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Ruled Surfaces with Stationary Width in Euclidean 3-Space $\mathbb{E}^3$

Maryam T. Aldossary[1,*] *and Rashad A. Abdel-Baky*[2]

[1]Department of Mathematics, College of Science, Imam Abdulrahman bin Faisal University, P.O. Box-1982 Damam, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, University of Assiut, 71516 Assiut, Egypt

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Abstract: In this paper, we define and study pairs of ruled surfaces in Euclidean 3-space $\mathbb{E}^3$, which have the following properties: Their bases curves are stationary width curves, and the corresponding rulings are parallels. Then, we find the condition of general helix as stationary width curve and construct ruled surfaces with stationary width.

Keywords: Constant breadth; General helix, Distribution parameter.

1 Introduction

A curve with stationary width is a closed simple curve in the plane whose width (the distance among parallel supporting lines) is stationary in all different directions, inspite of the slope of the parallel lines. Shapes that are restricted by a curve with stationary width at times named an orbiform [1]. Many mathematicians have shown an increased interest in the Euler work in the advantages of these curves [2-4]. Blaschke considered the curves of stationary width on the sphere [5]. In [6], Fujivara had raised a problem based on determining if there exist "space curves with stationary width" or not, and as a solution to the problem, the "width" notion for space curves was examined and these curves were plotted on a surface of stationary width.Reuleaux specified the curves with stationary width and indicated the procedure linked to these curves for the kinematics of machine [7,8]. Furthermore, theses types of curves were examined in the Euclidean space and Minkawski space [9-15].
However, In view of the mentioned references, there is no previous studies dealing with ruled surfaces with stationary width in Euclidean 3-space $\mathbb{E}^3$. The objective of this paper is to consider and study pair of ruled surfaces with stationary width and construct them. Finally, we support the results of the work by some examples.

2 Preliminaries

In this section we recall some basic notions about curves, and ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ which can be found in the textbooks on differential geometry [16].

Let $\mathbf{r} = \mathbf{r}(s)$ be a regular space curve parameterized by its arc-length $s \in I$, and $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ be its associated Serret–Frenet frame, then the Serret–Frenet formulae read:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \ ('= \frac{d}{ds}), \qquad (1)$$

$\kappa(s)$, and $\tau(s)$ are the natural curvature and torsion, respectively. Further, it is well-known that the planes which correspond to the subspaces $\mathrm{Sp}\{\mathbf{t}(s), \ \mathbf{n}(s)\}$, $\mathrm{Sp}\{\mathbf{n}(s), \ \mathbf{b}(s)\}$, and $\mathrm{Sp}\{\mathbf{t}(s), \ \mathbf{b}(s)\}$ are, respectively, called osculating plane, normal plane, and rectifying plane.

A ruled surface M in Euclidean 3-space $\mathbb{E}^3$ is a surface formed by moving a straight line L along a curve $\mathbf{r}(s)$. The various locations of L are named the generators of the surface. Such a surface constantly has a ruled form,

$$M : \mathbf{\Phi}(s, v) = \mathbf{r}(s) + v\mathbf{x}(s), \ v \in \mathbb{R},$$

The curve $\mathbf{r}(s)$ and the vector field $\mathbf{x}(s)$ are named the directrix curve (or a generating curve) and the

* Corresponding author e-mail: maaldosary@iau.edu.sa

director curve (or the director vector filed), respectively. The distribution parameter $\mu(s)$ of M is given by

$$\mu(s)=\frac{\det(\mathbf{r}^{'},\mathbf{x},\mathbf{x}^{'})}{\|\mathbf{x}^{'}\|^{2}}.$$

The $\mathbf{\Phi}'s$ tangent vectors are

$$\mathbf{\Phi}_{s}(s,v)=\frac{\partial\mathbf{\Phi}}{\partial s}=\mathbf{r}^{'}+v\mathbf{x}^{'},\ \mathbf{\Phi}_{v}(s,v)=\frac{\partial\mathbf{\Phi}}{\partial v}=\mathbf{x}. \tag{2}$$

The unit normal vector to the surface M is

$$\mathbf{u}(s,v)=\frac{\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}}{\|\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}\|}=\frac{\mathbf{r}^{'}\times\mathbf{x}+v\mathbf{x}^{'}\times\mathbf{x}}{\|\mathbf{r}^{'}\times\mathbf{x}+v\mathbf{x}^{'}\times\mathbf{x}\|}, \tag{3}$$

where $\times$ indicates the vector product in $\mathbb{E}^{3}$. The normal curvature κ_{n}, the geodesic curvature κ_{g}, and the geodesic torsion τ_{g} of the curve $\mathbf{r}(s)$ on the surface M can be defined as follows:

$$\left.\begin{array}{r}\kappa_{n}=<\mathbf{u},\mathbf{r}^{''}>,\\ \kappa_{g}=\det(\mathbf{u},\mathbf{t},\mathbf{t}^{'}),\\ \tau_{g}=\det(\mathbf{u},\mathbf{u}^{'},\mathbf{t}^{'}).\end{array}\right\} \tag{4}$$

The first fundamental form I is defined by

$$I(s,v)=g_{11}ds^{2}+2g_{12}dsdv+g_{22}dv^{2}, \tag{5}$$

where

$$\left.\begin{array}{l}g_{11}=<\mathbf{\Phi}_{s},\mathbf{\Phi}_{s}>,\\ g_{12}=<\mathbf{\Phi}_{s},\mathbf{\Phi}_{v}>,\\ g_{22}=<\mathbf{\Phi}_{v},\mathbf{\Phi}_{v}>.\end{array}\right\}$$

We define the second fundamental form II of M by

$$II(s,v)=h_{11}ds^{2}+2h_{12}dsdv+h_{22}dv^{2}, \tag{6}$$

where

$$\left.\begin{array}{l}h_{11}=<\mathbf{\Phi}_{ss},\mathbf{u}>,\\ h_{12}=<\mathbf{\Phi}_{sv},\mathbf{u}>,\\ h_{22}=<\mathbf{\Phi}_{vv},\mathbf{u}>.\end{array}\right\}$$

The Gaussian and mean curvatures, respectively, are

$$\begin{array}{r}K(s,v)=\dfrac{h_{11}h_{22}-h_{12}^{2}}{g_{11}g_{22}-g_{12}^{2}},\text{ and}\\ H(s,v)=\dfrac{h_{11}g_{11}-2h_{12}g_{12}+h_{22}g_{22}}{2(g_{11}g_{22}-g_{12}^{2})}\end{array} \tag{7}$$

3 Ruled surfaces with stationary width

In this section, we define and study pairs of ruled surfaces which have the following characterizations: The directrix curves are curves with stationary width, and the corresponding generators are parallels.

Let a unit vector

$$\mathbf{x}(s)=\mathbf{x}_{1}\mathbf{t}+\mathbf{x}_{2}\mathbf{n}+\mathbf{x}_{3}\mathbf{b}, \tag{8}$$

where the components $\mathbf{x}_{1}$, $\mathbf{x}_{2}$, and $\mathbf{x}_{3}$ are constant. Hence, as $\mathbf{x}$ moves along $\mathbf{r}(s)$ it generates a ruled surface has the ruled form,

$$\left.\begin{array}{l}M:\mathbf{\Phi}(s,v)=\mathbf{r}(s)+v\mathbf{x}(s),\ v\in\mathbb{R},\\ x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1,\mathbf{x}^{'}\neq\mathbf{0}.\end{array}\right\} \tag{9}$$

The distribution parameter of M is

$$\mu(s)=\frac{\left(1-x_{1}^{2}\right)\tau-x_{1}x_{3}\kappa}{x_{2}^{2}\left(\kappa^{2}+\tau^{2}\right)+\left(x_{1}\kappa-x_{3}\tau\right)^{2}}. \tag{10}$$

Since M is a developable surface, then

$$\left(1-x_{1}^{2}\right)\tau-x_{1}x_{3}\kappa=0\Rightarrow\frac{\tau}{\kappa}(s)=\frac{x_{1}x_{3}}{1-x_{1}^{2}}=const. \tag{11}$$

Corollary 1. M is a developable surface if $\mathbf{r}=\mathbf{r}(s)$ is a general helix with $\frac{\tau}{\kappa}(s)=\frac{x_{1}x_{3}}{1-x_{1}^{2}}$.

Furthermore, we have:

$$\left.\begin{array}{l}\mathbf{\Phi}_{s}=(1-vx_{2}\kappa)\mathbf{t}+v(x_{1}\kappa-x_{3}\tau)\mathbf{n}+v\mathbf{x}_{2}\tau\mathbf{b},\\ \mathbf{\Phi}_{v}=x_{1}\mathbf{t}+x_{2}\mathbf{n}+x_{3}\mathbf{b}.\end{array}\right\} \tag{12}$$

Then, we obtain the elements of the first fundamental form of M as follows:

$$\left.\begin{array}{l}g_{11}=(1-vx_{2}\kappa)^{2}+v^{2}(x_{1}\kappa-x_{3}\tau)^{2}+v^{2}\mathbf{x}_{2}^{2}\tau^{2},\\ g_{12}=x_{1},\ g_{22}=1.\end{array}\right\} \tag{13}$$

Also, we have

$$\begin{array}{r}\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}=v\left\{x_{3}(x_{1}\kappa-x_{3}\tau)-x_{2}^{2}\tau\right\}\mathbf{t}\\ +\left\{-x_{3}+vx_{2}(x_{3}\kappa+x_{2}\tau)\right\}\mathbf{n}\\ +\left\{x_{2}-v(1-x_{3}^{2})-vx_{3}\tau)\right\}\mathbf{b}.\end{array} \tag{14}$$

Thus, we can see that $\|\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}\|^{2}(s,0)=1-x_{3}^{2}$, that is, M is a regular surface along its dirctrix curve iff $1-x_{3}^{2}\neq0$. In this case, the unit normal of M along its dirctrix curve $\mathbf{r}(s)$ is given by

$$\mathbf{u}(s,0)=\frac{\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}}{\|\mathbf{\Phi}_{s}\times\mathbf{\Phi}_{v}\|}=\frac{x_{3}\mathbf{n}+x_{2}\mathbf{b}}{\sqrt{1-x_{3}^{2}}}. \tag{15}$$

Via Eqs. (4), and (15) we gain

$$\left.\begin{array}{l}\kappa_{n}(s)=\frac{\kappa x_{3}}{\sqrt{1-\mathbf{x}_{1}^{2}}},\\ \kappa_{g}(s)=\frac{\kappa x_{2}}{\sqrt{1-\mathbf{x}_{1}^{2}}},\ \tau_{g}(s)=\tau.\end{array}\right\} \tag{16}$$

Corollary 2. $\mathbf{r}(s)$ is a curvature line on M iff $\mathbf{r}(s)$ is a planar curve.

By a straightforward calculation, we get:

$$\begin{aligned} \Phi_{ss} &= -v\left\{x_2\kappa^{'}+\kappa(x_1\kappa-x_3\tau)\right\}\mathbf{t}+ \\ &\left\{\kappa+v\left[x_2\left(\tau^2-\kappa^2\right)+x_1\kappa^{'}-x_3\tau^{'}\right]\right\}\mathbf{n} \\ &+v\left\{x_2\tau^{'}-\tau\left(x_1\kappa-x_3\tau\right)\right\}\mathbf{b}, \\ \Phi_{sv} &= -vx_2\kappa\mathbf{t}+\left(x_1\kappa-x_3\tau\right)\mathbf{n}+x_2\tau\mathbf{b}, \\ \Phi_{vv} &= 0. \end{aligned} \tag{17}$$

Further, we have;

$$\left.\begin{aligned} h_{11}(s,0) &= \frac{-x_3\kappa}{\sqrt{1-\mathbf{x}_1^2}}, \\ h_{12}(s,0) &= \frac{-x_3(x_1\kappa-x_3\tau)+x_2^2\tau}{\sqrt{1-\mathbf{x}_1^2}}, \\ h_{22}(s,0) &= 0. \end{aligned}\right\} \tag{18}$$

Then at the point $(s,0)$, we have:

$$\left.\begin{aligned} K(s,0) &= -\frac{1}{\left(1-\mathbf{x}_1^2\right)^2}\left[\tau\left(1-\mathbf{x}_1^2\right)-\kappa x_1x_3\right]^2, \\ H(s,0) &= -\frac{\kappa x_3(1-2x_1^2)+2\tau x_1(1-\mathbf{x}_1^2)}{2\sqrt{(1-\mathbf{x}_1^2)}^3}. \end{aligned}\right\} \tag{19}$$

Hence, one can have the following corollary:

Corollary 3. At the point $(s,0)$ on M, one has the following:

1) M is a flat surface, that is, $K(s,0)=0$ iff $\mathbf{r}=\mathbf{r}(s)$ is general helix with $\frac{\tau}{\kappa}(s)=\frac{x_1x_3}{1-x_1^2}$,

2) M is a minimal surface, that is, $H(s,0)=0$ iff $\mathbf{r}=\mathbf{r}(s)$ is general helix with $\frac{\tau}{\kappa}=-\frac{x_3(1-2x_1^2)}{2x_1(1-\mathbf{x}_1^2)}$.

Definition 1. Let (C) be a regular curve with position vector $\mathbf{r}(s)$. If (C) possess parallel tangents in opposite directions at corresponding points $\mathbf{r}(s)$ and $\mathbf{r}^*(s^*)$ and the distance between these points is constantly stationary, then (C) is called a curve with stationary width. Furthermore, a pair of curves (C) and (C^*) for which the tangents at the corresponding points (C) and (C^*) are parallel and in opposite directions, and the distance among these points is always constant are named a curve pair with stationary width and indicated by (C,C^*).

Let (C,C^*) be two unit speed curves of class C^3 with non-vanishing curvature and torsion in the Euclidean 3-space $\mathbb{E}^3$. Then, we may write the equation of the curve (C^*) as follows:

$$\mathbf{r}^*(s)=\mathbf{r}(s)+\lambda_1(s)\mathbf{t}(s)+\lambda_2(s)\mathbf{n}(s)+\lambda_3(s)\mathbf{b}(s). \tag{20}$$

Here, the functions $\lambda_1(s)$, $\lambda_2(s)$, and $\lambda_3(s)$ are regular functions of s which is arc length of C. Derivative of Eq. (20) with regards to s and using the Serret–Frenet formulae we get

$$\begin{aligned} \frac{d\mathbf{r}^*}{ds} &:= \mathbf{t}^*\frac{ds^*}{ds}=(1-\lambda_1^{'}-\lambda_2\kappa)\mathbf{t}+ \\ &(\lambda_1\kappa+\lambda_2^{'}-\lambda_3\tau)\mathbf{n}+(\lambda_3^{'}+\lambda_2\tau)\mathbf{b}, \end{aligned} \tag{21}$$

where $\mathbf{t}^*$ is the unit tangent vector of C^*. Since $\mathbf{t}^*=-\mathbf{t}$ at the corresponding points of C, and C^*, then we have

$$\left.\begin{aligned} -\tfrac{ds^*}{ds} &= 1+\lambda_1^{'}+\lambda_2\kappa, \\ \lambda_1\kappa+\lambda_2^{'}-\lambda_3\tau &= 0, \\ \lambda_3^{'}+\lambda_2\tau &= 0. \end{aligned}\right\} \tag{22}$$

Definition 2. Let M, and M^* be a pair of non-developable ruled surfaces in Euclidean 3-space $\mathbb{E}^3$. If their corresponding rulings are parallel, and their directrix curves are with stationary width, then (M,M^*) is named a ruled surface pair with stationary width.

Let now (M,M^*) be two non-developable ruled surfaces with stationary width. Then, we may express the equation of M^* as follows:

$$M^*:\Phi^*(s,v)=\mathbf{r}^*(s)+v\mathbf{x}(s),\ v\in\mathbb{R}. \tag{23}$$

For ruled surface M^*, we have

$$\mu^*(s)=(1+\lambda_1^{'}-\lambda_2\kappa)\mu. \tag{24}$$

Proposition 1. Let (M,M^*) be a pair of ruled surface with stationary width in $\mathbb{E}^3$. If $\mu^*=\mu$, then the distance between the corresponding points on their directrix curves is fixed. Likewise, the converse is true.

Proof. Let $\mu^*=\mu$, then $\lambda_1^{'}-\lambda_2\kappa=0$, hence by differentiating the vector $\mathbf{C}=\lambda_1(s)\mathbf{t}+\lambda_2(s)\mathbf{n}+\lambda_3(s)\mathbf{b}$, and applying Eqs. (22) implies; $\mathbf{C}^{'}=\mathbf{0}$.

Furthermore, if $\mathbf{C}^{'}=\mathbf{0}$, we can conclude that Eq. (22) that $\lambda_1^{'}-\lambda_2\kappa=0$, so $\mu^*=\mu$ ■.

Alternately, since $\mathbf{t}^*=-\mathbf{t}$, we have $\mathbf{n}^*=-\mathbf{n}$, $\mathbf{b}^*=\mathbf{b}$. Then, the Serret–Frenet frames $\{\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s)\}$, and $\{\mathbf{t}^*(s^*),\mathbf{n}^*(s^*),\mathbf{b}^*(s^*)\}$ of (C) and (C^*), respectively, are equivalent at the corresponding points. As a result, we obtain

$$\frac{ds^*}{ds}=\frac{\kappa}{\kappa^*}=\frac{\tau}{\tau^*}. \tag{25}$$

Therefore, based on Eqs. (24), and (25), we have:

Corollary 4. Under the above notations, the following characterizations are satisfied:

1) If M is a developable surface, then M^* is a developable surface too,

2) If $\mathbf{r}(s)$ is a general helix, then $\mathbf{r}^*(s^*)$ is a general helix too.

Now, considering the fact that the curvature of $\mathbf{r}(s)$ is $\varphi^{'}(s)=\kappa(s)$, with $\varphi(s)$ is the angle of contingency, so Eqs. (22) can be written in the following form:

$$\left.\begin{aligned} \dot{\lambda}_1-\lambda_2+f(\varphi) &= 0, \\ \dot{\lambda}_2+\lambda_1-\lambda_3\rho\tau &= 0, \\ \dot{\lambda}_3+\lambda_2\rho\tau &= 0, \end{aligned}\right\} \tag{26}$$

where $f(\varphi)=\rho+\rho^*$, and dot indicates the differentiation with respect to φ. By excluding λ_3 and its derivatives from Eqs. (26), we get

$$\rho\tau\left(\ddot{\lambda}_2+\lambda_2-f(\varphi)\right)+\lambda_2(\rho\tau)^3-\left(\lambda_1+\dot{\lambda}_2\right)\frac{d}{d\varphi}(\rho\tau)=0. \tag{27}$$

If the distance among the opposite points of C and C^* is fixed, that is,

$$\|\mathbf{C}\|^2=\lambda_1^2+\lambda_2^2+\lambda_3^2=\text{const}.$$

Then

$$\lambda_1\dot{\lambda}_1+\lambda_2\dot{\lambda}_2+\lambda_3\dot{\lambda}_3=0. \tag{28}$$

Combining Eqs. (27) and (28), we get:

$$\lambda_1\left(\lambda_2-\dot{\lambda}_1\right)=0.$$

We have two main cases:

Case 1) Suppose $\lambda_2-\dot{\lambda}_1=0$, and $\lambda_1\neq 0$. Then $f(\varphi)=0$ in Eqs. (26), it follows that (27) becomes:

$$\rho\tau\left(\ddot{\lambda}_2+\lambda_2\right)+\lambda_2(\rho\tau)^3-\left(\lambda_1+\dot{\lambda}_2\right)\frac{d}{d\varphi}(\rho\tau)=0. \tag{29}$$

We consider λ_1 is non-zero constant and $\lambda_2=0$. Then $\frac{d}{d\varphi}(\rho\tau)=0\Leftrightarrow\rho\tau=const$. It shows that C is a general helix. Consequently, in light of Eq. (25), we have

$$\frac{\tau}{\kappa}(s)=\frac{\tau^*}{\kappa^*}(s^*)=const. \tag{30}$$

Proposition 2. Let (M,M^*) be a ruled surface pair with stationary width in $\mathbb{E}^3$. If the distance among the rectifying planes vanish ($\lambda_2=0$), and the distance among the normal planes is stationary (λ_1 is a non-zero constant), then their dirctrix curves are general helices.

Case 2) Suppose $\lambda_1=0$, the normal planes coincide. In this case, the Eqs. (26) becomes:

$$\left.\begin{array}{r}-\lambda_2+f(\varphi)=0,\\ \dot{\lambda}_2-\lambda_3\rho\tau=0,\\ \dot{\lambda}_3+\lambda_2\rho\tau=0.\end{array}\right\} \tag{31}$$

We may remark that $f(\varphi)=\rho+\rho^*=0\Leftrightarrow\|\mathbf{C}\|^2=\lambda_2^2+\lambda_3^2=$const. Hence, we have:

Proposition 3. Consider (M,M^*) be a ruled surface pair with stationary width in $\mathbb{E}^3$. If the directrix curves of M, and M^* have $\rho+\rho^*=0$, then the distance among the corresponding points is stationary. The converse is also true.

By eliminating λ_3 and its derivatives from Eqs. (31), we get:

$$\ddot{\lambda}_2-\frac{(\tau\dot{\rho}+\rho\dot{\tau})}{\rho\tau}\dot{\lambda}_2+(\rho\tau)^2\lambda_2=0.$$

Its general solution is given by

$$\lambda_2=a\cos\left(\int_0^{\varphi}\rho\tau d\varphi+b\right),$$

Via the second equation in Eqs. (31), we have

$$\lambda_3=-a\sin\left(\int_0^{\varphi}\rho\tau d\varphi+b\right).$$

Hence, we state the following theorem:

Theorem 1. Let (M,M^*) be a ruled surface pair with stationary width in $\mathbb{E}^3$. If the normal planes of their directrix curves are identical, then M^* is defined as

$$M^*:\mathbf{\Phi}^*(\varphi,v)=\mathbf{\Phi}(\varphi,v)+a\cos\left(\int_0^{\varphi}\rho\tau d\varphi+b\right)\mathbf{n}-\sin\left(\int_0^{\varphi}\rho\tau d\varphi+b\right)\mathbf{b}. \tag{32}$$

where a, and b being constants of integration.

It is clear that for these ruled surfaces, the distance among the corresponding points is a, and when M is given, then an infinite numbers of M^* can be constructed.

Example. We consider a unit speed circular helix defined by

$$\mathbf{r}(s)=\left(\frac{1}{\sqrt{2}}\cos s,\frac{1}{\sqrt{2}}\sin s,\frac{1}{\sqrt{2}}s\right),\ s\in[-\pi,\pi].$$

It is obvious that

$$\left.\begin{array}{r}\mathbf{t}(s)=\frac{1}{\sqrt{2}}(-\sin s,\cos,1),\\ \mathbf{n}(s)=(-\cos s,-\sin,0),\\ \mathbf{b}(s)=\frac{1}{\sqrt{2}}(\sin s,-\cos,1),\\ \kappa(s)=\tau(s)=\frac{1}{\sqrt{2}}.\end{array}\right\}$$

We choose $\lambda_1=1$, $\lambda_2=0$, and $\lambda_3=-1$. Then we can construct the curve of stationary width of the curve $\mathbf{r}(s)$ as

$$\mathbf{r}^*(s)=\left(\frac{1}{\sqrt{2}}\cos s-\sqrt{2}\sin s,\frac{1}{\sqrt{2}}\sin s+\sqrt{2}\cos s,\frac{1}{\sqrt{2}}s\right).$$

By simple calculations, the curve $\mathbf{r}^*(s)$ has $\kappa^*(s)=\frac{\sqrt{10}}{6}$, and $\tau^*(s)=\frac{\sqrt{2}}{6}$. Hence, the curve $\mathbf{r}^*(s)$ is a circular helix too (see Fig. 1).

In the sections that follow, we'll build the surfaces M, and M^* in special cases.

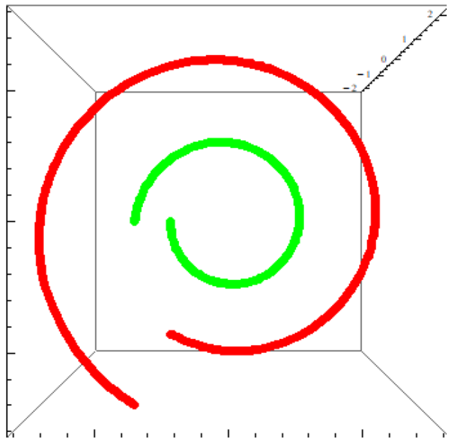

Fig. 1: Graph of **r** (in green), and $\mathbf{r}^*$ (in red).

Case 1. At $x_1 = x_3 = 0$, the ruled surfaces M, and M^*, respectively, are (Figs. 2, 3);

$$M : \mathbf{\Phi}(s,v) = (\frac{1}{\sqrt{2}}\cos s - v\cos s, \frac{1}{\sqrt{2}}\sin s - v\sin s, \frac{1}{\sqrt{2}}s),$$

with

$$K(s,0) = -\frac{1}{2}, H(s,0) = \kappa_n(s) = \tau_g(s) = 0, \quad \kappa_g(s) = \frac{1}{\sqrt{2}},$$

and

$$M^* : \mathbf{\Phi}^*(s,v) = (\frac{1}{\sqrt{2}}\cos s - \sqrt{2}\sin s - v\cos s, \frac{1}{\sqrt{2}}\sin s + \sqrt{2}\cos s - v\sin s, \frac{1}{\sqrt{2}}s),$$

with

$$K^*(s,0) = -\frac{1}{18}, H^*(s,0) = \kappa_n^*(s) = \tau_g^*(s) = 0, \quad \kappa_g^*(s) = \frac{\sqrt{10}}{6}.$$

Case 2. At $x_1 = x_2 = 0$, the ruled surfaces M, and M^*, respectively, are (Figs. 4, 5);

$$M : \mathbf{\Phi}(s,v) = \left(\frac{1}{\sqrt{2}}\cos s + \frac{v}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\sin s - \frac{v}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}(s+v)\right),$$

with

$$K(s,0) = -\frac{1}{2}, H(s,0) = -\frac{1}{2\sqrt{2}}, \quad \kappa_g(s) = \tau_g(s) = 0, \ \kappa_n(s) = \frac{1}{\sqrt{2}},$$

and

$$M^* : \mathbf{\Phi}^*(s,v) = \left(\frac{1}{\sqrt{2}}\cos s + \left(-\sqrt{2} + \frac{v}{\sqrt{2}}\right)\sin s, \frac{1}{\sqrt{2}}\sin s + \left(\sqrt{2} - \frac{v}{\sqrt{2}}\right)\cos s, \frac{1}{\sqrt{2}}(s+v)\right),$$

with

$$K^*(s,0) = -\frac{1}{18}, H^*(s,0) = -\frac{\sqrt{10}}{12}, \quad \kappa_g^*(s) = \tau_g^*(s) = 0, \ \ \kappa_n^*(s) = \frac{\sqrt{10}}{2}.$$

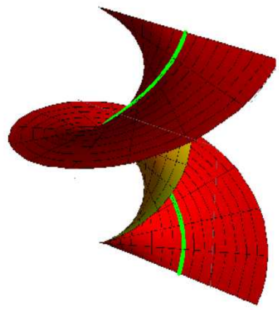

Figure 2. M.

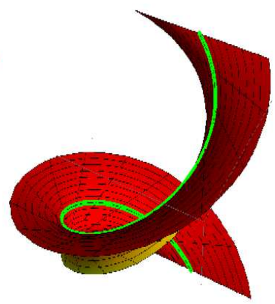

Figure 3. M*.

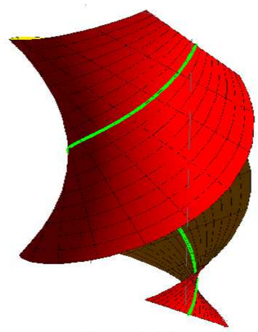

Figure 4. M.

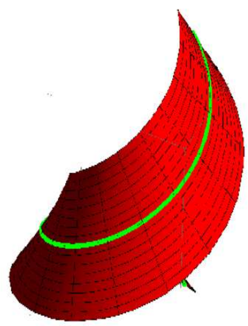

Figure 4. M*.

Case 3. At $x_2 = x_3 = 0$, the ruled surfaces M, and M^*, accordingly, given as (Figs. 6, 7);

$$M : \mathbf{\Phi}(s,v) = \left(\frac{1}{\sqrt{2}}\cos s - \frac{v}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\sin s + \frac{v}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}(s+v)\right),$$

with

$$K(s,0)=H(s,0)=0,\ \kappa_g(s)=\tau_g(s)=\kappa_n(s)=0,$$

and

$$M^*:\mathbf{\Phi}^*(s,v)=\left(\frac{1}{\sqrt{2}}\cos s-\left(\frac{v}{\sqrt{2}}+\sqrt{2}\right)\sin s,\ \frac{1}{\sqrt{2}}\sin s+\left(\frac{v}{\sqrt{2}}+\sqrt{2}\right)\cos s,\ \frac{1}{\sqrt{2}}(s+v)\right),$$

with

$$K^*(s,0)=H^*(s,0)=0,\ \kappa_g^*(s)=\tau_g^*(s)=\kappa_n^*(s)=0.$$

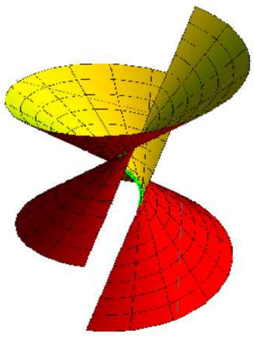

Figure 6. M.

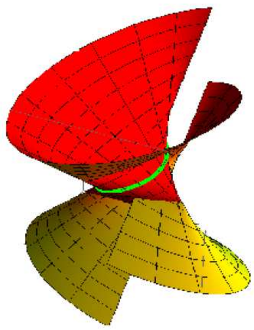

Figure 7. M*.

4 Conclusion

In this paper, we investigate ruled surfaces with stationary width in Euclidean 3-Space $\mathbb{E}^3$ and give some characteristics of these ruled surfaces. There are several possibilities for the further work. An analogue of the problem discussed in this article can be the consideration of surfaces in Minkowski 3-space. We will investigate this issue in the future.

Data Availability: All of the data are available within the paper.

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M. T. Aldossary, works currently as an assistant professor in the mathematics department at Imam Abdulrahman bin Faisal university, KSA. She published several papers in the field of differential geometry. Her field of interest is the classical geometry of curves and surfaces and the geometry of motion. Dr. Maryam received her Ph.D. from IAU in 2007.

R. A. Abdel-baky is currently a sciences faculty in the department of mathematics at Assiut University, Egypt. He received his BSc and MSc from university Assiut. His research interest includes geometry of motion and the theory of curves and surfaces. Prof. Rashad received his Ph.D. in mathematics from Ankara University in 1994, Ankara Turkey.