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Relationship between Illicit Drug Users and Bandits in a Population: Mathematical Modelling Approach

John O. Akanni[1,2,*] *and Afeez Abidemi* [3]

[1]Department of Mathematical and Computing Sciences, Koladaisi University Ibadan, Oyo State, Nigeria
[2]Department of Mathematics, Universitas Airlangga, Kampus C Mulyorejo Surabaya 60115, Indonesia
[3]Department of Mathematical Sciences, Federal University of Technology, Akure, P.M.B. 704, Ondo State, Nigeria

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Abstract: Among the major global health and social problems facing the world today is the use of illicit drugs and the act of banditry. The two problems have resulted in the loss of precious lives and possessions and even devastating effects on the economy of some countries where such acts were being practised. Of interest in this work is to study the global stability of illicit drug use spread dynamics with banditry compartments using a dynamical system theory approach. Illicit drug use and banditry reproduction number, which measures the potential spread of illicit drug use and banditry in the population, is evaluated analytically. The system exhibits supercritical bifurcation property, telling us that the local stability of an illicit drug and banditry-present equilibrium exists and is unique. In addition, the illicit drug and banditry-free and illicit drug and banditry-present equilibria are shown to be globally asymptotically stable; this was achieved by constructing suitable Lyapunov functions. Sensitivity analysis is carried out to know the impact of each parameter on the dynamic spread of illicit drug use and banditry in a population. Numerical simulations validate the quantitative results and examine the effects of some key parameters on the system. It has been discovered that to reduce the burden of banditry in the population, stringent control measures must be implemented to reduce the use of illicit drugs. Control measures are recommended to use in curtailing the menace of illicit drug use and banditry.

Keywords: Illicit drug use and banditry model, Illicit drug use and banditry reproduction number, Bifurcation analysis, Sensitivity analysis.

1 Introduction

The world is facing many problems today, which can be classified as follows; poverty, religious conflict, political polarization, government accountability (political scandals), education (access to schooling), food and water scarcity, poor health facilities in developing nations, lack of access to credit, discrimination, physical fitness and the host of others [1]. Two of the elements of poor health facilities and poverty are the use of illicit drugs and banditry [2]. The production and abuse of illicit drugs and the act of banditry have increased sporadically in Africa in the last decades [3]. This increase has impacted the public health system's costs and increased the spread of other epidemics, such as HIV, Gonorrhea among other sexually transmitted diseases [4]. Illicit drug use remains a serious problem; with a countless number of health risks, an increase in the rate of social vice and also jack-up the expenses of the government [4,5].

The term Illicit drug use, also known as substance abuse, as defined by Nutt *et al.* [6] as excessive use of a drug or the mode of usage of a drug which is detrimental to users or people around him/her. One of the substances abused among humans today is marijuana [2,7,8,9,10]. Marijuana is a dehydrated leaf which has a chemical component that alters the mind and makes the consumers behave irrationally [9,10,11,12]. Alcohol is another type of substance abuse, which causes harm to the health of the one that takes it [9,13,14]. When a pregnant woman takes alcohol, it has a great negative effect on the unborn baby (fetus), and can even cause miscarriage [9,15].

According to [15,16], the rate of methamphetamine consumption in the world today, particularly in Africa, is on the high side. The population of those that use methamphetamine abnormally are youths [9,17]. The

* Corresponding author e-mail: jide28@gmail.ocm

effects of illicit drug use are serious in African countries like South Africa, Nigeria and Ghana; it has a serious and great effect on their populations, particularly among their youth [15,16]. The report by SACENDU [18] between January and June 2013 shows that 76%, 71%, 59%, 67% and 59% are illicit drug use patients that were men, coloured people, jobless, single and between the ages of 15 and 29 years, respectively. The population of illicit drug users varies from place to place, even in Africa, for example, the Western Cape Province has a high rate of illicit drug use consumption compared to some other provinces in South Africa [19,22]. These provinces were the ones causing trouble in South Africa, and the substance being abused are alcohol, methamphetamine and marijuana [23]. Also, Tik is another substance abuse in Cape Town [9,18].

Banditry can be defined as a type of organized crime committed by outlaws, typically involving the threat or use of violence. A person who engages in banditry is known as a bandit [24]. According to [24], the causes of banditry can be ecological and climate change, consistent shifts in the human and livestock populations, weak state capacity and the provision of security, proliferation of small arms and light weapons and the host of others. Examples of banditry are the sexual assault of women and girls, theft, and attacks on banks, markets and school hostels and so on [10,24].

Many researchers have used the idea of mathematical modelling to provide solutions to the problem of illicit drug use and banditry, either separately or together as a complex problem. In particular, Pang *et al.* [25] examined the effect of tobacco on a country being victimized by its consumption, where China was the concerned country. It was noted that this country was seriously affected by smoking-associated sicknesses. The work is major in the issue of curtailing smoking in China. Kalula and Nyabadza [26] aimed at the qualitative investigation of the dynamics of substance abuse and predicting drug abuse trends. The analysis of the model was presented in terms of the substance abuse epidemic threshold $\mathscr{R}_0$. Simulations were performed to fit the model to available data for methamphetamine use in the Western Cape using the least squares curve fitting method and determine the role played by some key parameters and methamphetamine users who entered rehabilitation. It is important to note that their model exhibits a backward bifurcation, pointing out that it is insufficient to reduce $\mathscr{R}_0$ below unity to effectively control the substance abuse epidemic.

In the works of [6,26], one of their findings was that the illicit drug use problem can be eradicated if light drug users are targeted with appropriate controls and the treatment rate for drug addicts is also beefed up. One of the problems associated with Methamphetamine drug use is an abnormal desire for sex which may likely increase sexually transmitted infections (STIs) and HIV [27]. In [10], a study on substance abuse which used a mathematical model to gain an insight into the dynamics spread of drug abuse and banditry in a population was carried out. It is worthy of note that the study exhibits backward bifurcation, pointing out that it is not only sufficient to lower the $\mathscr{R}_0$ below unity but other things must be put into consideration. Also, sensitivity analysis was done to determine the impact of the model's parameters on the spread of drug abuse and banditry menace in the population.

This study considers the concomitant of illicit drug use and banditry, the presence of relapse of illicit drug users and the bandit and the quitters' compartments, which is less or not concerned in the existing literature. This formed the major purpose of this work. Hence, a new mathematical model is used to explore the co-problematic population of illicit drug users and bandits, the presence of relapse illicit drug users and bandits and the quitters' compartments. The rest of the study is sectionalized in the following order: Section 2 is for the model formation and analysis of the basic properties. In Section 3, stability and sensitivity analyses of the illicit drug use and banditry model are carried out. Concluding remarks are given in Section 4.

2 Model Formation

The dynamics of the concomitant illicit drug use and banditry (IDUB) in the population are considered with the total population denoted by $N(t)$ at time t, which is sub-divided into seven well-defined classes, susceptible individuals $S(t)$ (those who are in the company of illicit drug users or/and bandits but who are neither illicit drug users nor bandits), illicit drug users $I(t)$ (individuals who abuse drugs or those who depend on drugs wrongly), suspected bandits $B(t)$ (people who commit banditry or involve in bandit act), detainees $D(t)$ (illicit drug users or/and suspected bandits who are in police custody), prisoners $P(t)$ (illicit drug users or/and suspected bandits that are imprisoned) and rehabilitation population $R(t)$ (illicit drug users or/and bandits who are undergoing rehabilitation) and quitters $Q(t)$ (illicit drug users or/and bandits who quit either one or both of these activities). The population is not a constant-size population because the removal (natural death) rate does not balance the recruitment (birth) due to the induced death rate, denoted by δ. Figure 1 represents the schematic diagram for the concomitant of illicit drug use and banditry population dynamics, in which the system of nonlinear ordinary differential equations (1) is the governed system.

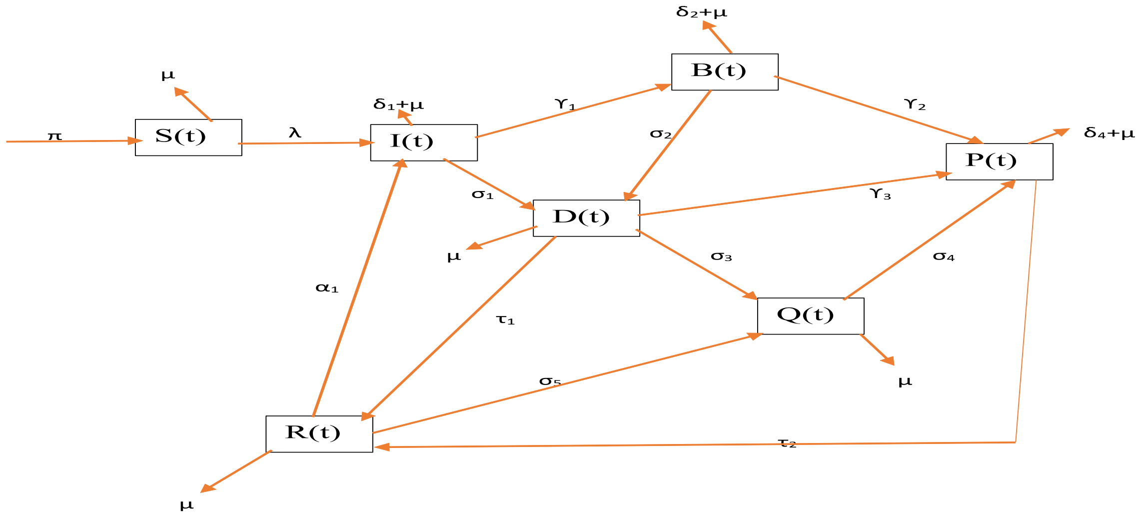


Fig. 1: Schematic diagram of the model (1)

$$
\begin{aligned}
\frac{dS}{dt} &= \pi - \beta\frac{I(t)+\eta B(t)}{N(t)}S(t) - \mu S(t),\\
\frac{dI}{dt} &= \beta\frac{I(t)+\eta B(t)}{N(t)}S(t) + \alpha_1 R(t) - (\sigma_1+\gamma_1+\delta_1+\mu)I(t),\\
\frac{dB}{dt} &= \gamma_1 I(t) - (\sigma_2+\gamma_2+\delta_2+\mu)B(t),\\
\frac{dD}{dt} &= \sigma_1 I(t) + \sigma_2 B(t) - (\sigma_3+\gamma_3+\tau_1+\mu)D(t),\\
\frac{dP}{dt} &= \gamma_2 B(t) + \gamma_3 D(t) - (\sigma_4+\tau_2+\delta_4+\mu)P(t),\\
\frac{dR}{dt} &= \tau_1 D(t) + \tau_2 P(t) - (\sigma_5+\alpha_1+\mu)R(t),\\
\frac{dQ}{dt} &= \sigma_3 D(t) + \sigma_4 P(t) + \sigma_5 R(t) - \mu Q(t),
\end{aligned} \tag{1}
$$

with initial conditions when $t = 0$:

$$
\begin{aligned}
&S(0) = S_0,\ B(0) = B_0,\ I(0) = I_0,\ D(0) = D_0,\\
&P(0) = P_0,\ R(0) = R_0,\ Q(0) = Q_0.
\end{aligned} \tag{2}
$$

The following assumptions guide the dynamics of the formulated model: the interaction of susceptible humans with illicit drug users and bandits first results in illicit drug users, then progresses to the bandit population. Also, rehabilitated individuals return to illicit drug users for one reason or another during treatments. Furthermore, the illicit drug users and bandit populations move to detention before being transferred to prison. Moreover, the detainees can either move to prison or rehabilitation centres. For limpidity, the definition of variables and parameters used in a model (1) are given in Table 1.

Table 1: Definition of variables and parameters of model (1)

Variable/Parameter	Definition
S	susceptible individuals
I	illicit drug users
B	suspected bandits
D	detainees
P	prisoners
R	rehabilitate individual
Q	quitters
π	recruitment rate into the susceptible population
α_1	movement of rehabilitate to illicit drug user
η	modification parameter for suspected bandits
σ_1	detention rate of illicit drug users
γ_1	progression rate of illicit drug users to suspected bandits
δ_1	induce death rate of illicit drug users
β	effective influence rate
δ_4	induce death rate of those in prisoners
μ	natural rate
σ_2	detention rate of suspected bandits
γ_2	movement of suspected bandits to prisoners
δ_2	induced death rate of suspected bandits
σ_3	quitting rate of detainee
γ_3	movement of detainee to prison
τ_1	rate of rehabilitate detainee
σ_4	quitting rate of individuals in the prison
τ_2	rehabilitating rate of those in prison
σ_5	treatment rate of those in rehabilitation center

Looking through the system of equations (1), one can notice that Q only appears in the last equation of the system. On this note, for theoretical analysis, the Q compartment of system (1) is silenced [28].

The equation of the total population, which is obtained by adding all the equations of (1), is given below:

$$\frac{dN}{dt} \leq \pi - \mu N. \tag{3}$$

2.1 Basic Qualitative Properties

The basic properties of the model can now be investigated.

2.1.1 Positivity and boundedness of solutions

Since model (1) monitors the human population, all the parameters are non-negative. Therefore, it is important to prove that all the state variables are also non-negative for all time $t > 0$.

Theorem 1. *The state variables, $S(t)$, $I(t)$, $B(t)$, $D(t)$, $P(t)$, $R(t)$ and $Q(t)$, of model (1), with the initial data (2), remain non-negative for all $t > 0$.*

Proof. Noting that

$$\lambda = \frac{\beta(I(t) + \eta B(t))}{N(t)},$$

one sees from the first equation of (1) that

$$\frac{dS}{dt} \geq -(\lambda + \mu)S(t). \tag{4}$$

Simplifying (4) gives

$$\frac{d}{dt}\left(S(t)\exp\left(\mu t + \int_0^t \lambda(\varpi)d\varpi\right)\right) \geq 0. \tag{5}$$

Solving (5) yields

$$S(t) \geq S(0)\exp\left(-\left(\mu t + \int_0^t \lambda(\varpi)d\varpi\right)\right) > 0, \forall\, t > 0.$$

The remaining state variables $I(t)$, $B(t)$, $D(t)$, $P(t)$, $R(t)$ and $Q(t)$, can be proved to be positive for all $t > 0$ when the same approach used for $S(t)$ is employed.

Next, consider the biologically feasible region, defined by $\Gamma \subset \mathbb{R}^7_+$, where

$$\Gamma = \left\{(S,I,B,D,P,R,Q) \in \mathbb{R}^7_+ : N \leq \frac{\pi}{\mu}\right\}.$$

Γ is shown to be positively invariant in a certain region.

The total population is given by

$$\frac{dN}{dt} \leq \pi - \mu N, \tag{6}$$

which results in the solution $N(t) = N(0)\exp(-\mu t) + \frac{\pi}{N}(1 - \exp(-\mu t))$. It follows that $N(t) \to \frac{\pi}{\mu}$ as $t \to \infty$ in particular, $N(t) \leqslant \frac{\pi}{\mu}$ if $N(0) \leqslant \frac{\pi}{\mu}$ concerning the illicit drug use and banditry model (1). Hence, it suffices to consider the dynamics of the model in Γ. The illicit drug use and banditry model in this region can be mathematically and biologically well-posed [29].

3 Stability and Sensitivity Analysis

3.1 Illicit drug use and banditry-free Equilibrium ($\mathscr{D}_0$)

Illicit drug use and banditry-free equilibrium, denoted by $\mathscr{D}_0$, is the equilibrium point where illicit drug use and banditry are absent in the population. At this point, $I = B = 0$ and so, setting the vector field of (1) to zero gives

$$\mathscr{D}_0 = (S_0, I_0, B_0, D_0, P_0, R_0) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0, 0\right). \tag{7}$$

The illicit drug use and banditry reproduction number $\mathscr{R}_0$ is established next.

3.2 Illicit drug use and banditry threshold ($\mathscr{R}_0$)

The IDUB reproduction number, also noted as IDUB threshold, $\mathscr{R}_0$, given by (8), is the criterion of the IDUB spread in a complete naive population. Also, $\mathscr{R}_0$ can be defined as the average number of new cases of illicit drug users and bandits influenced by a typical illicit drug user and bandit in a naive population.

The next generation matrix method [30] is adopted to theoretically compute the IDUB reproduction number, $\mathscr{R}_0$, as follows:

$$F = \begin{pmatrix} \beta & \beta\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and let $d_1 = \sigma_1 + \gamma_1 + \delta_1 + \mu$, $d_2 = \sigma_2 + \gamma_2 + \delta_2 + \mu$, $d_3 = \sigma_3 + \gamma_3 + \tau_1 + \mu$, $d_4 = \sigma_4 + \tau_2 + \delta_4 + \mu$, $d_5 = \sigma_5 + \alpha_1 + \mu$, so that

$$V = \begin{pmatrix} d_1 & 0 & 0 & 0 & -\alpha_1 \\ -\gamma_1 & d_2 & 0 & 0 & 0 \\ -\sigma_1 & -\sigma_2 & d_3 & 0 & 0 \\ 0 & -\gamma_2 & -\gamma_3 & d_4 & 0 \\ 0 & 0 & -\tau_1 & -\tau_2 & d_5 \end{pmatrix}.$$

Therefore, the spectral radius of the matrix FV^{-1}, which is also $\mathscr{R}_0$, is obtained in the sense of [10,19,20,21,33,37] as

$$\mathscr{R}_0 = \frac{\beta d_3 d_4 d_5 (d_2 + \eta \gamma_1)}{K_7}, \tag{8}$$

where $K_7 = d_1 d_2 d_3 d_4 d_5 - ((d_4 \tau_1 \alpha_1 + \tau_2 \alpha_1 \gamma_3)(d_2 \sigma_1 + \gamma_1 \sigma_2) + d_3 \tau_2 \alpha_1 \gamma_1 \gamma_2)$. Algebraic simplification shows that

$$d_1 d_2 d_3 d_4 d_5 > (d_4 \tau_1 \alpha_1 + \tau_2 \alpha_1 \gamma_3)(d_2 \sigma_1 + \gamma_1 \sigma_2) + d_3 \tau_2 \alpha_1 \gamma_1 \gamma_2.$$

To study the local asymptotic stability (LAS) of $\mathscr{D}_0$ given by (7), the illicit drug use and banditry reproduction number, $\mathscr{R}_0$ given by (8), is a major key needed for this analysis. The following theorem establishes the local asymptotically stability of $\mathscr{D}_0$.

Theorem 2. *The illicit drug use and banditry-free equilibrium, $\mathscr{D}_0$, of the system (1) is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

Proof. Since the illicit drug use and banditry reproduction number, $\mathscr{R}_0$, is obtained using the next-generation matrix method by [30], it suffices to proof that model (1) is locally asymptotically stable when $\mathscr{R}_0 < 1$.

The significance of Theorem 2 shows that the problem of the IDUB model governed by (1) will be wiped out from the population if the initial sizes of the illicit drug user and bandit sub-populations are in the basin of attraction of $\mathscr{D}_0$. However, illicit drug users and bandit obliteration are independent of the initial sizes if the $\mathscr{D}_0$ is globally asymptotically stable. The global stability result is established in the following theorem.

Theorem 3. *The IDBFE of the model (1), given by (7), is globally asymptotically stable (GAS) in the region Γ whenever $\mathscr{R}_0 \leq 1$.*

Proof. Consider the linear Lyapunov function $\mathscr{L} : \Gamma \to \mathbb{R}$ defined by

$$\mathscr{L} = S(t) + \frac{\alpha_1[\sigma_2(\gamma_3\tau_2+\tau_1 d_4)+d_3\gamma_2\tau_2]}{d_2 d_3 d_4 d_5} B(t) + \frac{\gamma_3\tau_2\alpha_1+\alpha_1\tau_1 d_4}{d_3 d_4 d_5} D(t) + \frac{\tau_2\alpha_1}{d_4 d_5} P(t) + \frac{\alpha_1}{d_5} R(t), \tag{9}$$

The time derivative of (9) along the solution path of the system (1) is given by

$$\begin{aligned}
\dot{\mathscr{L}} &= [\lambda S + \alpha_1 R - d_1 I] + \tfrac{\alpha_1[\sigma_2(\gamma_3\tau_2+\tau_1 d_4)+d_3\gamma_2\tau_2]}{d_2 d_3 d_4 d_5} \\
&\quad [\gamma_1 I - d_2 B] + \tfrac{\gamma_3\tau_2\alpha_1+\alpha_1\tau_1 d_4}{d_3 d_4 d_5} [\sigma_1 I + \sigma_2 B - d_3 D] \\
&\quad + \tfrac{\tau_2\alpha_1}{d_4 d_5} [\gamma_2 B + \gamma_3 D - d_4 P] + \tfrac{\alpha_1}{d_5} [\tau_1 D + \tau_2 P - d_5 R], \\
&= \left[\tfrac{\mu(d_2+\eta\gamma_1)}{\pi}\beta S - d_1 + \tfrac{(\alpha_1\sigma_2(\gamma_3\tau_2+\tau_1 d_4)+\alpha_1 d_3\gamma_2\tau_2)\gamma_1}{d_2 d_3 d_4 d5}\right] I \\
&\quad + \left[\tfrac{(\gamma_3\alpha_1\tau_2+\alpha_1\tau_1 d_4)}{d_3 d_4 d_5}\right] I, \\
&\leq \left[\tfrac{\beta(d_2+\eta\gamma_1)}{d_2} - \tfrac{K_7}{d_2 d_3 d_4 d_5}\right] I(t), \text{ since } S \leq \tfrac{\pi}{\mu} \text{ in } \Gamma, \\
&= \tfrac{d_2 d_3 d_4 d5}{K_7}[1 - \mathscr{R}_0] I.
\end{aligned}$$

Thus, $\dot{\mathscr{L}} \leq 0$ if $\mathscr{R}_0 \leq 1$ with $\dot{\mathscr{L}} = 0$ if and only if $I(t) = 0$. This shows that as $t \to \infty$, then $(S(t), I(t), B(t), D(t), P(t), R(t)) \to (\frac{\pi}{\mu}, 0, 0, 0, 0, 0)$. It follows that the largest compact invariant set in $\{(S(t), I(t), B(t), D(t), P(t), R(t)) \in \Gamma : \dot{\mathscr{L}} = 0\}$ is the singleton $\{\mathscr{D}_0\}$. Therefore, by LaSalle's Invariance Principle [31], the illicit drug use and banditry-free equilibrium given by $\mathscr{D}_0$ is GAS in Γ if $\mathscr{R}_0 \leq 1$.

The deduction from Theorem 4 proves that the diminution or evacuation of IDUB does not depend on the initial sizes of the population's illicit drug users and bandits. The stability property is shown in Figure 2, where all the solutions meet illicit drug use and banditry-free equilibrium. Hence, illicit drug use and banditry can be evacuated, if the associated illicit drug use and banditry reproduction number is less than one.

3.3 Illicit drug use and banditry-present equilibrium

The steady-state solution of model (1) when all the state variables are positive is referred to as the illicit drug use and banditry-present equilibrium point (IDBPE) denoted and given by

$$\mathscr{D}^* = (S^*, I^*, B^*, D^*, P^*, R^*). \tag{10}$$

Then, setting the right hand sides of (1) to zero, the following expressions are obtained in terms of λ:

$$\begin{aligned} S^* &= \frac{\pi}{\lambda+\mu}, \\ I^* &= \frac{\pi\lambda\alpha_1 A_1}{d_1[k_4 - A_8\alpha_1(\lambda+\mu)]}, \\ B^* &= \frac{\pi\lambda(A_7-(A_5+A_6)\alpha_1(\lambda+\mu))}{k_9[k_4-A_8\alpha_1(\lambda+\mu)]}, \\ D^* &= \frac{\pi\lambda k_2(k_4-\alpha_1(A_2-A_3(\lambda+\mu))}{k_7[k_4-A_8\alpha_1(\lambda+\mu)]}, \\ P^* &= \frac{\pi\lambda A_4(k_4+\alpha_1 A_5(\lambda+\mu))}{k_8[k_4-A_8\alpha_1(\lambda+\mu)]}, \\ R^* &= \frac{\pi\lambda A_2}{k_4-A_8\alpha_1(\lambda+\mu)}, \end{aligned} \tag{11}$$

where
$k_1 = d_1\tau_1 + \tau_2\gamma_2$, $k_2 = \sigma_1 d_2 + \sigma_2\gamma_1$, $k_3 = \tau_2\gamma_1\gamma_2 d_3$, $k_4 = d_1d_2d_3d_4d_5$, $k_5 = \tau_1 d_4 + \gamma_1\gamma_2 d_3 + \gamma_3\tau_2$, $k_6 = \gamma_1\gamma_2 d_3$,t $k_7 = d_1d_2d_3$, $k_8 = d_1d_2d_3d_4$, $k_9 = d_1d_2$, $A_1 = k_4 - k_2k_6 + k_3$, $A_2 = k_1k_2 + k_3$, $A_3 = k_1k_2 + k_6$, $A_4 = k_6 + \gamma_3 k_2$, $A_5 = k_6(\tau_2 - k_2)$, $A_6 = k_1k_2(\gamma_1 - 1)$, $A_7 = \gamma_1 k_4$, $A_8 = k_2k_5\alpha_1$. Now, at steady states, the force of influence λ^* becomes

$$\lambda^* = \frac{\beta(I^* + \eta B^*)}{N^*}, \tag{12}$$

and

$$N^* = S^* + I^* + B^* + D^* + P^* + R^*. \tag{13}$$

Substituting (11) and (13) into equation (12), and after some algebraic manipulations give the following results:

$$\lambda^* = 0 \text{ or } \lambda^* = \mu(\mathscr{R}_0 - 1). \tag{14}$$

When $\lambda^* = 0$ from system (11), it means illicit drug use and banditry-present equilibrium does not exist, but we have illicit drug use and banditry-free equilibrium, and when $\lambda^* = \mu(\mathscr{R}_0 - 1)$, we have unique illicit drug use and banditry-present equilibrium if $\mathscr{R}_0 > 1$. Simplifying (11), the following unique illicit drug use and banditry-present equilibrium $\mathscr{D}^*$ is obtained in terms of $\mathscr{R}_0$ as follows:

$$\begin{aligned} S^* &= \frac{\pi}{\mu\mathscr{R}_0}, \\ I^* &= \frac{\pi\mu(\mathscr{R}_0-1)\alpha_1 A_1}{d_1[k_4-A_8\alpha_1\mu\mathscr{R}_0]}, \\ B^* &= \frac{\mu\pi(\mathscr{R}_0-1)(A_7-(A_5+A_6)\alpha_1\mu\mathscr{R}_0)}{k_9[k_4-A_8\alpha_1\mu\mathscr{R}_0]}, \\ D^* &= \frac{\mu\pi(\mathscr{R}_0-1)k_2(k_4-\alpha_1(A_2-A_3\mu\mathscr{R}_0)}{k_7[k_4-A_8\alpha_1\mu\mathscr{R}_0)]}, \\ P^* &= \frac{\pi\mu(\mathscr{R}_0-1)A_4(k_4+\alpha_1 A_5\mu\mathscr{R}_0)}{k_8[k_4-A_8\alpha_1\mu\mathscr{R}_0]}, \\ R^* &= \frac{\pi\mu A_2(\mathscr{R}_0-1)}{k_4-A_8\alpha_1\mu\mathscr{R}_0}. \end{aligned} \tag{15}$$

Also, when $\mathscr{R}_0 = 1$ from system (15), it means that we have illicit drug use and banditry-free equilibrium but illicit drug use and banditry-present equilibrium does not exist. Similarly, we have a unique illicit drug use and banditry-present equilibrium when $\mathscr{R}_0 > 1$.

The bifurcation analysis is explored next to examine its stability when $\mathscr{R}_0 = 1$.

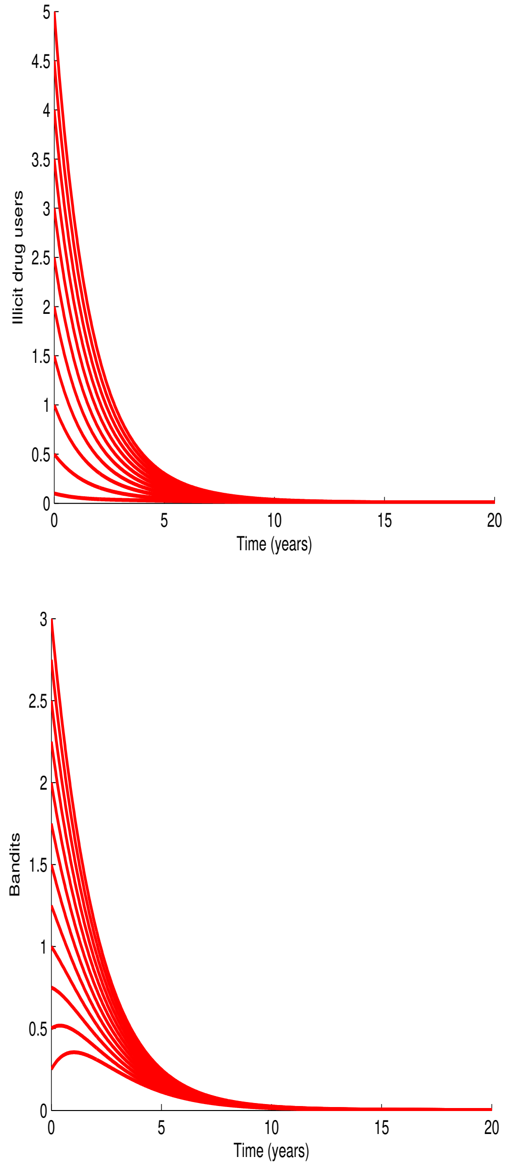


Fig. 2: Simulated results of model (1) illustrating the global dynamics of illicit drug use and banditry-free with different initial sizes. Parameter values used are given in Table 3 except: $\beta = 0.04$ and $\sigma_5 = 0.7$, so that,$\mathscr{R}_0 < 1$.

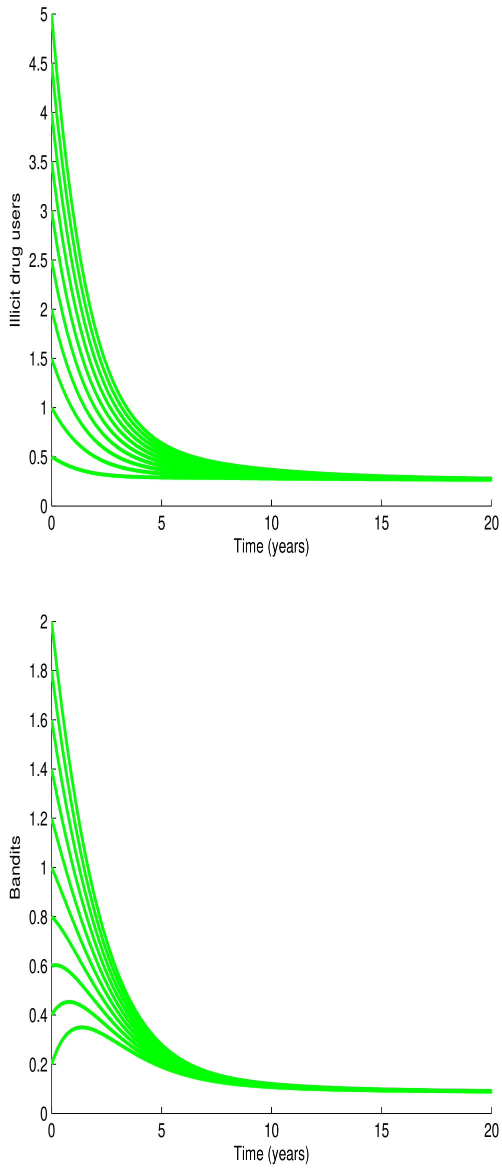


Fig. 3: Simulated results of model (1) illustrating the convergence of solution trajectories with different initial sizes to illicit drug use and banditry-present equilibrium. Parameter values used are given in Table 3, such that $\mathscr{R}_0 > 1$.

3.4 Forward Bifurcation

To study the bifurcation of the model (1), the centre manifold theory described in Castillo-Chavez and Song [32] is employed. The illicit drug use and banditry model (1) is written in vector form for this purpose as

$$\frac{dX}{dt} = F(X),$$

where $X = (x_1,x_2,x_3,x_4,x_5,x_6)^T$ and F $= (f_1,f_2,f_3,f_4,f_5,f_6)^T$ with $S = x_1, I = x_2, B = x_3, D = x_4, P = x_5, R = x_6$. Then, model (1) becomes

$$\begin{aligned}
f_1 &= \frac{dx_1}{dt} = \pi - \lambda x_1 - \mu x_1,\\
f_2 &= \frac{dx_2}{dt} = \lambda x_1 + \alpha_1 x_6 - (\sigma_1 + \gamma_1 + \delta_1 + \mu)x_2,\\
f_3 &= \frac{dx_3}{dt} = \gamma_1 x_2 - (\sigma_2 + \gamma_2 + \delta_2 + \mu)x_3,\\
f_4 &= \frac{dx_4}{dt} = \sigma_1 x_2 + \sigma_2 x_3 - (\sigma_3 + \gamma_3 + \tau_1 + \mu)x_4,\\
f_5 &= \frac{dx_5}{dt} = \gamma_2 x_3 + \gamma_3 x_4 - (\sigma_4 + \tau_2 + \delta_4 + \mu)x_5,\\
f_6 &= \frac{dx_6}{dt} = \tau_1 x_4 + \tau_2 x_5 - (\sigma_5 + \alpha_1 + \mu)x_6.
\end{aligned} \tag{16}$$

At $\mathscr{R}_0 = 1$ in (8), the bifurcation parameter β^* can be obtained as

$$\beta^* = \frac{K_7}{d_3 d_4 d_5 (d_2 + \eta\gamma_1)}. \tag{17}$$

The linearized matrix of the system (16) around $\mathscr{D}_0$ and evaluated at β^* is given by

$$\mathscr{J}_{(\mathscr{D}_0,\beta^*)} = \begin{pmatrix} -\mu & -\beta^* & -\beta^*\eta & 0 & 0 & 0\\ 0 & \beta^* - d_1 & \beta^*\eta & 0 & 0 & \alpha_1\\ 0 & \gamma_1 & -d_2 & 0 & 0 & 0\\ 0 & \sigma_1 & \sigma_2 & -d_3 & 0 & 0\\ 0 & 0 & \gamma_2 & \gamma_3 & -d_4 & 0\\ 0 & 0 & 0 & \tau_1 & \tau_2 & -d_5 \end{pmatrix}.$$

The eigenvalues λ of $\mathscr{J}_{(\mathscr{D}_0,\beta^*)}$ given by the matrix above are the roots of the characteristic equation of the form

$$(\lambda + \mu)\mathscr{P}(\lambda) = 0, \tag{18}$$

where $\mathscr{P}(\lambda)$ is a polynomial of degree four whose roots are all negative except one zero eigenvalues. The left eigenvector, $v = (v_1, v_2, ..., v_6)$, corresponding to the simple zero eigenvalue of (16) is obtained from $v\mathscr{J}_{(\mathscr{D}_0,\beta^*)} = 0$ as

$$\begin{aligned}
&v_1 = 0,\ v_2 = \frac{d_5}{\alpha_1}v_6,\ v_4 = \frac{\tau_2\gamma_3 + \tau_1 d_4}{d_3 d_4}v_6,\\
&v_3 = \frac{\beta^*\eta d_3 d_4 d_5 + \sigma_2\alpha_1(\tau_2\gamma_3 + \tau_1 d_4) + \gamma_2\tau_2\alpha 1d3}{\alpha 1 d_2 d_3 d_4}v_6,\\
&v_5 = \frac{\tau_2}{d_4}v_6.
\end{aligned} \tag{19}$$

Further, the right eigenvector, $w = (w_1, w_2, ..., w_6)^T$, associated with this simple zero eigenvalue can be obtained from $w\mathcal{J}_{(\mathcal{D}_0,\beta^*)} = 0$. As a result, we have

$$w_1 = \frac{-\beta^*(d_2+\eta\gamma_1)}{\mu d_2}w_2,\ w_3 = \frac{\gamma_1}{d_2}w_2,$$
$$w_4 = \frac{\sigma_1 d_2+\sigma_2\gamma_1}{d_2 d_3}w_2,\ w_5 = \frac{\gamma_1\gamma_2 d_3+\gamma_3(\sigma_1 d_2+\sigma_2\gamma_1)}{d_2 d_3 d_4}w_2,$$
$$w_6 = \frac{\tau_1 d_4(\sigma_1 d_2+\sigma_2\gamma_1)+\tau_2(\gamma_1\gamma_2 d_3+\gamma_3(\sigma_1 d_2+\sigma_2\gamma_1))}{d_2 d_3 d_4 d_5}w_2. \quad (20)$$

It should be noted that the components of w and v are obtained so that $v.w = 1$ as required in [32]. All the second-order partial derivatives of $f_i, i = i = 1,2...,6$, from the system (16) are zero at point $(\mathcal{D}_0,\beta^*)$ except the following:

$$\begin{aligned}\frac{\partial^2 f_1}{\partial x_1\partial x_2} &= \frac{\partial^2 f_1}{\partial x_2\partial x_1} = \frac{-\beta^*\mu}{\pi},\\ \frac{\partial^2 f_1}{\partial x_1\partial x_3} &= \frac{\partial^2 f_1}{\partial x_3\partial x_1} = \frac{-\beta^*\mu\eta}{\pi},\\ \frac{\partial^2 f_2}{\partial x_1\partial x_2} &= \frac{\partial^2 f_2}{\partial x_2\partial x_1} = \frac{\beta^*\mu}{\pi},\\ \frac{\partial^2 f_2}{\partial x_1\partial x_3} &= \frac{\partial^2 f_2}{\partial x_3\partial x_1} = \frac{\beta^*\mu\eta}{\pi},\end{aligned} \quad (21)$$

with

$$\frac{\partial^2 f_1}{\partial x_2\partial\beta} = -1,\quad \frac{\partial^2 f_1}{\partial x_3\partial\beta} = -\eta,$$
$$\frac{\partial^2 f_2}{\partial x_2\partial\beta} = 1,\quad \frac{\partial^2 f_2}{\partial x_3\partial\beta} = \eta. \quad (22)$$

The direction of the bifurcation at $\mathcal{R}_0 = 1$ is determined by the signs of the bifurcation coefficients **a** and **b**, define as follow:-

$$\mathbf{a} = \sum_{k,i,j=1}^{6} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i\partial x_j}(\mathcal{D}_0,\beta^*) \quad (23)$$

and

$$\mathbf{b} = \sum_{k,i=1}^{6} v_k w_i \frac{\partial^2 f_k}{\partial x_i\partial\beta}(\mathcal{D}_0,\beta^*). \quad (24)$$

Since $v_1 = 0$, then (23) and (24) can be written as

$$\mathbf{a} = v_2 w_1 w_2\frac{\partial^2 f_2}{\partial x_1\partial x_2} + v_2 w_1 w_3\frac{\partial^2 f_2}{\partial x_1\partial x_3} \quad (25)$$

and

$$\mathbf{b} = v_2 w_2\frac{\partial^2 f_2}{\partial x_2\partial\beta} + v_2 w_3\frac{\partial^2 f_2}{\partial x_3\partial\beta}. \quad (26)$$

Substituting (17), (19), (20), (21) and (22) into (25) and (26) gives

$$\mathbf{a} = \frac{-d_5\mu^2}{\pi\alpha_1}w_2^2 v_6 \quad (27)$$

and

$$\mathbf{b} = \frac{d_5(d_2+\gamma_1\eta)}{\alpha_1 d_2}. \quad (28)$$

As we can see, $a < 0$ and $b > 0$. It follows that the IDUB model (1) exhibits a forward bifurcation, and $\mathcal{D}^*$ is locally stable. This result is claimed as follows.

Theorem 4. *The IDUB model governed by (1) exhibits a forward bifurcation at the threshold $\mathcal{R}_0 = 1$ (or, equivalently, there exists an illicit drug use and banditry-present equilibrium, $\mathcal{D}^*$, which is locally asymptotically stable whenever $\mathcal{R}_0 > 1$, but near $\mathcal{R}_0 = 1$).*

The epidemiological importance of the above result is that a small inflow of illicit drug users and bandits into a completely susceptible population will cause a raise in the spread of illicit drug use and banditry within the community whenever $\mathcal{R}_0 > 1$. Nevertheless, the initial sizes of the illicit drug users and bandits in the population are critical factors in this result. Hence, to show that the elimination and persistence of illicit drug use and banditry do not depend on the initial sizes of the illicit drug users and bandits, we established the global stability of IDBPE in the next Section.

3.5 Global stability of IDBPE

Theorem 5. *The unique IDBPE of the illicit drug use and banditry model (1), given by (15), is globally asymptotically stable (GAS) in $\Gamma\setminus\Gamma_0$ when $\mathcal{R}_0 > 1$, $DR^{**} \le D^{**}R, DP^{**} \le D^{**}P$ and $PR^{**} \le P^{**}R$.*

Proof. Consider the non-linear Lyapunov function $\mathfrak{F} : \Gamma\setminus\Gamma_0 \to \mathbb{R}$ defined by

$$\begin{aligned}\mathfrak{F} &= S - S^{**} - S^{**}\ln\frac{S}{S^{**}} + \left(I - I^{**} - I^{**}\ln\frac{I}{I^{**}}\right)\\ &+\frac{\alpha_1\tau_1(\gamma_3+d_4)}{d_3 d_4 d_5}\left(D - D^{**} - D^{**}\ln\frac{D}{D^{**}}\right)\\ &+\frac{d_3 d_4 d_5(d_1+\beta S^{**})+\sigma_1\alpha_1\tau_1(\gamma_3+d_4)}{d_3 d_4 d_5\gamma_1}\\ &\left(B - B^{**} - B^*\ln\frac{B}{B^{**}}\right)\\ &+\frac{\alpha_1\tau_1}{d_4 d_5}\left(P - P^{**} - P^{**}\ln\frac{P}{P^{**}}\right)\\ &+\frac{\alpha_1}{d_5}\left(R - R^{**} - R^{**}\ln\frac{R}{R^{**}}\right),\end{aligned} \quad (29)$$

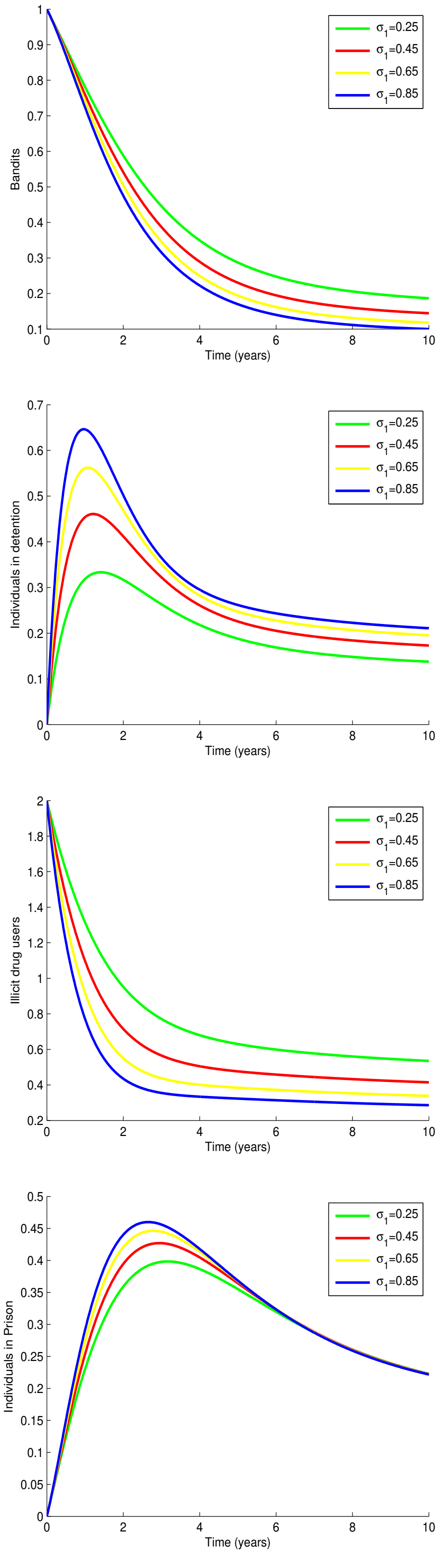


Fig. 4: Simulation results of model (1) with the effect of σ_1 on (a) Individuals in detention; (b) Bandits; (c) Illicit drug users and (d) Individuals in prison

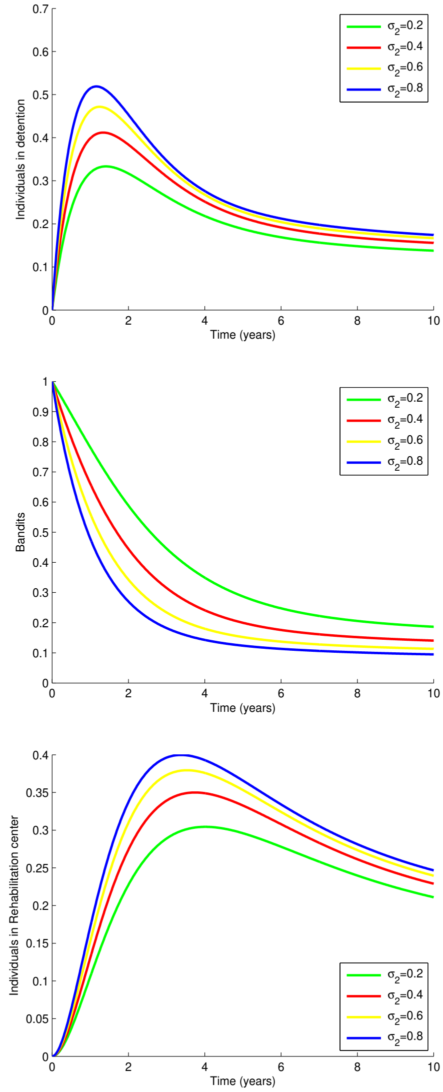


Fig. 5: Simulation results of model (1) with the effect of σ_2 (a) Individuals in detention; (b) Bandits and (c) Individuals in rehabilitation center.

which is of Goh-Volterra type (see, e.g., [33,34,35,38]). The Lyapunov derivative is given by

$$\frac{d\mathfrak{F}}{dt} = \frac{dS}{dt} - \frac{S^{**}}{S}\frac{dS}{dt} + \left(\frac{dI}{dt} - \frac{I^{**}}{I}\frac{dI}{dt}\right) + Z_1\left(\frac{dB}{dt} - \frac{B^{**}}{B}\frac{dB}{dt}\right)$$
$$+ \frac{\alpha_1\tau_1(\gamma_3 + d_4)}{d_3 d_4 d_5}\left(\frac{dD}{dt} - \frac{D^{**}}{D}\frac{dD}{dt}\right)$$
$$+ \frac{\alpha_1\tau_1}{d_4 d_5}\left(\frac{dP}{dt} - \frac{P^{**}}{P}\frac{dP}{dt}\right)$$
$$+ \frac{\alpha_1}{d_5}\left(\frac{dR}{dt} - \frac{R^{**}}{R}\frac{dR}{dt}\right). \tag{30}$$

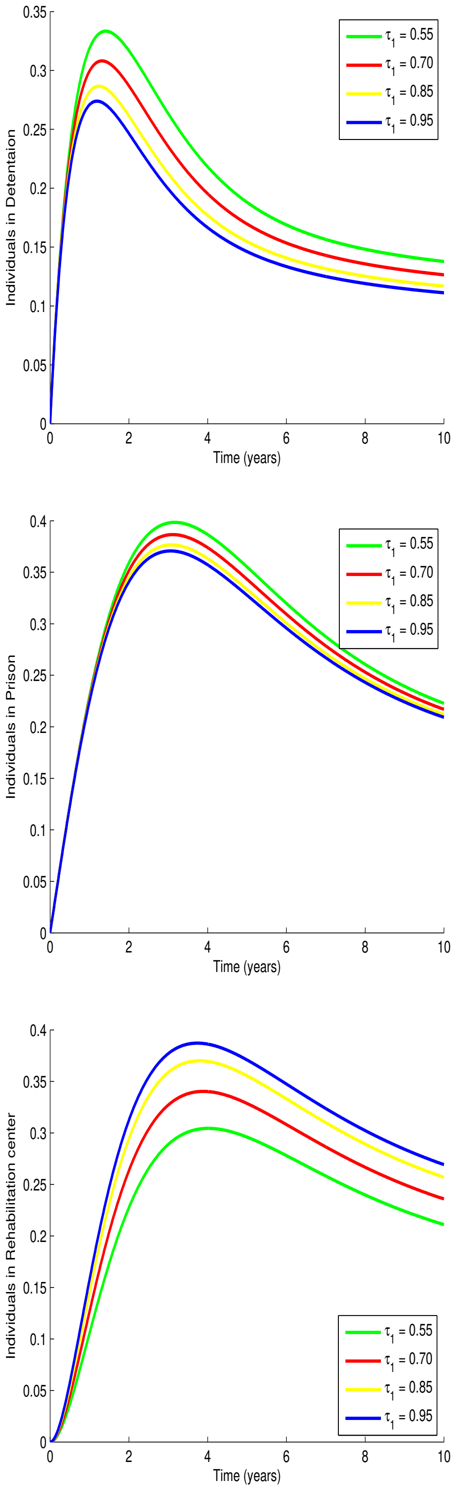


Fig. 6: Simulation results of model (1) with the effect of τ_1 on (a) Individuals in detention; (b) Individuals in prison and (c) Individuals in the rehabilitation center.

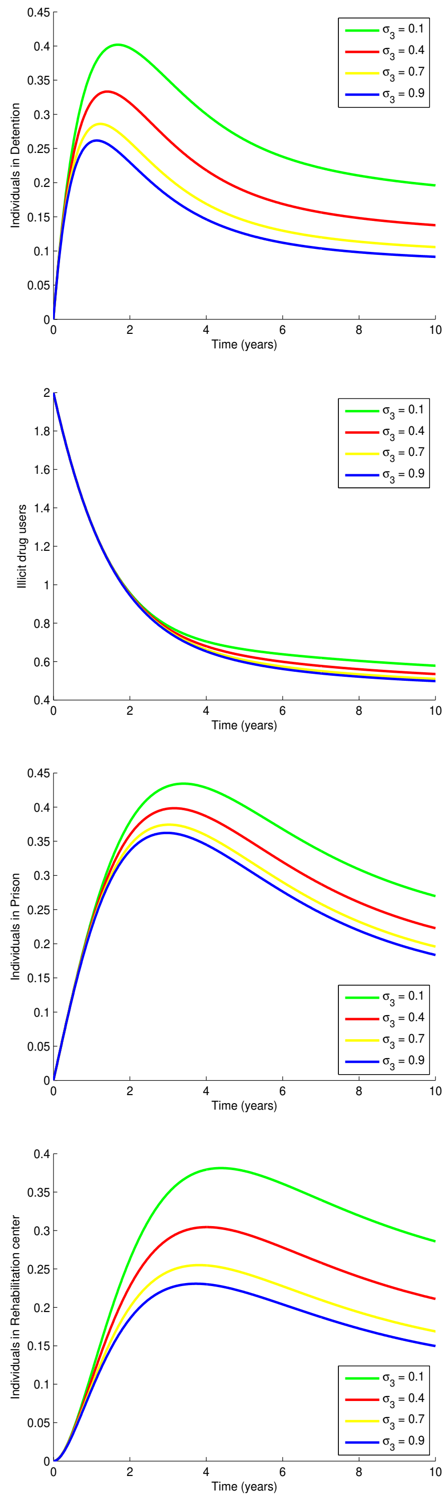


Fig. 7: Simulation results of model (1) with the effect of σ_3 on (a) Individuals in detention; (b) Illicit drug users; (c) Individuals in prison and (d) Individuals in the rehabilitation centre.

where $Z_1 = \dfrac{d_3d_4d_5(d_1+\beta S^{**})+\sigma_1\alpha_1\tau_1(\gamma_3+d_4)}{d_3d_4d_5\gamma_1}$,
$Z_2 = (\sigma_5+\alpha_1+\mu)$.

Let $\frac{\beta(I(t)+\eta B(t))}{N(t)} = \tilde{\beta}(I(t)+\eta B(t))$, then putting the appropriate equations of the system (1) into (30), leads

$$\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \left(1-\frac{S^{**}}{S}\right)(\pi-\tilde{\beta}(I(t)+\eta B(t))S(t)-\mu S(t))\\
&+\left(1-\frac{I^{**}}{I}\right)(\tilde{\beta}(I(t)+\eta B(t))S(t)+\alpha_1 R(t)-\\
&(\sigma_1+\gamma_1+\delta_1+\mu)I(t))+Z_1\left(1-\frac{B^{**}}{B}\right)\\
&(\gamma_1 I(t)-(\sigma_2+\gamma_2+\delta_2+\mu)B(t))\\
&+\frac{\alpha_1\tau_1(\gamma_3+d_4)}{d_3d_4d_5}\left(1-\frac{D^{**}}{D}\right)(\sigma_1 I(t)+\sigma_2 B(t)\\
&-(\sigma_3+\gamma_3+\tau_1+\mu)D(t))\\
&+\frac{\alpha_1\tau_1}{d_4d_5}\left(1-\frac{P^{**}}{P}\right)(\gamma_2 B(t)+\gamma_3 D(t)\\
&-(\sigma_4+\tau_2+\delta_4+\mu)P(t))+\frac{\alpha_1}{d_5}\\
&\left(1-\frac{R^{**}}{R}\right)(\tau_1 D(t)+\tau_2 P(t)-Z_2 R(t)).
\end{aligned} \tag{31}$$

.

At the illicit drug use and banditry-present equilibrium, the following relations hold from the system (1):

$$\begin{aligned}
\pi &= \tilde{\beta}(I^{**}+\eta B^{**})S^{**}+\mu S^{**},\\
\sigma_1+\gamma_1+\delta_1+\mu &= \tfrac{\tilde{\beta}(I^{**}+\eta B^{**})}{I^{**}}+\tfrac{\alpha_1 R^{**}}{I^{**}},\\
\sigma_2+\gamma_2+\delta_2+\mu &= \tfrac{\gamma_1 I^{**}}{B^{**}},\\
\sigma_3+\gamma_3+\tau_1+\mu &= \tfrac{\sigma_1 I^{**}+\sigma_2 B^{**}}{D^{**}},\\
\sigma_4+\tau_2+\delta_4+\mu &= \tfrac{\gamma_2 B^{**}+\gamma_3 D^{**}}{P^{**}},\\
\sigma_5+\alpha_1+\mu &= \tfrac{\tau_1 D^{**}+\tau_2 P^{**}}{R^{**}}.
\end{aligned} \tag{32}$$

Using the relations (32) in (31) and simplifying yields

$$\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \tilde{\beta}S^{**}(I^{**}+\eta B^{**})\left(1-\frac{S^{**}}{S}\right)+\mu S^{**}\\
&\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)+\alpha_1 R^{**}\left(1-\frac{RI^{**}}{R^{**}I}\right)\\
&+\tilde{\beta}S^{**}(I+\eta B)\left(1-\frac{SI^{**}}{S^{**}I}\right)+\tfrac{\sigma_1\alpha_1\tau_1\gamma_3}{d_3d_4d_5}I^{**}\\
&\left(2-\frac{IB^{**}}{I^{**}B}-\frac{ID^{**}}{I^{**}D}\right)+(d_1+\tilde{\beta}S^{**})\left(1-\frac{IB^{**}}{I^{**}B}\right)\\
&+\tfrac{\sigma_1\alpha_1\tau_1}{d_3d_5}I^{**}\left(2-\frac{IB^{**}}{I^{**}B}-\frac{ID^{**}}{I^{**}D}\right)+\tfrac{\sigma_2\alpha_1\tau_1}{d_3d_5}B^{**}\\
&\left(1-\frac{BD^{**}}{B^{**}D}\right)+\tfrac{\sigma_2\alpha_1\tau_1\gamma_3}{d_3d_4d_5}B^{**}\left(1-\frac{BD^{**}}{B^{**}D}\right)\\
&+\tfrac{\gamma_2\alpha_1\tau_1}{d_4d_5}B^{**}\left(1-\frac{BP^{**}}{B^{**}P}\right)+\tfrac{\gamma_3\alpha_1\tau_1}{d_4d_5}D^{**}\left(1-\frac{DP^{**}}{D^{**}P}\right)\\
&+\tfrac{\alpha_1\tau_1}{d_5}D^{**}\left(1-\frac{DR^{**}}{D^{**}R}\right)+\tfrac{\alpha_1\tau_2}{d_5}P^{**}\left(1-\frac{PR^{**}}{P^{**}R}\right).
\end{aligned} \tag{33}$$

Since $DR^{**}\le D^{**}R$, $DP^{**}\le D^{**}P$ and $PR^{**}\le P^{**}R$ then $1-\dfrac{D^{**}R}{DR^{**}}\le 0$, $1-\dfrac{D^{**}P}{DP^{**}}\le 0$ and $1-\dfrac{P^{**}R}{PR^{**}}\le 0$ with equality if $D^{**}R = DR^{**}$, $D^{**}P = DP^{**}$ and $P^{**}R = PR^{**}$. Consequently, (33) becomes

$$\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \tilde{\beta}S^{**}(I^{**}+\eta B^{**})\left(1-\frac{S^{**}}{S}\right)+\mu S^{**}\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)\\
&+\alpha_1 R^{**}\left(1-\frac{RI^{**}}{R^{**}I}\right)+\tilde{\beta}S^{**}(I+\eta B)\left(1-\frac{SI^{**}}{S^{**}I}\right)\\
&+\tfrac{\sigma_1\alpha_1\tau_1\gamma_3}{d_3d_4d_5}I^{**}\left(2-\frac{IB^{**}}{I^{**}B}-\frac{ID^{**}}{I^{**}D}\right)\left(1+\frac{\gamma_3}{d_4}\right)\\
&+(d_1+\tilde{\beta}S^{**})\left(1-\frac{IB^{**}}{I^{**}B}\right)+\tfrac{\sigma_2\alpha_1\tau_1}{d_3d_5}B^{**}\left(1-\frac{BD^{**}}{B^{**}D}\right)\\
&\left(1+\frac{\gamma_3}{d_4}\right)+\tfrac{\gamma_2\alpha_1\tau_1}{d_4d_5}B^{**}\left(1-\frac{BP^{**}}{B^{**}P}\right).
\end{aligned} \tag{34}$$

Using Arithmetic Mean -Geometric Mean inequality [36], the following ineqaulities hold:

$$\left(1-\frac{S}{S^{**}}\right)\le 0,\quad \left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)\le 0,$$

$$\left(2-\frac{IB^{**}}{I^{**}B}-\frac{ID^{**}}{I^{**}D}\right)\le 0,\quad \left(1-\frac{BP^{**}}{B^{**}P}\right)\le 0,$$

$$\left(1-\frac{RI^{**}}{R^{**}I}\right)\le 0,\quad \left(1-\frac{SI^{**}}{S^{**}I}\right)\le 0,\quad \left(1-\frac{IB^{**}}{I^{**}B}\right)\le 0,$$

$$\left(1-\frac{BD^{**}}{B^{**}D}\right)\le 0.$$

Table 2: Sensitivity sign of each parameter of model (1)

Parameter	Sign
β	Positive
α_1	Positive
τ_1	Positive
η	Positive
δ_4	Positive
σ_1	Negative
δ_1	Negative
μ	Negative
σ_2	Negative
γ_2	Negative
δ_2	Negative
σ_3	Negative
γ_3	Negative
γ_1	Negative
σ_5	Negative

Table 3: Values of the parameters of model (1)

Parameter	Range	Baseline value	Source
π	40 - 60	50	[19,22]
β	0.2 - 0.5	0.64	[17,26]
η	0.6 - 0.9	0.7	[41]
μ	0.009 - 0.04	0.02	[41]
α_1	0.6 - 0.9	0.75	[19,42]
σ_1	0.5 - 0.8	0.65	[22,41]
γ_1	0.5 - 0.8	0.6	Assumed
δ_1	0.09 - 0.2	0.14	[19]
σ_2	0.5 - 0.8	0.6	[19,41]
γ_2	0.5 - 0.8	0.65	[17]
δ_2	0.09 - 0.2	0.14	[19]
σ_3	0.6 - 0.9	0.7	[17]
γ_3	0.6 - 0.9	0.7	[17]
τ_1	0.7 - 0.95	0.85	[19]
σ_4	0.1 - 0.3	0.2	[17]
τ_2	0.05 - 0.2	0.1	[4,19]
δ_4	0.08 - 0.2	0.14	[19]
σ_5	0.4 - 0.9	0.55	Assumed

Moreover, since all the model parameters are non-negative, it follows from (34) that $\frac{d\mathfrak{F}}{dt} \leq 0$ with equality if and only if $S = S^{**}$, $I = I^{**}$, $B = B^{**}$, $D = D^{**}$, $P = P^{**}$, $R = R^{**}$. Hence, by LaSalle's invariance principle [31], $(S,I,B,D,P,R) \to \mathscr{D}_0$ as $t \to \infty$.

Literally, Theorem 5 means that IDUB will remain, regardless of the initial sizes of illicit drug users and bandits in the population, whenever $\mathscr{R}_0 > 1$. This stability property is shown in Figure 3, where all the solutions tend to illicit drug use and banditry-present equilibrium (IDBPE). This now leads us to sensitivity analysis.

3.6 Sensitivity Analysis

Following the idea in [9,10,33,39,40], we perform a sensitivity analysis of the model (1) to determine the contributory effects of the model parameters on the transmission and spread of the illicit drug and banditry menace in a population. The normalized forward-sensitivity index of a variable, v, that depends differentiably on a parameter, p, is defined as

$$\Upsilon_p^v = \frac{\partial v}{\partial p} \times \frac{p}{v}. \tag{35}$$

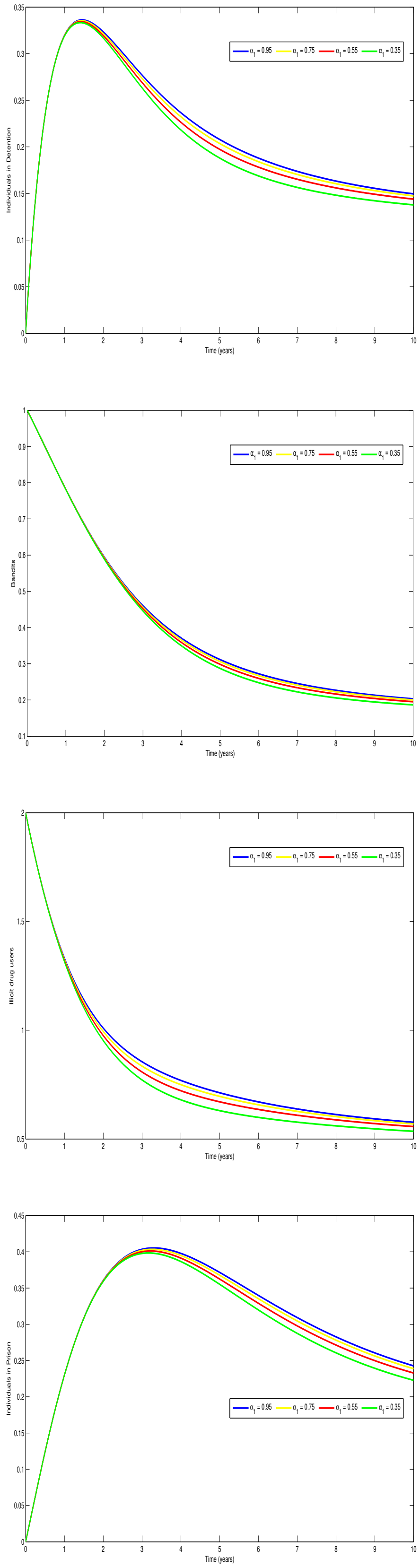
In particular, sensitivity indices of the basic reproduction number, $\mathscr{R}_0$, with respect to the model parameters are computed and the summary is given in Table 2.

The sign of the sensitivity index plays a key role in determining how the model's parameters relate to the basic reproduction number, $\mathscr{R}_0$, of the model.

In Table 2, the parameters with positive signs have a direct relationship with $\mathscr{R}_0$ while the ones with negative sensitivity sign have an inverse relation with $\mathscr{R}_0$. This means that increasing any parameter with the positive sign will increase the value of $\mathscr{R}_0$ (i.e the menace will persist) and vice versa, while increasing any parameter with the negative sign will decrease the value of $\mathscr{R}_0$ (i.e the menace will fade out) and the vice versa. With sensitivity analysis, one can get insight into the appropriate intervention strategies to prevent and control the effect and spread of IDUB in the population. It means that an effect should be made to reduce the value of parameters with a positive sign, in like manner, the value of parameters with a negative sign must be increased at all costs.

In addition, the graphs of the IDUB model (1) are plotted, using the value of parameters in Table 3, to study the behavioural effects of detention rate of illicit drug users σ_1, treatment rate of individuals in rehabilitation center σ_5, detention rate of bandits σ_2, rehabilitation rate of detainee τ_1, quitting rate of detainee σ_3, loss of determination of individuals in rehabilitation center α_1, effective influence rate β, progression rates of the illicit drug user to bandits γ_1 and rates at which bandits are imprisoned γ_2, as well as the rate at which the detainees are imprisoned γ_3 on the dynamics of the population.

From Figure (4), it can be observed that as σ_1 increases (decreases), the population of bandits decreases (increases), illicit drug user decreases (increases) with time while as σ_1 increases (decreases), the population of detainee and individuals in prison increases (decreases) with time. It shows that σ_1 has a positive effect on the population of bandits and illicit drug users. The physical meaning of this is that as the detention rate of illicit drug users increases, the number of illicit drug users and bandits in the population decreases. Thus, σ_1 is a good corrective measure that can reduce the spread of IDUB in the population.



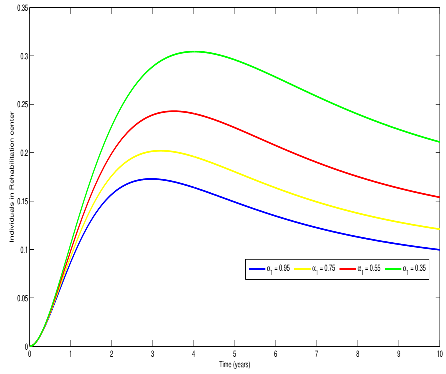
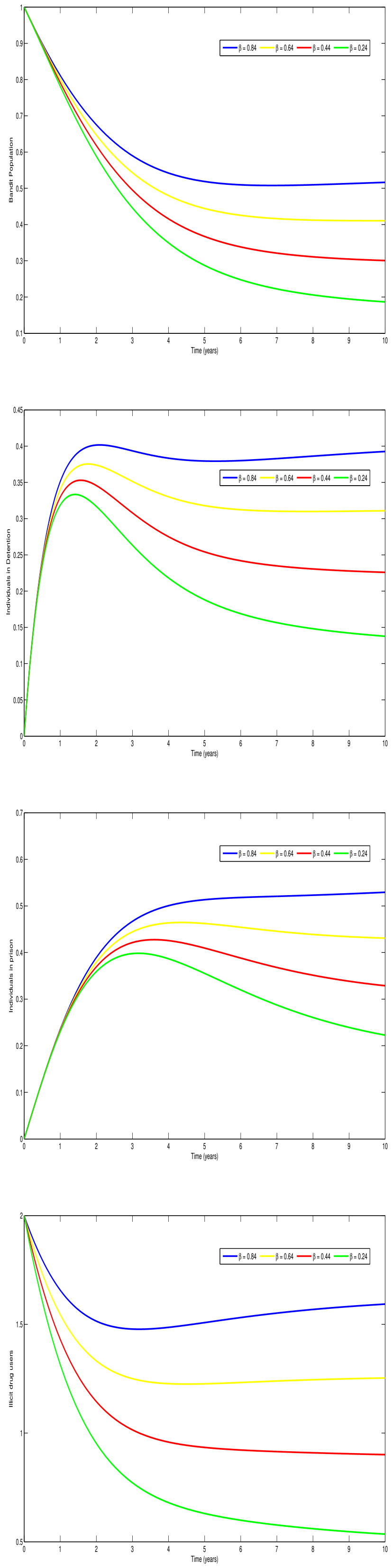


Fig. 8: Simulation results of model (1) with the effect of α_1 on (a) Individuals in detention; (b) Bandits; (c) Illicit drug users (d) Individuals in prison and (e) Individuals in the rehabilitation centre

In a similar manner, Figure 5 reveals the behaviour of detention rate of bandits σ_2 on detainees, bandits and individuals in the rehabilitation center. It is observed that the populations of detainees and individuals in rehabilitation center increase (decrease) as σ_2 increases (decrease) but the population of bandits decreases. It shows that σ_2 positively affects the bandits' population. Literally, the higher the detention rate of bandits, the lesser the number of bandits in the population. This shows that σ_2 is also a good corrective measure that can reduce the spread of banditry in the population. Also, Figure 6 shows the effects of τ_1 on the population of detainees, prisoners and individuals in the rehabilitation center. More so, as τ_1 rises, the population of detainees and prisoners falls while the population of individuals in rehabilitation center increases. It shows that τ_1 has a positive effect on populations of detainees and prisoners. Figuratively, the higher the rehabilitation rate of detainees, the lesser the number of detainees in the population, which means illicit drug use and banditry acts will reduce. This also shows that τ_1 is a good corrective measure that can reduce the spread of illicit drug use and banditry in the population.

In addition, the populations of detainees, illicit drug users, prisoners and individuals in rehabilitation center increase (decrease) as quitting rate of detainees decreases (increases) (Figure 7). It shows that all efforts must be put in place to encourage detainees to quit the act of illicit drug use or banditry, as the case may be. Literally, this means that decreasing the quitting rate of detainees increases the number of detainees in the population, which means illicit drug use and banditry will reduce. This also shows that σ_3 is a good corrective measure that can reduce the spread of IDUB in the population. In another development, Figure 8 shows the effects of the relapsed rate of individuals in a rehabilitation center, α_1,



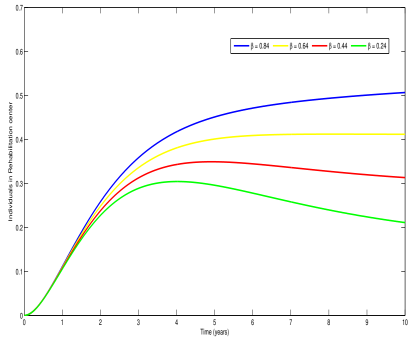


Fig. 9: Simulation results of model (1) with the effect of β on (a) Bandits; (b) Individuals in detention; (c) Individuals in prison (d) Illicit drug users and (e) Individuals in the rehabilitation center.

on the detainees, bandits, illicit drug users, prisoners and individuals in the rehabilitation center. It is observed that detainees, bandits, illicit drug users and prisoners increase (decrease) as α_1 increases (decreases), while the population of individuals in rehabilitation center decreases (increases). It shows that α_1 has a negative effect on the population of illicit drug users and bandits. The physical meaning of this is that the higher the relapsed rate of individuals in a rehabilitation center, the number of illicit drug users and bandits population increases and that increasing α_1 will have a positive effect on the dynamics of IDUB. This shows that rehabilitated individuals should be monitored closely to avoid relapsed or re-occurrence of IDUB in the population.

It can be observed in Figure 9 that as β increases (decreases), the populations of bandits, detainees, prisoners, illicit drug users and individuals in rehabilitation center increase (decrease) with time. The physical meaning of this is that the number of bandits, detainees, prisoners, illicit drug users and individuals in rehabilitation center increases as the effective influence rate of IDUB increases. This connotes that control measures must be targeted at effective influence rate to inhibit the spread of IDUB in the population. Similarly, it is seen that the populations of bandits and prisoner increase (decrease) as the movement rate of the illicit drug users to bandit increases (decreases) while the population of illicit drug users decreases (increases) as the movement rate of the illicit drug user to bandit increases (decreases) as shown in Figure 10. It shows that γ_1 has a positive effect on the act of illicit drug use. In the physical sense, the higher the movement rate of the illicit drug users to the bandit, the lesser the movement of the illicit drug user. Also, It is observed in Figure 11 that the population of bandits decreases (increases) as γ_2 increases (decreases) while the population of prisoners and detainee increase (decreases) with time. It shows that γ_2 positively affects the bandits' population. The meaning of this is that

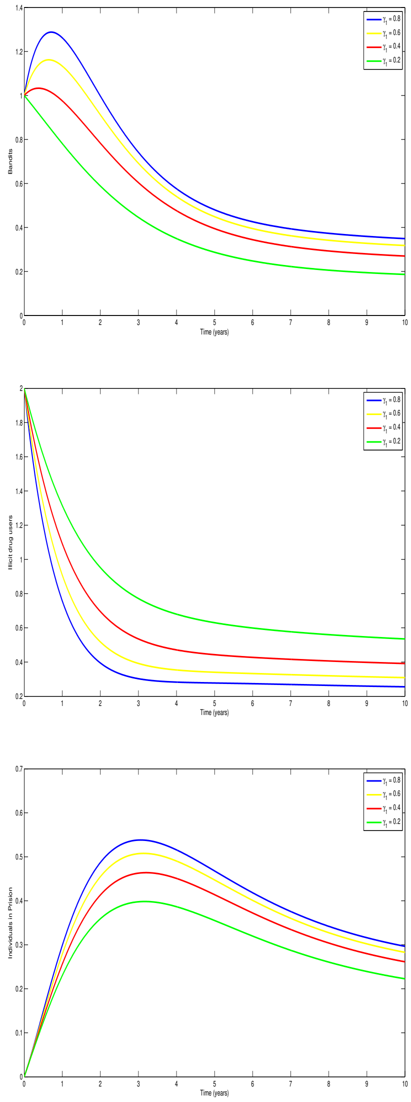


Fig. 10: Simulation results of model (1) with the effect of γ_1 on (a) Bandits; (b) Illicit drug users (c) Individuals in prison.

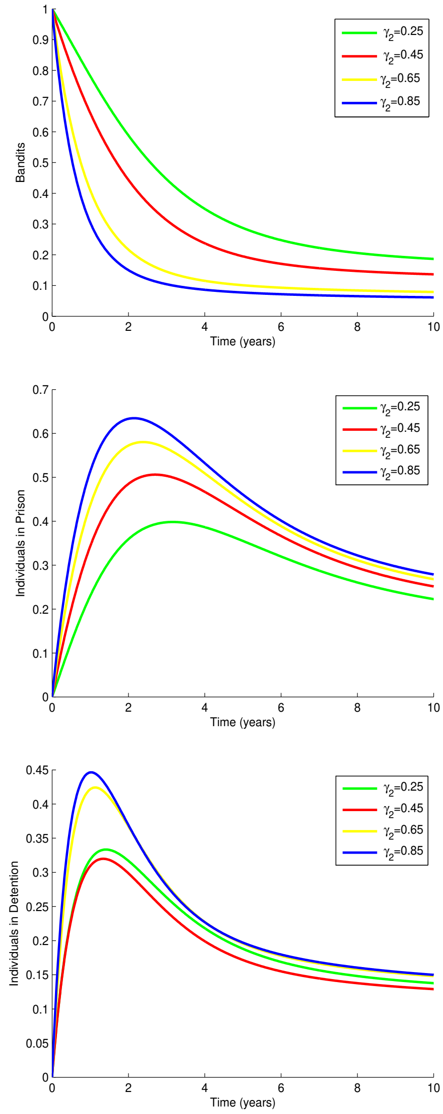


Fig. 11: Simulation results of model (1) with the effect of γ_2 on (a) Bandits, (b) Individuals in prison and (c) Individuals in detention.

the higher the number of movements of bandits to prison, the lesser the number of bandits in the population, but good corrective measures must be put in place to compliment the work of γ_2. This shows that increasing γ_2 will increase the population's banditry spread.

Finally, in Figure 12, it is observed that γ_3 increases (decreases) as the population of detainees decreases (increases) while the population of prisoners increases (decreases). It shows that γ_3 has a positive effect on the population of detainees. The physical interpretation is that the higher the progression rate of detainees to prison, the fewer the number of detainees in the population, which means that illicit drug use and banditry will reduce. Still, good corrective measures must be implemented in prison to complement the work of γ_3.

4 Conclusion

This work has presented and analysed a suitable compartmental deterministic model for IDUB population dynamics. The illicit drug and banditry model was formulated and analysed to study the effect of illicit drug use and banditry on population dynamics. The existence of illicit drug and banditry-free and illicit drug and banditry-present equilibria was shown. The illicit drug

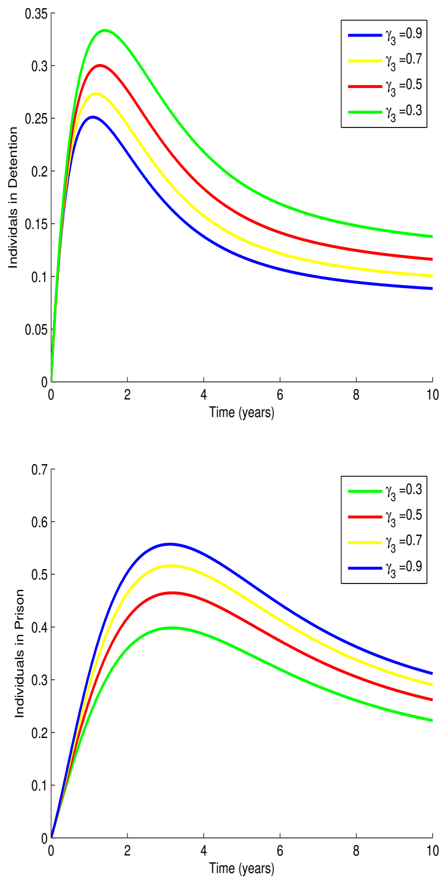


Fig. 12: Simulation results of model (1) with the effect of γ_3 (a) Individuals in detention and (b) Individuals in prison.

and banditry reproduction number was computed, and the banditry-free equilibrium and local stability were shown. The formulated model was proved to have global asymptotic stability when the illicit drug use and banditry-free equilibrium is less than unity by constructing a suitable linear Lyapunov function. Also, the model was shown to have unique illicit drug use and banditry-present equilibrium whenever the associated illicit drug use and banditry reproduction number exceeds unity.

Further, the unique illicit drug use and banditry-present equilibrium were shown to be globally asymptotically stable by a suitably constructed nonlinear Lyapunov function of the Goh-Volterra type. Bifurcation analysis of the mode was carried out on the illicit drug and banditry-present equilibrium and proved the same to be a forward bifurcation. The sensitivity analysis of the model was done to know the contributory effects of each parameter on the dynamic spread of illicit drug use and banditry in the population. Finally, corrective measures like detention, rehabilitation and quitting rates should be beefed up for illicit drug use and a banditry-free population. This points out that if the law against illicit drug use and banditry are strengthened, and good corrective measure is put in place in the prison, it will reduce the number of illicit drug users and bandits in the population.

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John O. Akanni is currently a Postdoctoral fellow at Universitas Airlangga, Surabaya, Indonesia. He finished his PhD in Applied Mathematics at the Pure and Applied Mathematics Department, Ladoke Akintola University of Technology, Ogbomoso, Nigeria, in August 2021. In addition, he holds a Masters of Technology (MTech.) degree in Applied Mathematics from the same institution in July 2017. He works at the Department of Mathematical and Computing Sciences, Faculty of Applied Sciences, Koladaisi University Ibadan. Akanni does research in Applied Mathematics. His current project revolves around the 'Mathematical model of dynamical systems, infectious disease and behavioural systems'. He has published a few articles and a book; similarly, he is a reviewer of many International and national mathematical journals.

Afeez Abidemi is a Lecturer in Mathematics at the Federal University of Technology Akure (FUTA), Ondo State, Nigeria. He received his PhD in Mathematics (Biomathematics) from Universiti Teknologi Malaysia (UTM) in December 2020. His areas of research expertise are Biomathematics, fractional calculus, mathematical modelling, optimal control theory and ordinary differential equations among others. As scientific contributions to the body of knowledge, he has coauthored over 20 publications in reputable journals. He has served as a reviewer for more than 10 international journals in the frame of applied mathematics.