol{\eta})$ is a smooth mapping.
(2) $\forall\ t \geq 0$, there is a real value v satisfying the inequality:

$$\int_0^t \mathbf{v}^T(\varsigma)\boldsymbol{\lambda}(\varsigma)d\varsigma \geq v, \tag{6}$$

or, there exist a constant $\varepsilon > 0$ and a real-valued constant v which satisfy the inequality:

$$\int_0^t \mathbf{v}^T(\varsigma)\boldsymbol{\lambda}(\varsigma)d\varsigma + v \geq \varepsilon \int_0^t \boldsymbol{\lambda}^T(\varsigma)\boldsymbol{\lambda}(\varsigma)d\varsigma. \quad (7)$$

Definition 4 [46]: The system (5) is said to be passive if there is a nonnegative function $V : \boldsymbol{\eta} \longrightarrow \mathbb{R}$, called storage function, which satisfies $V(0) = 0$, such that:

$$V(\boldsymbol{\eta}) - V(\boldsymbol{\eta}(\mathbf{0})) \leq \int_0^t \boldsymbol{\lambda}^T(\varsigma)\mathbf{v}(\varsigma)d\varsigma. \quad (8)$$

The physical meaning of passive system is that the energy of the nonlinear system can be increased only through the supply from an external source. In other words, a passive system cannot store more energy than that supplied externally.

Remark 3: The parametric version of the normal form for the system (5) which satisfying definition 1 is:

$$\begin{cases} \dot{\boldsymbol{\mu}} = \boldsymbol{\varphi}_0(\boldsymbol{\mu}) + \boldsymbol{\chi}(\boldsymbol{\mu},\boldsymbol{\lambda})\boldsymbol{\lambda}, \\ \dot{\boldsymbol{\lambda}} = \boldsymbol{\Omega}(\boldsymbol{\mu},\boldsymbol{\lambda}) + \boldsymbol{\kappa}(\boldsymbol{\mu},\boldsymbol{\lambda})\mathbf{v}, \end{cases} \quad (9)$$

where a new coordinate of the system (5) is $(\boldsymbol{\mu},\boldsymbol{\lambda})$, locally defined in the neighborhood of the origin, $\boldsymbol{\mu} \in \mathbb{R}^{2n-2m}$ and $\boldsymbol{\kappa}(\boldsymbol{\mu},\boldsymbol{\lambda})$ is nonsingular for all $(\boldsymbol{\mu},\boldsymbol{\lambda})$ in the neighborhood of the origin. By designing a suitable controller $\mathbf{v}$, the system (9) may be passive. Thus, the equilibrium point of the system (9) can be asymptotically stabilized by applying the nonlinear controller $\mathbf{v}$ [47]. If the system (9) describes an error dynamical system with time lag, then the synchronization between the drive and response systems is achieved.

Remark 4: Setting $\boldsymbol{\lambda} = \mathbf{0}$ in system (9), yields the zero dynamic system:

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\varphi}_0(\boldsymbol{\mu}). \quad (10)$$

The system (9) is called minimum phase, if its zero dynamics is asymptotically stable.

Theorem 1 [47]: *If the system (5) has a relative degree [1, 1, ..., 1] at $\boldsymbol{\eta} = 0$ and system (5) is a minimum phase system, the system (9) will be equivalent to a passive system and asymptotically stabilized at an equilibrium point if we let the local feedback control as follows:*

$$\mathbf{v} = \boldsymbol{\kappa}^{-1}(\boldsymbol{\mu},\boldsymbol{\lambda})[-\boldsymbol{\Omega}(\boldsymbol{\mu},\boldsymbol{\lambda}) - \left(\frac{\partial W(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\boldsymbol{\chi}(\boldsymbol{\mu},\boldsymbol{\lambda})\right)^T - \varepsilon\boldsymbol{\lambda} + \boldsymbol{\xi}], \quad (11)$$

where the Lyapunov function of $\boldsymbol{\varphi}_0(\boldsymbol{\mu})$ is $W(\boldsymbol{\mu})$, ε is a positive real value and $\boldsymbol{\xi}$ is an external signal vector that is connected with the reference input.

4 LS via passive control

Let us now investigate the LS of system (3), which has been introduced in [31], based on the passivity theory.

The drive and the response systems are thus defined, respectively, as follows:

$$\begin{aligned} \dot{z}_1^d &= \alpha(z_2^d - z_1^d) + iz_4^d, \\ \dot{z}_2^d &= \gamma z_1^d - z_2^d - z_1^d z_3^d + iz_4^d, \\ \dot{z}_3^d &= 1/2\left(\overline{z}_1^d z_2^d + z_1^d \overline{z}_2^d\right) - \beta z_3^d, \\ \dot{z}_4^d &= 1/2\left(\overline{z}_1^d z_2^d + z_1^d \overline{z}_2^d\right) - \sigma z_4^d, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \dot{z}_1^r &= \alpha(z_2^r - z_1^r) + iz_4^r, \\ \dot{z}_2^r &= \gamma z_1^r - z_2^r - z_1^r z_3^r + iz_4^r + v_1, \\ \dot{z}_3^r &= 1/2(\overline{z}_1^r z_2^r + z_1^r \overline{z}_2^r) - \beta z_3^r + v_2, \\ \dot{z}_4^r &= 1/2(\overline{z}_1^r z_2^r + z_1^r \overline{z}_2^r) - \sigma z_4^r + v_3, \end{aligned} \quad (13)$$

where $z_1^d = \eta_{11}^d + i\eta_{12}^d$, $z_2^d = \eta_{21}^d + i\eta_{22}^d$, $z_3^d = \eta_{31}^d$, $z_4^d = \eta_{41}^d$, $z_1^r = \eta_{11}^r + i\eta_{12}^r$, $z_2^r = \eta_{21}^r + i\eta_{22}^r$, $z_3^r = \eta_{31}^r$, $z_4^r = \eta_{41}^r$, $v_1 = v_{11} + iv_{12}$, $v_2 = v_{21}$, $v_3 = v_{31}$ are complex and real control functions, respectively, which we need to determine.
In order to obtain the complex and real control signals v_1, v_2, v_3, the complex error dynamical system takes the form:

$$\begin{aligned} \dot{\Delta}_1 &= \alpha(\Delta_2 - \Delta_1) + i\Delta_4, \\ \dot{\Delta}_2 &= \gamma\Delta_1 - \Delta_2 + i\Delta_4 - z_1^r(t)z_3^r(t) \\ &\quad + z_1^d(t-\tau)z_3^d(t-\tau) + v_1, \\ \dot{\Delta}_3 &= -\beta\Delta_3 + 1/2(\overline{z}_1^r(t)z_2^r(t) + z_1^r(t)\overline{z}_2^r(t) \\ &\quad - \overline{z}_1^d(t-\tau)z_2^d(t-\tau) - z_1^d(t-\tau)\overline{z}_2^d(t-\tau)) + v_2, \\ \dot{\Delta}_4 &= -\sigma\Delta_4 + 1/2(\overline{z}_1^r(t)z_2^r(t) + z_1^r(t)\overline{z}_2^r(t) \\ &\quad - \overline{z}_1^d(t-\tau)z_2^d(t-\tau) - z_1^d(t-\tau)\overline{z}_2^d(t-\tau)) + v_3, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \Delta_1 &= z_1^r(t) - z_1^d(t-\tau),\ \Delta_2 = z_2^r(t) - z_2^d(t-\tau), \\ \Delta_3 &= z^r(t) - z^d(t-\tau),\ \Delta_4 = z_4^r(t) - z_4^d(t-\tau), \end{aligned} \quad (15)$$

and $\Delta_1 = \Delta_{11} + i\Delta_{12}$, $\Delta_2 = \Delta_{21} + i\Delta_{22}$ and $\Delta_3 = \Delta_{31}$, $\Delta_4 = \Delta_{41}$ are complex and real errors functions, respectively.
System (14) in the real form:

$$\begin{aligned} \dot{\Delta}_{11} &= \alpha(\Delta_{21} - \Delta_{11}),\ \dot{\Delta}_{12} = \alpha(\Delta_{22} - \Delta_{12}) + \Delta_{41}, \\ \dot{\Delta}_{21} &= \gamma\Delta_{11} - \Delta_{21} - \Delta_{11}\eta_{31}^d(t-\tau) \\ &\quad - \Delta_{31}\eta_{11}^r(t) + v_{11}, \\ \dot{\Delta}_{22} &= \gamma\Delta_{12} - \Delta_{22} + \Delta_{41} - \Delta_{12}\eta_{31}^d(t-\tau) \\ &\quad - \Delta_{31}\eta_{12}^r(t) + v_{12}, \\ \dot{\Delta}_{31} &= -\beta\Delta_{31} + \Delta_{11}\eta_{21}^d(t-\tau) + \Delta_{12}\eta_{22}^d(t-\tau) \\ &\quad + \Delta_{21}\eta_{11}^r(t) + \Delta_{22}\eta_{12}^r(t) + v_{21}, \\ \dot{\Delta}_{41} &= -\sigma\Delta_{41} + \Delta_{11}\eta_{21}^d(t-\tau) + \Delta_{12}\eta_{22}^d(t-\tau) \\ &\quad + \Delta_{21}\eta_{11}^r(t) + \Delta_{22}\eta_{12}^r(t) + v_{31}, \end{aligned} \quad (16)$$

where $\Delta_{jl} = \eta_{jl}^r(t) - \eta_{jl}^d(t-\tau)$, $j = 1, 2, 3, 4$, $l = 1, 2$.

Theorem 2: *The error dynamical system (16) is minimum phase system.*

Proof:

First we can compute the matrix $\mathbf{L}_{\boldsymbol{\Phi}}\mathbf{H}(\mathbf{0})$ as [46,47]:

$$\mathbf{L}_{\boldsymbol{\Phi}}\mathbf{H}(\mathbf{0}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (17)$$

since the matrix $\mathbf{L}_{\Phi}\mathbf{H}(\mathbf{0})$ is nonsingular and according to the definition 1 system (16) has relative degree [1, 1, ..., 1] at $\boldsymbol{\Delta}=\mathbf{0}$, we let $\boldsymbol{\mu}=\left[\mu_1\ \mu_2\right]^T = \left[\Delta_{11}\ \Delta_{12}\right]^T$, $\boldsymbol{\lambda}=\left[\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\right]^T = \left[\Delta_{21}\ \Delta_{22}\ \Delta_{31}\ \Delta_{41}\right]^T$, thus system (16) has the form:

$$\begin{aligned}
\dot{\mu}_1 &= \alpha(\lambda_1-\mu_1),\\
\dot{\mu}_2 &= \alpha(\lambda_2-\mu_2)+\lambda_4,\\
\dot{\lambda}_1 &= \gamma\mu_1-\lambda_1-\mu_1\eta_{31}^d(t-\tau)-\lambda_3\eta_{11}^r(t)+v_{11},\\
\dot{\lambda}_2 &= \gamma\mu_2-\lambda_2+\lambda_4-\mu_2\eta_{31}^d(t-\tau)\\
&\quad -\lambda_3\eta_{12}^r(t)+v_{12},\\
\dot{\lambda}_3 &= -\beta\lambda_3+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)\\
&\quad +\lambda_1\eta_{11}^r(t)+\lambda_2\eta_{12}^r(t)+v_{21},\\
\dot{\lambda}_4 &= -\sigma\lambda_4+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)\\
&\quad +\lambda_1\eta_{11}^r(t)+\lambda_2\eta_{12}^r(t)+v_{31}.
\end{aligned} \tag{18}$$

System (18) can be represented in the following form:

$$\begin{cases} \dot{\boldsymbol{\mu}} = \boldsymbol{\varphi}_0(\boldsymbol{\mu})+\boldsymbol{\chi}(\boldsymbol{\mu},\boldsymbol{\lambda})\boldsymbol{\lambda},\\ \dot{\boldsymbol{\lambda}} = \boldsymbol{\Omega}(\boldsymbol{\mu},\boldsymbol{\lambda})+\boldsymbol{\kappa}(\boldsymbol{\mu},\boldsymbol{\lambda})\mathbf{v}, \end{cases} \tag{19}$$

where

$$\boldsymbol{\varphi}_0(\boldsymbol{\mu}) = \begin{bmatrix}-\alpha\mu_1\\ -\alpha\mu_2\end{bmatrix},\ \boldsymbol{\chi}(\boldsymbol{\mu},\boldsymbol{\lambda}) = \begin{bmatrix}\alpha & 0\\ 0 & \alpha\\ 0 & 0\\ 0 & 1\end{bmatrix}^T,$$

$$\boldsymbol{\kappa}(\boldsymbol{\mu},\boldsymbol{\lambda}) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$\boldsymbol{\Omega}(\boldsymbol{\mu},\boldsymbol{\lambda}) = \begin{bmatrix}\gamma\mu_1-\lambda_1-\mu_1\eta_{31}^d(t-\tau)-\lambda_3\eta_{11}^r(t)\\ \gamma\mu_2-\lambda_2+\lambda_4-\mu_2\eta_{31}^d(t-\tau)-\lambda_3\eta_{12}^r(t)\\ -\beta\lambda_3+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)+\rho\\ -\sigma\lambda_4+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)+\rho\end{bmatrix},$$

where $\rho = \lambda_1\eta_{11}^r(t)+\lambda_2\eta_{12}^r(t)$.
Choose a storage function as:

$$\begin{aligned}
V(\mu,\ \boldsymbol{\lambda}) &= W(\mu)+\frac{1}{2}\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\lambda},\\
&= W(\mu)+\frac{1}{2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2\right),
\end{aligned} \tag{20}$$

where $W(\boldsymbol{\mu}) = \frac{1}{2}\left(\mu_1^2+\mu_2^2\right)$, $W(\mathbf{0}) = 0$.
The zero dynamics of the system (19) describes the internal dynamics and occur when $\boldsymbol{\lambda} = \mathbf{0}$, i.e.

$$\dot{\boldsymbol{\mu}}=\boldsymbol{\varphi}_0(\boldsymbol{\mu}). \tag{21}$$

Differentiating $W(\boldsymbol{\mu})$ respect to t, we get:

$$\begin{aligned}
\dot{W}(\boldsymbol{\mu}) &= \frac{\partial W(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}}\dot{\boldsymbol{\mu}}=\frac{\partial W(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}}\boldsymbol{\varphi}_0(\boldsymbol{\mu}),\\
&= \left[\mu_1\ \mu_2\right]\left[-\alpha\mu_1\ -\alpha\mu_2\right]^T,\\
&= -(\alpha\mu_1^2+\alpha\mu_2^2)\le 0.
\end{aligned} \tag{22}$$

Since $W(\boldsymbol{\mu}) > 0$ and $\dot{W}(\boldsymbol{\mu}) < 0$, it can be concluded that $W(\boldsymbol{\mu})$ is the Lyapunov function of $\boldsymbol{\varphi}_0(\boldsymbol{\mu})$ and $\boldsymbol{\varphi}_0(\boldsymbol{\mu})$ is globally asymptotically stable which means that the error system (16) is *minimum phase* system. So system (16) can be equivalent to a passive system, and theorem 2 is proved.
Because our system (16) is minimum phase system and has vector relative degree [1, ..., 1]. So system (16) is appropriate to apply theorem 1. We can derive the controller as:

$$\begin{aligned}
v_{11} &= -\alpha\mu_1-(\gamma\mu_1-\lambda_1-\mu_1\eta_{31}^d(t-\tau)\\
&\quad -\lambda_3\eta_{11}^r(t))-\varepsilon\lambda_1+\xi_1,\\
v_{12} &= -\alpha\mu_2-(\gamma\mu_2-\lambda_2+\lambda_4-\mu_2\eta_{31}^d(t-\tau)\\
&\quad -\lambda_3\eta_{12}^r(t))-\varepsilon\lambda_2+\xi_2,\\
v_{21} &= -(-\beta\lambda_3+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)\\
&\quad +\lambda_1\eta_{11}^r(t)+\lambda_2\eta_{12}^r(t))-\varepsilon\lambda_3+\xi_3,\\
v_{31} &= -\mu_2-(-\sigma\lambda_4+\mu_1\eta_{21}^d(t-\tau)+\mu_2\eta_{22}^d(t-\tau)\\
&\quad +\lambda_1\eta_{11}^r(t)+\lambda_2\eta_{12}^r(t))-\varepsilon\lambda_4+\xi_4.
\end{aligned} \tag{23}$$

The controller in the final form is:

$$\begin{aligned}
v_{11} &= -\alpha\Delta_{11}-(\gamma\Delta_{11}-\Delta_{21}-\Delta_{11}\eta_{31}^d(t-\tau)\\
&\quad -\Delta_{31}\eta_{11}^r(t))-\varepsilon\Delta_{21}+\xi_1,\\
v_{12} &= -\alpha\Delta_{12}-(\gamma\Delta_{21}-\Delta_{22}+\Delta_{41}-\Delta_{12}\eta_{31}^d(t-\tau)\\
&\quad -\Delta_{31}\eta_{12}^r(t))-\varepsilon\Delta_{22}+\xi_2,\\
v_{21} &= -(-\beta\Delta_{31}+\Delta_{11}\eta_{21}^d(t-\tau)+\Delta_{12}\eta_{22}^d(t-\tau)\\
&\quad +\Delta_{21}\eta_{11}^r(t)+\Delta_{22}\eta_{12}^r(t))-\varepsilon\Delta_{31}+\xi_3,\\
v_{31} &= -\Delta_{12}-(-\sigma\Delta_{41}+\Delta_{11}\eta_{21}^d(t-\tau)+\Delta_{12}\eta_{22}^d(t-\tau)\\
&\quad +\Delta_{21}\eta_{11}^r(t)+\Delta_{22}\eta_{12}^r(t))-\varepsilon\Delta_{41}+\xi_4.
\end{aligned} \tag{24}$$

The analytical formula of the controller (24), which has been calculated by using our scheme, is used with system (16) to achieve the LS of hyperchaotic attractors of our example.

5 Numerical results

Numerical simulations are conducted in this section to illustrate the effectiveness of the designed controller (24). We solve systems (12) and (13) with (24) numerically for $\alpha = 20$, $\beta = 5$, $\gamma = 40$ and $\sigma = 13$ for which hyperchaotic attractor exists [33] and with different initial conditions $t_0 = 0$, $\eta_{11}^d(0) = 1$, $\eta_{12}^d(0) = 2$, $\eta_{21}^d(0) = 3$, $\eta_{22}^d(0) = 4$, $\eta_{31}^d(0) = 5$, $\eta_{41}^d(0) = 6$ and $\eta_{11}^r(0) = 6$, $\eta_{12}^r(0) = 8$, $\eta_{21}^r(0) = 3$, $\eta_{22}^r(0) = 4$, $\eta_{31}^r(0) = 8, \eta_{41}^r(0) = 1$. We choose $\tau = 0.2$, $\varepsilon = 15$ and $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$. The variables' states during the LS process between systems 12 and 13 are shown in Figure 2. From it, one can see that each $\eta_{jl}^r(t)$ converge to $\eta_{jl}^d(t)$, $j = 1,\ 2,\ 3,\ 4,\ l = 1,\ 2$ but with positive time lagged $\tau = 0.2$. Figure 2 shows LS is achieved after small time interval. The LS errors are plotted in Figure 3, and as expected from the above analytical considerations the LS errors Δ_{jl} converge to zero as $t \longrightarrow \infty$ after small value of t. Comparing the numerical results in this research, emerging from the Figures 2 and 3, with those in [31]. In [31] LS was achieved after large time and this unlike our

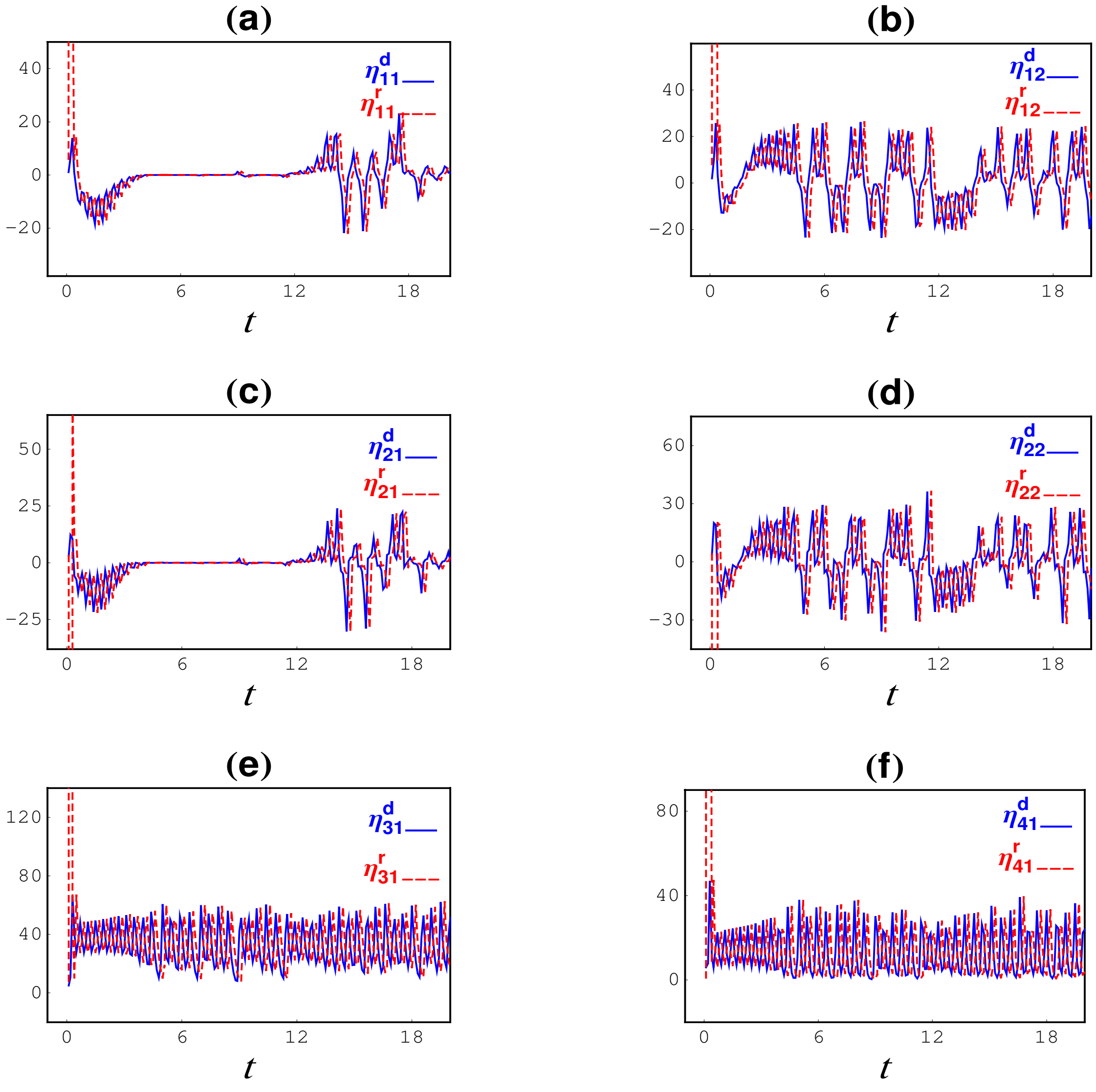

Fig. 2: LS of systems (12) and (13) with (24):
(a) $\eta^d_{11}(t)$ and $\eta^r_{11}(t)$ versus t, (b) $\eta^d_{12}(t)$ and $\eta^r_{12}(t)$ versus t, (c) $\eta^d_{21}(t)$ and $\eta^r_{21}(t)$ versus t, (d) $\eta^d_{22}(t)$ and $\eta^r_{22}(t)$ versus t, (e) $\eta^d_{31}(t)$ and $\eta^r_{31}(t)$ versus t, (f) $\eta^d_{41}(t)$ and $\eta^r_{41}(t)$ versus t.

results. We solve systems (12) and (13) with the same parameters and initial conditions in [31]. But the number of control functions in our paper is less than those used in [31]. This shows the effectiveness of controller (24) which has been calculated by using theorem 1 in our scheme.

6 Conclusion

In engineering applications, time lag always exists. For example in the telephone communication system, the voice one hears on the receiver side at time t is the voice from the transmitter side at time $t-\tau$. So, strictly speaking, it is not reasonable to require the drive system to synchronize the response system at exactly the same time. Therefore, recently, much attention has been given to the LS, in which the state of the response system at time t is asymptotically synchronous with the drive system at time $t-\tau$.
Unique to this paper is to study LS of hyperchaotic complex nonlinear systems by using passive control theory. LS of two identical hyperchaotic complex Lorenz systems is achieved by applying the passivity therory on the error dynamical system with time lag. We have shown that the error dynamical system is passive system and the

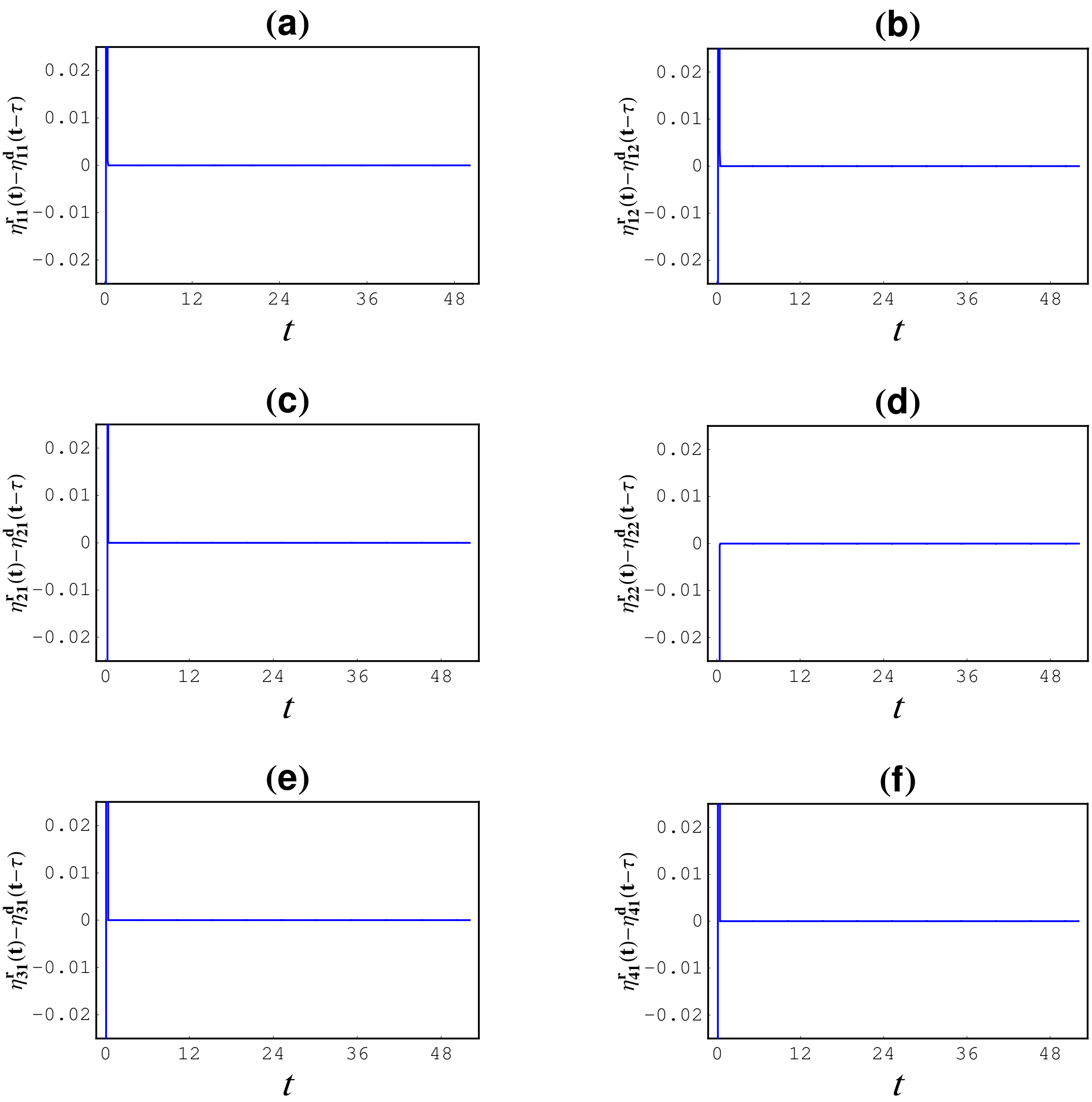

Fig. 3: LS errors: (a) (Δ_{11},t) diagram, (b) (Δ_{12},t) diagram, (c) (Δ_{21},t) diagram, (d) (Δ_{22},t) diagram, (e) (Δ_{31},t) diagram, (f) (Δ_{41},t) diagram.

controller is derived by applying theorem 1 [47]. All the theoretical results are verified by numerical simulation of our example. An excellent agreement is found as shown in Figures 2 and 3. LS occurs after reasonable value of t as shown in Figure 2. Figure 3 displays the error dynamical systems. These errors approach zero after small values of t which shows the effectiveness of the controller. Our results in this paper are better than those published in the [31]. Although the number of control functions in our paper is less than those used in [31].

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