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New Special Surfaces in de Sitter 3-Space

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The setting of this work is the de Sitter 3-space S_1^3 and the study of space-like surfaces based on space-like curves. Moreover, a study of Bertrand curves in S_1^3, will be explored, as well as, developable and normal surfaces of a space-like curve. Further, singularities of these surfaces are discussed.

Keywords: Space-like ruled surfaces, de Sitter 3-space, Bertrand curve, singularities.

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1 Introduction

Let R_1^4 denote the 4-dimensional Minkowski space-time, i.e., the Euclidean space R^4 with the standard flat metric given by [6]

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where $(x_1, \dots, x_4)$ is a rectangular coordinate system of R_1^4.

For any $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4) \in R^4$, the Lorentz metric on R^4 is defined as

$$< a, b >= a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

The representation of $<\quad>$ in the matrix form with respect to the standard basis of R_1^4 is $\mu = diag(1, 1, 1, -1)$.

Since g is indefinite metric, nonzero vectors x in R_1^4 are classified as one of three causal characters space-like, time-like and null (light-like) according to whether [9] $g(x, x) > 0$ or $x = 0$, $g(x, x) < 0$ and $g(x, x) = 0$.

For simplicity, we take the vector $\overline{0}$ to be space-like.

The norm of a vector x is given by $\|x\| = \sqrt{\mid g(x, x) \mid}$. Therefore, x is a unit vector if $g(x, x) = \pm 1$. The definition of norm is valid only for space-like vectors because $< x, x >< 0$ for a time-like vector x.

In physics, for a time-like vector x the norm $\|x\|$ can be also defined as $\|x\| = \sqrt{-<x,x>}$. This actually has a physical meaning. If $x(t)$ is a time-like curve in S_1^3, then $\|x(t)'\| = \sqrt{-<x'(t),x'(t)>}$ is the actual time elapsed by the moving particle. This is called proper time in relativity. Here after, vectors a and b are said to be orthogonal if $g(a,b) = 0$.

For any three vectors $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$, $c = (c_1, c_2, c_3, c_4) \in R^4$, the Lorentzian vector product is defined by [1,4]

$$a \times b \times c = \begin{vmatrix} i & j & k & -l \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

where $\times$ is the cross-product of S_1^3 and (i,j,k,l) is the canonical basis of R_1^4.

In this case, it is easy to check that $(<e, <a \times b \times c>) = det(e,a,b,c)$ for any vector e in R_1^4.

Now, it is important to note the following definitions:

Definition 1.1. A surface in S_1^3 is called a time-like surface if the induced metric on the surface is a Lorentz metric, i.e., the normal on the surface is a space-like vector [8].

Definition 1.2. A surface in S_1^3 is called a space-like surface if the induced metric on the surface is a Riemannian metric. This is equivalent to the condition that the tangent plane T_pM of M is a space-like plane (i.e., consists of space-like vectors) for any point $p \in M$. In this case, the normal space N_pM is a time-like plane (i.e., Lorentz plane).

2 Basic Facts on Geometry of Space-Like Curves in Minkowski S_1^3-Space

It is well-known that the Lorentzian space form with a positive curvature, more precisely, a positive sectional curvature is called de Sitter space. We define de Sitter 3-space by

$$S_1^3 = \{x \in R_1^4 \mid <x,x>= 1\}.$$

In this section, we study space-like, Bertrand curves as curves on space-like surfaces. So, we introduce the basic geometrical tools and some definitions, theorems which we need for this study. A detailed description can be found in [6].

Let $\eta : I \subset R \to S_1^3$, $t \to \eta(t) = (\eta_1(t), \eta_2(t), \eta_3(t))$ be a smooth regular curve in the space S_1^3 (i.e., $\eta'(t) > 0$ for any $t \in I$), where I is an open interval. It can locally be space-like, time-like or null, if respectively the tangent vector at every point of the curve η satisfies $<\eta', \eta'>> 0, <\eta', \eta'>< 0$ or $<\eta', \eta'>= 0$.

The arc-length of a space-like curve η, measured from $\eta(t_0)$, $t_0 \in I$ is

$$S(t) = \int_{t_0}^{t} \|\eta'(t)\| dt. \tag{2.1}$$

and it is determined such that $\|\eta'(s)\| = 1$, where $\eta'(s) = d\eta/ds$. Therefore, we say that a space-like curve η is parameterized by arc-length if it satisfies$\|\eta'(s)\| = 1$. Moreover, η is a unit speed curve if $g(\eta'(s), \eta'(s)) = \pm 1$.

It is well-known that to each unit speed space-like curve $\eta : I \to S_1^3$, one can associate a pseudo orthonormal frame $\{\eta(s), T(s), N(s), B(s)\}$. Denote by $T(s), N(s), B(s)$ the space-like tangent vector, the space-like principal normal vector, and the time-like binomial vector, respectively.

In this situation, the Frenet-Serret equations satisfied by the Frenet vectors T, N, B formally given by [7]

$$\begin{aligned} \eta'(s) &= T(s), \\ T'(s) &= -\eta(s) + k(s)N(s), \\ N'(s) &= k(s)\delta(\eta(s))T(s) + \tau(s)B(s), \\ B'(s) &= \tau(s)N(s), \end{aligned} \tag{2.2}$$

where $\delta(\eta(s)) = -sign(N(s))$, $k(s), \tau(s)$ are the curvature and the torsion of a curve η at s respectively and given by

$$k(s) = \|T'(s) + \eta(s)\|, \tag{2.3}$$

$$\tau(s) = \frac{\delta(\eta(s))}{K^2(s)} det(\eta(s), \eta'(s), \eta''(s), \eta'''(s)) \tag{2.4}$$

with $K(s) \neq 0$

The vectors T, N, and B satisfy the equations

$$g(T, T) = g(N, N) = 1, \quad g(B, B) = -1.$$

Since $B(s)$ is the unique time-like unit vector perpendicular to $\{T, N\}$, it follows

$$B = \frac{\eta(s) \times T \times N}{\|\eta \times T \times N\|},$$

where $\|\eta \times T \times N\| = - < \eta(s) \times T(s) \times N(s), \eta(s) \times T(s) \times N(s) >$, and $T(s) = \eta'$ is the tangent.

In the case of $< T'(s), T'(s) >> 1$, we have a unit vector

$$N(s) = \frac{T'(s) + \eta(s)}{\|T'(s) + \eta\|}.$$

Here, it is easy to see that

$$\eta(s) \wedge \eta'(s) \wedge \eta''(s) \quad = \quad \eta(s)\Lambda T(s) \wedge (-\eta(s) + k(s)N(s))$$

$$= k(s)\eta(s) \wedge T(s) \wedge N(s))$$
$$= k(s)B(s).$$

Definition 2.1. Let η_1 and η_2 be two regular curves with $k_1(s) \neq 0$, $k_2(s) \neq 0$, $s \in I$. Let (T_1, N_1, B_1) and (T_2, N_2, B_2) be the Frenet frames of $T_{\eta_1(s)}S_1^3$, $T_{\eta_2(s)}S_1^3$, the tangent space of S_1^3 at $\eta_1(s)$ and the tangent space of S_1^3 at $\eta_2(s)$ respectively. If the principal normal lines of η_1 and η_2 at $s \in I$ are equal, then the curve η_1 is called a Bertrand curve. In this case, the other curve η_2 is called Bertrand mate of η_1 and it writes

$$\eta_2(s) = \frac{1}{\alpha}\eta_1(s) + \lambda N_1(s), \forall s \in I, \quad \alpha \text{ is constant }, \alpha \neq 0, 1. \tag{2.5}$$

The mate of Bertrand curve is denoted by (η_1, η_2) [2].

Under the above definition, one can give the following theorems.

Theorem 2.1. [2] *If (η_1, η_2) is a mate of Bertrand curve in S_1^3. Then λ is a constant and is defined by Eq. (2.5).*

Theorem 2.2. [2] *Let η_1 and η_2 be two regular curves of S_1^3. Then (η_1, η_2) is a mate of Bertrand curve if and only if there exists a linear relation in the form of*

$$pk_1(s) + q\tau_1(s) = 1, \tag{2.6}$$

where p, q are nonzero constants and $k_1(s)$ and $\tau_1(s)$ are the curvature and the torsion of η_1, respectively.

Theorem 2.3. [2] *Consider (η_1, η_2) be a mate of Bertrand curve in S_1^3. Then the product of torsions τ_1 and τ_2 at the corresponding points of the Bertrand curve is constant, where τ_1 and τ_2 are the torsions of the curves η_1 and η_2, respectively.*

Now, consider the following corollary

Corollary 2.1. *Consider $\eta_1 : I \subset R \to S_1^3$ be a space-like curve with $k_1(s) \neq 0$ and $\tau_1(s) \neq 0$. Then η_1 is a Bertrand curve if and only if there exists a real number $p \neq 0$ such that*

$$p(\tau_1'(s)k_1(s) - k_1'(s)\tau_1(s)) - \tau_1'(s) = 0.$$

The Bertrand mate of η_1 is then given by

$$\eta_2(s) = \frac{1}{\alpha}\eta_1(s) + pN_1(s), \quad \alpha \neq 0, 1$$

Proof. By the use of Theorems 2.2 and 2.3, it follows that a space-like curve η_1 is a Bertrand curve if and only if there exists a real number $p \neq 0$ and q such that $pk_1(s) + q\tau_1(s) = 1$. In other words, it means that there exists a real number $p \neq 0$ such that $(1 - pk_1(s))/\tau_1(s)$ is constant.

Differentiating both sides of the last equality, we have

$$p(\tau_1'(s)k_1(s) - k_1'(s)\tau_1(s)) = \tau_1'(s). \tag{2.7}$$

The converse assertion is also true. □

3 Special Space-Like Ruled Surfaces with Space-Like Curves

The study of space-like surfaces represents one of the interesting subjects in the extrinsic differential geometry and in the theory of relativity [6].

In this section, we give some geometric properties of new special space-like surfaces associated to space-like curves. Singularities of these surfaces are discussed.

Let $\eta : I \to S_1^3$ be a unit speed differentiable space-like curve in S_1^3 parameterized by arc-length s.

In this case, when a director curve moves along the curve η, we get a 2-dimensional space-like ruled surface $M(s, v) : I \times R \to S_1^3$. it is parametrization as follows

$$M : \phi_{(\eta,L)}(s, v) = \eta(s) + vL(s), \text{ for all } (s, v) \in I \times R, \quad v \in R$$

We call a space-like curve $\eta(s)$ the base curve and L the director curve [8].

Consider now the following definition:

Definition 3.1. A space-like surface $\phi_{(\eta,N)}(s, v)$ defined by

$$\phi_{(\eta,N)}(s, v) = \eta(s) + vN(s) \tag{3.1}$$

is called the principal normal surface of a space-like curve η.

Taking the derivatives of ϕ with respect to s and v, we have

$$\phi_s = \eta' + vN', \qquad \phi_v = N(s).$$

Not that

$$rank[\phi_s, \phi_v] = rank[\eta'(s) + vN'(s), N(s)].$$

In details

$$\phi_s = (1 + vk(s)\delta(\eta(s)))T(s) + v\tau(s)B(s), \tag{3.2}$$

$$\phi_v = N(s). \tag{3.3}$$

The vectors (3.2) and (3.3) are linearly dependent if and only if

$$(1 + vk(s)\delta(\eta(s)) = 0. \tag{3.4}$$

From (3.4) we see that the wedge product (i.e., the oriented tangent plane generated by tangent vectors ϕ_s and ϕ_v) is given by

$$\frac{\partial\phi}{\partial s} \wedge \frac{\partial\phi}{\partial v} = (1 + vk(s)\delta(\eta(s))T(s) \wedge N(s) + v\tau(s)B(s) \wedge N(s). \qquad (3.5)$$

By comparing with the equation (3.2), one can immediately see that

(i) if ϕ is cylindrical, then $k(s_0)(\delta\eta(s_0)) = \tau(s_0) = 0$ and if non-cylindrical then $k(s_0)(\delta\eta(s_0)) = \tau(s_0) \neq 0$

The singular point of the surface (3.1) can be obtained by the use of Frenet-Serret formula as

$$(1 + vk(s_0)\delta(\eta(s_0)))T(s_0) \wedge N(s_0) + v\tau(s_0)B(s_0) \wedge N(s_0) = 0. \qquad (3.6)$$

In this case,

(ii) (s_0, v_0) is a singular point if and only if $\tau(s_0) = 0$.

So, the principal normal surface $\phi_{(\eta,N)}$ is non-singular whenever $\tau(s_0) \neq 0$.

(iii) The singularities of ϕ is given by the set

$$\{(s,v) : v = -\frac{1}{k(s)\delta(\eta(s)))}, \quad s \in I\}, k(s) \neq 0. \qquad (3.7)$$

Now, for any unit speed space-like curve $\eta : I \to S_1^3$, we can define two vector fields E and $\overline{E}$ as

$$E = -\tau(s)T + k(s)\delta(\eta(s))B(s), \qquad (3.8)$$

$$\overline{E} = -(\frac{\tau(s)}{k(s)\delta(\eta(s))})T + B(s) \qquad (3.9)$$

along a space-like curve $\eta(s)$ under the condition that $k(s) \neq 0$.

We call the vectors E and $\overline{E}$ the Darboux and the modified Darboux vector fields of $\eta(s)$ respectively [5].

Definition 3.2. A space-like surface $\psi_{(\eta,\overline{E})}(s,v)$ defined by

$$\psi : (s,v) = \eta(s) + v\overline{E}(s)$$

is called a rectifying developable of space-like curve $\eta(s)$ [3]. From Eq. (3.9), we get

$$\overline{E}'(s) = \frac{\eta}{\delta(\eta(s))}\frac{\tau}{k} - \frac{1}{\delta(\eta(s))}(\frac{\tau}{k})'T.$$

Therefore (s_0, v_0) is a singular point of $\psi_{(\eta,\overline{E})}$ if and only if

$$\frac{1}{\delta(\eta(s_0))}(\frac{\tau}{k})'(s_0) \neq 0 \ \ (\text{i.e., } \frac{\tau k' - k\tau'}{k^2\delta(\eta(s_0))})(s_0) \neq 0)$$

and it is equal to $\delta(\eta(s_0)[(\tau/k)'(s_0)]^{-1}$.

Under the above definition, one may consider the following proposition.

Proposition 3.1. *For a space-like curve $\eta : I \to S_1^3$ with $k(s) \neq 0$, the following are equivalent*

(i) *The rectifying developable $\psi_{(\eta,\bar{E})} : I \times R \to S_1^3$ of a space-like curve η is a non-singular surface.*
(ii) *A space-like curve η is a cylindrical helix.*
(iii) *The rectifying developable $\psi_{(\eta,\bar{E})}$ of a space-like curve η is a cylindrical surface.*

Proof. It is easy to see that $\psi_{(\eta,\bar{E})}$ is non-singular at any point in $I \times R$ with the use of the previous calculation if and only if

$$\frac{1}{\delta(\eta(s))}(\frac{\tau}{k})'(s_0) = 0.$$

This means that a space-like curve η is a cylindrical helix. On the other hand, as we have seen before

$$E'(s) = \frac{\eta}{\delta(\eta(s))}(\frac{\tau}{k}) - \frac{1}{\delta(\eta(s))}(\frac{\tau}{k})'(s)T(s).$$

The rectifying developable $\psi_{(\eta,\bar{E})}(s,v)$ is cylindrical if and only if $E'(s) = 0$, so that condition (ii) is equivalent to condition (iii), which completes the proof. □

Consider now the following proposition

Proposition 3.2. *Suppose that $\eta : I \to S_1^3$ is a space-like curve which is a Bertrand curve. The principal normal surface $\phi_{(\eta,N)}$ has a singular point if and only if η is a plane curve. In this case the image of $\phi_{(\eta,N)}$ is a plane in S_1^3.*

Proof. If there exists a point $s_0 \in I$ such that $\tau(s_0) = 0$, then η is a plane curve. On the other hand, the singular point of $\phi_{(\eta,N)}$ corresponds to the point $s_0 \in I$ with $\tau(s_0) = 0$. This completes the proof of the last assertion of the proposition. □

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