Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100307

From Calculus to α- Calculus

Mohammed Shehata

Bilbeis Higher Institute For Engineering, Sharqia, Egypt

Received: 2 May 2023, Revised: 18 Jun. 2023, Accepted: 20 Jun. 2023
Published online: 1 Jul. 2024

Abstract: In the previous definitions of fractional ($\alpha-$) calculus, there were a mismatch in some properties to classical calculus. This is because these definitions were built in an unusual way, they were built from the definition of integral to derivative. For example, in the Riemann-Liouville definition of derivative, the derivative of a constant may not be zero. In this paper, we will overcome these incompatibilities, by accurately constructing $\alpha-$ derivative and $\alpha-$ integral by usual way, so it coincides with the classical ones. We also generalized some basic formulas and theorems.

Keywords: Fractional calculus, Riemann-Liouville definition, Caputo fractional definition.

1 Introduction

Fractional calculus deals with derivatives and integrals of any order. Fractional calculus dates back to roughly the same time as classical calculus. It was first referenced in a letter from Leibniz to l'Hospital in 1695, where the concept of the fractional derivative was introduced. Many famous mathematicians, such as Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, and Abell, mentioned fractional calculus on formal considerations which can be found chronologically in [1]. Since then, many forms of fractional calculus have emerged, Riemann-Liouville, Riesz, and Caputo fractional derivatives [2]-[4] and more recent conceptions of [5]-[8]. Fractional calculus is nowadays the realm of physicists and mathematicians, who investigate the usefulness of such non-integer order derivatives and integrals in different areas of physics and mathematics (see, e.g.,[9]-[11]). Fractional calculus has applications in both classical and quantum mechanics, Appli field theories, variational calculus, and optimal control (see, e.g., [12]-[14]).

The two most widely used definitions are Riemann-Liouville and Caputo operators.

Definition 1*(Riemann-Liouville fractional- derivatives 1847). Let $f(.)$ be a continuous function in $[a,b]$, $0<\alpha\leq 1$. The left Riemann-Liouville fractional derivative of order α is given by*

$$ {}_aD_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau\right),\quad t\in[a,b]. $$

The right Riemann-Liouville fractional derivative of order α is given by

$$ {}_tD_b^\alpha f(t)=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau\right),\quad t\in[a,b]. $$

Definition 2*(Caputo's fractional derivatives 1967). Let $f(.)$ be a continuous function in $[a,b]$, $0<\alpha\leq 1$. The left Caputo fractional derivative of order α is given by*

$$ {}_a^CD_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\frac{d}{dt}f(\tau)d\tau,\quad t\in[a,b]. $$

and the right Caputo fractional derivative derivative of order α is given by

$$ {}_t^CD_b^\alpha f(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b (\tau-t)^{-\alpha}\frac{d}{dt}f(\tau)d\tau,\quad t\in[a,b]. $$

* Corresponding author e-mail: mashehata_math@yahoo.com, mashehata_math@bhie.edu.eg

But this definitions gives rise to some problems. For example if one makes $f(t) = K =$ constant in Riemann-Liouville formula, one finds that its α^{th} derivative is $\dfrac{Kt^{\alpha}}{\Gamma(1-\alpha)}$, that is to say, is different from zero. This is because they depended on the integral on the definition of the derivative.

In [15], a fractional derivative was defined by

$$\lim_{h\to 0} \frac{f(t+ht^{\alpha-1})-f(t)}{h}$$

for only positive real number t.

In this paper, we will build the definitions of $\alpha-$ derivative and $\alpha-$ integral by usual way, we will define $\alpha-$ derivative by the limits, and $\alpha-$ integral by limits of Riemann. Also, we will generalized formulas and theorems of derivatives and integrals. Through this paper $\alpha \in (0,1]$ and for a real number t (positive or negative), we denote by $(t)^{\alpha}$ to the $\alpha-$ principal root of t which is defined by

$$(t)^{\alpha} := |t|^{\alpha}.\begin{cases} 1 & t \geq 0 \\ e^{i\pi\alpha} & t < 0 \end{cases}$$

2 α- differentiable function

Definition 3(*α- Differentiable function) Let $f(t)$ be a real valued function defined in an open interval containing a real number a and let a^{α}, $\alpha \in (0,1]$, denotes to the principal root of the real number a. The function $f(t)$ is α- differentiable at a if*

$$\frac{df}{dt^{\alpha}}(a) := \lim_{t\to a} \frac{f(t)-f(a)}{t^{\alpha}-a^{\alpha}}$$

exists as a complex number.

More generally, a function f is said to be $\alpha-$ differentiable on an open set Ω if it is $\alpha-$ differentiable at every point in Ω. A function that $\dfrac{df}{dt^{\alpha}}$ exists on its domain is said to be $\alpha-$ differentiable.

Example If $f(t) = t$. Find $\dfrac{df}{dt^{\frac{1}{2}}}(t)$ at $t = 1, t = -1$ and $t = 0$.

$$\begin{aligned}
\frac{df}{dt^{\frac{1}{2}}}(a) &= \lim_{\tau\to a} \frac{\tau - a}{\tau^{\frac{1}{2}} - a^{\frac{1}{2}}} \\
&= \lim_{\tau\to a} \frac{\tau - a}{|\tau|^{\frac{1}{2}} - |a|^{\frac{1}{2}}}.\begin{cases} 1 & a \geq 0 \\ e^{\frac{-i\pi}{2}} & a < 0 \end{cases} \\
&= \lim_{\tau\to a} \frac{\tau - a.\left(|\tau|^{\frac{1}{2}} + |a|^{\frac{1}{2}}\right)}{|\tau| - |a|}.\begin{cases} 1 & a \geq 0 \\ -i & a < 0 \end{cases} \\
&= \begin{cases} 2|a|^{\frac{1}{2}} & a \geq 0 \\ 2i|a|^{\frac{1}{2}} & a < 0 \end{cases}
\end{aligned}$$

So,

$$\left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=1} = 2, \quad \left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=0} = 0 \text{ and } \left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=-1} = 2i.$$

Remark 1*If $\dfrac{df}{dt^{\alpha}}$ exists on its domain, then it is a complex-valued function given by*

$$\frac{df}{dt^{\alpha}} = f_{\alpha}(t).s_{\alpha}(t),$$

Table 1: List of some α differentiable functions

$\frac{d}{dt^\alpha}(k)=0$	$\frac{d}{dt^\alpha}(t^\beta)=\frac{\beta}{\alpha}t^{\beta-\alpha}$
$\frac{d}{dt^\alpha}\left(e^t\right)=\frac{t^{1-\alpha}e^t}{\alpha}$	$\frac{d}{dt^\alpha}(\ln\|t\|)=\frac{1}{\alpha t^\alpha},\ t>0$
$\frac{d}{dt^\alpha}(\sin t)=\frac{t^{1-\alpha}\cos t}{\alpha}$	$\frac{d}{dt^\alpha}(\cos t)=\frac{-t^{1-\alpha}\sin t}{\alpha}$
$\frac{d}{dt^\alpha}(\tan t)=\frac{t^{1-\alpha}\sec^2 t}{\alpha}$	$\frac{d}{dt^\alpha}(\cot t)=\frac{-t^{1-\alpha}\csc^2 t}{\alpha}$
$\frac{dt}{dt^\alpha}(\sec t)=\frac{t^{1-\alpha}\sec t\tan t}{\alpha}$	$\frac{d}{dt^\alpha}(\csc t)=\frac{-t^{1-\alpha}\csc t\cot t}{\alpha}$
$\frac{d}{dt^\alpha}\left(\sin^{-1}t\right)=\frac{t^{1-\alpha}}{\alpha\sqrt{1-t^2}}$	$\frac{d}{dt^\alpha}\left(\tan^{-1}t\right)=\frac{t^{1-\alpha}}{\alpha\left(1+t^2\right)}$
$\frac{d}{dt^\alpha}\left(\csc^{-1}t\right)=\frac{t^{1-\alpha}}{\alpha t\sqrt{t^2-1}}$	

where $f_\alpha(t)$ *is a real valued function defined by*

$$f_\alpha(t):=\frac{df}{d|t|^\alpha}=\lim_{\tau\to t}\frac{f(\tau)-f(t)}{|\tau|^\alpha-|t|^\alpha}$$

and s_α *is a step complex valued function defined by*

$$s_\alpha(t)=\begin{cases}1 & t\geq 0,\\ e^{-i\alpha\pi} & t<0,\end{cases}$$

Remark 2*For* $0<\alpha<1$, *If* $\frac{df}{dt^\alpha}$ *is exist at zero, then*

$$\frac{df}{dt^\alpha}(0)=constant.$$

Remark 3*If* $0<\alpha_2<\alpha_1\leq 1$ *and* $f(t)$ *is* α_1- *differentiable at* a, *then it is* α_2- *differentiable at* a, *moreover*

$$\frac{df}{dt^{\alpha_2}}(a)=\frac{\alpha_1 a^{\alpha_1-\alpha_2}}{\alpha_2}\frac{df}{dt^{\alpha_1}}(a).$$

So, if $f(t)$ *is differentiable(* $1-$ *differentiable) at* a, *then it is* $\alpha-$ *differentiable at* a, *and*

$$\frac{df}{dt^\alpha}(a)=\frac{a^{1-\alpha}}{\alpha}\frac{df}{dt}(a).$$

The converse is not true. For example $f(t)=(t)^{\frac{1}{2}}$ *is* $\frac{1}{2}-$ *differentiable at* 0 *but not differentiable at* 0.

From remark 3, we can get table 1:
From the definition 3, we can easy prove the following theorems:

Theorem 1*Let* $f(t)$ *and* $g(t)$ *be* $\alpha-$ *differentiable functions and* k *be a constant. Then each of the following equations holds.*

(1) $\frac{d}{dt^\alpha}[k]=0$

(2) $\frac{d}{dt^{\alpha}}[kf(t)] = k\frac{d}{dt^{\alpha}}[f(t)]$
(3) $\frac{d}{dt^{\alpha}}[f(t)\pm g(t)] = \frac{d}{dt^{\alpha}}[f(t)]\pm\frac{d}{dt^{\alpha}}[g(t)]$
(4) $\frac{d}{dt^{\alpha}}[f(t).g(t)] = \frac{df(t)}{dt^{\alpha}}.g(t)+f(t).\frac{dg(t)}{dt^{\alpha}}$
(5) $\frac{d}{dt^{\alpha}}\left[\frac{f(t)}{g(t)}\right] = \frac{1}{(g(t))^2}\left[\frac{df(t)}{dt^{\alpha}}.g(t)-f(t).\frac{dg(t)}{dt^{\alpha}}\right], \quad g(t)\neq 0$

Theorem 2*If $y=f(x)$ is differentiable at x and $x=g(t)$ is $\alpha-$ differentiable at t, then*

$$\frac{dy}{dt^{\alpha}} = \frac{dy}{dx}.\frac{dx}{dt^{\alpha}}.$$

Theorem 3*Let $f(t)$ be a function and a be in its domain. If $f(t)$ is $\alpha-$ differentiable at a, then it is continuous at a.*

*Proof.*If $f(t)$ is $\alpha-$ differentiable at a, then $\frac{d}{dt^{\alpha}}(a)$ exists and

$$\frac{d}{dt^{\alpha}}(a) = \lim_{t\to a}\frac{f(t)-f(a)}{t^{\alpha}-a^{\alpha}}.$$

Now

$$\begin{aligned}\lim_{t\to a}f(t) &= \lim_{t\to a}(f(t)-f(a)+f(a))\\ &= \lim_{t\to a}\left(\frac{f(t)-f(a)}{t^{\alpha}-a^{\alpha}}.(t^{\alpha}-a^{\alpha})+f(a)\right)\\ &= \lim_{t\to a}\frac{f(t)-f(a)}{t^{\alpha}-a^{\alpha}}.\lim_{t\to a}(t^{\alpha}-a^{\alpha})+\lim_{t\to a}f(a)\\ &= \frac{d}{dt^{\alpha}}(a).0+f(a)=f(a).\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{t\to a}f(t)=f(a)$ we conclude that f is continuous at a.

3 Application of $\alpha-$ differentiable

Definition 4*Let f be a real valued function defined over an interval I and let $c\in I$. We say f has an absolute maximum on I at c if*

$$f(c)\geq f(t) \quad \text{for all } t\in I.$$

We say f has an absolute minimum on I at c if

$$f(c)\leq f(t) \quad \text{for all } t\in I.$$

If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.

Theorem 4*(Extreme value theorem) If f is a real valued continuous function over the closed, bounded interval $[t_0,t_f]$, then there is a point in $[t_0,t_f]$ at which f has an absolute maximum over $[t_0,t_f]$ and there is a point in $[t_0,t_f]$ at which f has an absolute minimum over $[t_0,t_f]$.*

Definition 5*We say that a function f has a local maximum on an open interval I at $t=c$ if*

$$f(c)\geq f(t) \quad \text{for all } t\in I \text{ that are near } c$$

Similarly, we say that a function f has a local minimum on an open interval I at $t=c$ if

$$f(c)\leq f(t) \quad \text{for all } t\in I \text{ that are near } c$$

We say that a function f has a local extremum at $t=c$ if f has a local maximum at c or f has a local minimum at c.

Theorem 5(*Fermat's - $\alpha-$ Theorem) If f is $\alpha-$ differentiable at a point c and has a local extremum at c, then* $\frac{df}{dt^{\alpha}}(c) = 0$ *(i.e $f_{\alpha}(c)=0$).*

Proof. 1.For $\alpha = 1$ (this is a classical case $\frac{df}{dt}(c) = 0$).

2.For $0<\alpha<1$ and $c=0$ (from remark 1, $\frac{df}{dt^{\alpha}}(c) = 0$).

3.For $0<\alpha<1$ and $c\neq 0$. Assume f has a local extremum at c and is $\alpha-$differentiable at c. We need to prove that $f_{\alpha}(c)=0$. To demonstrate this, we shall show that $f_{\alpha}(c)\geq 0$ and $f_{\alpha}(c)\leq 0$, implying that $f_{\alpha}(c)=0$. Because f has a local extremum at c, therefore it has a local maximum or local minimum at c. Assume f has a local minimum at c. Similarly, we can handel the case when f has a local maximum at c. There is an open interval I where $f(c)\leq f(t)$ for all $t\in I$. Because f is $\alpha-$differentiable at c,

$$f_{\alpha}(c) = \lim_{t\to c}\frac{f(t)-f(c)}{|t|^{\alpha}-|c|^{\alpha}}.$$

Because this limit exists, both sided limits also exist and equal $f_{\alpha}(c)$. Therefore,

$$f_{\alpha}(c) = \lim_{t\to c^{+}}\frac{f(t)-f(c)}{|t|^{\alpha}-|c|^{\alpha}} \tag{1}$$

and

$$f_{\alpha}(c) = \lim_{t\to c^{-}}\frac{f(t)-f(c)}{|t|^{\alpha}-|c|^{\alpha}}. \tag{2}$$

Because $f(c)$ is a local minimum, we see that $f(t)-f(c)\geq 0$ for t near c. Therefore, for t near c, but $t>c$, we have $\frac{f(t)-f(c)}{|t|^{\alpha}-|c|^{\alpha}}\geq 0$. From (1), we conclude that $f_{\alpha}(c)\geq 0$. Similarly, it can be shown that $f_{\alpha}(c)\leq 0$. Therefore, $f_{\alpha}(c)=0$.

Theorem 6(*Rolle's- $\alpha-$ theorem) Let f be a real valued continuous on $[t_0,t_f]$ and $\alpha-$ differentiable over (t_0,t_f) such that $f(t_0)=f(t_f)$. There is at least one $c\in(t_0,t_f)$ such that* $\frac{d}{dt^{\alpha}}(c) = 0$ *($f_{\alpha}(c)=0$).*

*Proof.*Let $k=f(t_0)=f(t_f)$. We consider three cases:

1.$f(t)=k$ for all $t\in(t_0,t_f)$.
2.There exists $t\in(t_0,t_f)$ so that $f(t)>k$.
3.There exists $t\in(t_0,t_f)$ so that $f(t)<k$.

Case 1:If $f(t)=k$ for all $t\in(t_0,t_f)$, then $f_{\alpha}(t)=0$ for all $t\in(t_0,t_f)$.

Case 2:Because f is continuous over $[t_0,t_f]$, According to the extreme value theorem, it has an absolute maximum. Also, since there is a point $t\in(t_0,t_f)$ such that $f(t)>k$, the absolute maximum is greater than k. As a result, the absolute maximum is not reached at either endpoint, the absolute maximum must occur at an interior point $c\in(t_0,t_f)$. Because f has a maximum at an interior point c, and is $\alpha-$differentiable at c, by Fermat's- $\alpha-$ theorem, $f_{\alpha}(c)=0$.

Case 3:The scenario when there exists a point $t\in(t_0,t_f)$ such that $f(t)<k$is similar to case 2, but maximum replaced by minimum.

Theorem 7(*Mean value- $\alpha-$ theorem) Let f be a continuous real valued function on $[t_0,t_f]$ and $\alpha-$ differentiable over (t_0,t_f). Then, there is at least one point $c\in(t_0,t_f)$ such that*

$$f_{\alpha}(c) = \frac{f(t_f)-f(t_0)}{|t_f|^{\alpha}-|t_0|^{\alpha}}.$$

*Proof.*Let $g(t)=f(t)-rt^{\alpha}$, where r is a constant. Because f is continuous on $[t_0,t_f]$ and $\alpha-$ differentiable on (t_0,t_f), the same is true for g. We now need to find r so that g meets the conditions of Rolle's- $\alpha-$ theorem. Namely

$$\begin{aligned} g(t_0)=g(t_f) &\Leftrightarrow f(t_0)-r|t_0|^{\alpha} = f(t_f)-r|t_f|^{\alpha} \\ &\Leftrightarrow r\left(|t_f|^{\alpha}-|t_0|^{\alpha}\right) = f(t_f)-f(t_0) \\ &\Leftrightarrow r = \frac{f(t_f)-f(t_0)}{|t_f|^{\alpha}-|t_0|^{\alpha}}. \end{aligned}$$

Table 2: List of indefinite α integral of some functions

$\int k\,dt^{\alpha}=c+kt^{\alpha}$	$\int t^{\beta}\,dt^{\alpha}=c+\frac{\alpha}{\beta}t^{\beta+\alpha}$
$\int t^{1-\alpha}e^{t}\,dt^{\alpha}=c+\alpha e^{t}$	$\int \frac{1}{t^{\alpha}}\,dt^{\alpha}=c+\alpha\ln\lvert t\rvert,\ t>0$
$\int \left(t^{1-\alpha}\sin t\right)dt^{\alpha}=c-\alpha\cos t$	$\int \left(t^{1-\alpha}\cos t\right)dt^{\alpha}=c+\alpha\sin t$
$\int \left(t^{1-\alpha}\sec^{2}t\right)dt^{\alpha}=c+\alpha\tan t$	$\int \left(t^{1-\alpha}\csc^{2}t\right)dt^{\alpha}=c-\alpha\cot t$
$\int \left(t^{1-\alpha}\sec t\tan t\right)dt^{\alpha}=c+\alpha\sec t$	$\int \left(t^{1-\alpha}\csc t\cot t\right)dt^{\alpha}=c-\alpha\csc t$
$\int \left(\frac{t^{1-\alpha}}{\sqrt{1-t^{2}}}\right)dt^{\alpha}=c+\alpha\sin^{-1}t$	$\int \left(\frac{t^{1-\alpha}}{1+t^{2}}\right)dt^{\alpha}=c-\alpha\tan^{-1}t$
$\int \left(\frac{t^{1-\alpha}}{t\sqrt{t^{2}-1}}\right)dt^{\alpha}=c+\alpha\sec^{-1}t$	

By Rolle's-$\alpha-$ theorem, since g is $\alpha-$ differentiable and $g(t_0)=g(t_f)$, there is some c in (t_0,t_f) for which $g_{\alpha}(c)=0$ and it follows from the equality $g(t)=f(t)-r|t|^{\alpha}$ that,

$$g_{\alpha}(t)=f_{\alpha}(t)-r=0$$
$$\Rightarrow f_{\alpha}(c)=r=\frac{f(t_f)-f(t_0)}{|t_f|^{\alpha}-|t_0|^{\alpha}}.$$

4 Indefinite $\alpha-$ integral

If $F(t)$ is $\alpha-$ differentiable and $\frac{dF}{dt^{\alpha}}=f(t)$, then

$$\frac{dF}{dt^{\alpha}}=f(t)\Leftrightarrow dF=f(t)\,dt^{\alpha}$$
$$\Leftrightarrow F(t)=\int f(t)\,dt^{\alpha}$$

and in general

$$\int f(t)\,dt^{\alpha}=F(t)+c.$$

From remark 3, if $F(t)$ is differentiable, then

$$dt^{\alpha}=\alpha t^{\alpha-1}dt$$

So we can construct table 2:

5 $\alpha-$ Integrable function

Suppose f is a continuous on the closed bounded interval $[t_0,t_f]$. Given a partition $P=\{t_0,t_1,t_2,...,t_n\}$, of $[t_0,t_f]$, with $t_0<t_1<t_2<...<t_n=t_f$ and $||p||=\max_{1\le i\le n}|t_i-t_{i-1}|$.

$$\text{Let}\qquad R_{\alpha,P}:=\sum_{i=1}^{n}f(t_i^{*})\left(t_i^{\alpha}-t_{i-1}^{\alpha}\right),\quad t_i^{*}\in[t_{i-1},t_i],$$

$$R_{\alpha}:=\lim_{||P||\to 0}R_{\alpha,P}.$$

Definition 6*Let f be a continuous function defined on $[t_0,t_f]$. The function $f(t)$ is said to be $\alpha-$ integrable on $[t_0,t_f]$ if*

$$\int_{t_0}^{t_f} f(t)\,dt^{\alpha} := \lim_{n\to\infty} R_{\alpha}$$

exists as a complex number. By convention we define

$$\int_{t_0}^{t_0} f(t)\,dt^{\alpha} := 0, \quad \int_{t_0}^{t_f} f(t)\,dt^{\alpha} := -\int_{t_f}^{t_0} f(t)\,dt^{\alpha}.$$

Remark 4*If f is $\alpha-$ integrable, then it is $1-$ integrable (integrable). and*

$$\begin{aligned}\int_{t_0}^{t_f} f(t)\,dt^{\alpha} &= \alpha \int_{t_0}^{t_f} f(t)\,t^{\alpha-1}\,dt\\ &= \alpha\,\overline{s_\alpha}(t).\int_{t_0}^{t_f} f(t)\,|t|^{\alpha-1}\,dt\end{aligned}$$

$$\begin{aligned}\textit{differentiable } \Rightarrow \alpha - \textit{differentiable } &\Rightarrow \textit{continuous}\\ &\Rightarrow \alpha - \textit{integrable } \Rightarrow \textit{integrable}.\end{aligned}$$

Then, we can construct the fundamental theorems for $\alpha-$ calculus

Theorem 8*(Fundamental $\alpha-$ theorem of calculus, part 1) If $f(t)$ is continuous on $[t_0,t_f]$, and the functions $F_1(t), F_2(t)$ are defined by*

$$F_1(t) = \int_{t_0}^{t} f(\tau)\,dt\tau^{\alpha}, \quad F_2(t) = \int_{t}^{t_f} f(\tau)\,dt\tau^{\alpha}$$

then $\dfrac{dF_1(t)}{dt^{\alpha}} = f(t), \quad \dfrac{dF_2(t)}{dt^{\alpha}} = -f(t)$ *over* $[t_0,t_f]$

Theorem 9*(Fundamental $\alpha-$ theorem of calculus, Part 2) If $f(t)$ is continuous on $[t_0,t_f]$ and $\dfrac{dF(t)}{dt^{\alpha}} = f(t)$, then*

$$\int_{t_0}^{t_f} f(t)\,dt^{\alpha} = F(t_f) - F(t_0).$$

From the fundamental theorem and Leibniz integral rule, we can prove

Theorem 10*($\alpha-$Leibniz integral rule) If $f(t,\tau)$ is continuous function and its partial derivatives exist and are themselves continuous function and the limits of integration $a(t), b(t)$ are continuous and differentiable functions of t, then*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)d\tau^{\alpha} = f(t,b(t)).\frac{db}{dt} - f(t,a(t)).\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,\tau)d\tau^{\alpha}$$

and by applying $\alpha-$Leibniz integral rule $n+1$ times to $\displaystyle\int_{t_0}^{t}(t-\tau)^n$ and $\int_{t_0}^{t_f}(\tau-t)^n$ respectively, we obtain the $\alpha-$ Couchy formulaes for repeated integration:

Theorem 11*($\alpha-$ Cauchy formulaes) If $f(t)$ is continuous function, then*

$$1. \int_{t_0}^{t} (t-\tau)^n f(\tau)d\tau^{\alpha} = n! \underbrace{\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{n-1}}}_{n+1\,times} f(\tau_n)d\tau_n^{\alpha} d\tau_{n-1}\ldots d\tau_1 d\tau,$$

$$2. \int_{t}^{t_f} (\tau-t)^n f(\tau)d\tau^{\alpha} = n! \underbrace{\int_{t}^{t_f}\int_{\tau}^{t_f}\int_{\tau_1}^{t_f}\cdots\int_{\tau_{n-1}}^{t_f}}_{n+1\,times} f(\tau_n)d\tau_n^{\alpha} d\tau_{n-1}\ldots d\tau_1 d\tau.$$

6 $\alpha-$ Operators of first extension

For a given time horizon $[t_0,t_f]$, we define the $\alpha-$ derivatives and integrals operators of continuous $\alpha-$ differentiable real valued function $x=f(t)$ by

First extension derivatives

$$\frac{d^{\alpha_1}f}{dt^{\alpha_2}}:=\Gamma(\alpha_1+1)\frac{df}{dt^{\alpha_2}}, \tag{3}$$

$$x^{(\alpha)}=f^{(\alpha)}=D^{\alpha}f:=\frac{d^{\alpha}f}{dt^{\alpha}}=\Gamma(\alpha+1)\frac{df}{dt^{\alpha}} \tag{4}$$

and if $f(t)$ is differentiable

$$\frac{d^{\alpha_1}f}{dt^{\alpha_2}}=\frac{\Gamma(\alpha_1+1)}{\alpha_2}t^{1-\alpha_2}\frac{df}{dt}, \tag{5}$$

$$x^{(\alpha)}=f^{(\alpha)}=D^{\alpha}f=\frac{d^{\alpha}f}{dt^{\alpha}}=\Gamma(\alpha)t^{1-\alpha}\frac{df}{dt}. \tag{6}$$

First extension integral

$$x^{(-\alpha)}=f^{(-\alpha)}=I^{\alpha}f:=\frac{1}{\Gamma(\alpha+1)}\int f(t)\,dt^{\alpha}. \tag{7}$$

Lower and upper integral operators are defined by

$$\underline{I}^{\alpha}f:=\frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}f(t)\,dt^{\alpha}=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}t^{\alpha-1}f(t)\,dt \tag{8}$$

$$\overline{I}^{\alpha}f:=\frac{1}{\Gamma(\alpha+1)}\int_{t}^{t_f}f(t)\,dt^{\alpha}=\frac{1}{\Gamma(\alpha)}\int_{t}^{t_f}t^{\alpha-1}f(t)\,dt \tag{9}$$

Remark 5 *Note that*

1. $D^{\alpha}(t^{\alpha})=\Gamma(\alpha+1)$
2. $D^{\alpha}(I^{\alpha}f(t))=D^{\alpha}(\underline{I}^{\alpha}f(t))=f(t)$
3. $D^{\alpha}\left(\overline{I}^{\alpha}f(t)\right)=-f(t)$
4. $I^{\alpha}(D^{\alpha}f(t))=f(t)+constant$
5. $\underline{I}^{\alpha}(D^{\alpha}f(t))=f(t)-f(t_0)$
6. $\overline{I}^{\alpha}(D^{\alpha}f(t))=f(t_f)-f(t)$

7 $\alpha-$ Operators of second extension

In this section, we extend the notation to second extension,

Second extension derivatives

$$\frac{d^{\alpha_1,\alpha_3}f}{dt^{\alpha_2,\alpha_4}}:=\frac{d^{\alpha_1}}{dt^{\alpha_2}}\left(\frac{d^{\alpha_3}f}{dt^{\alpha_4}}\right)=\Gamma(\alpha_1+1)\Gamma(\alpha_3+1)\frac{d}{dt^{\alpha_2}}\left(\frac{df}{dt^{\alpha_4}}\right), \tag{10}$$

$$\begin{aligned}x^{(\alpha_1,\alpha_2)}=f^{(\alpha_1,\alpha_2)}=D^{\alpha_1,\alpha_2}f:=D^{\alpha_1}(D^{\alpha_2}f)&=\frac{d^{\alpha_1}}{dt^{\alpha_1}}\left(\frac{d^{\alpha_2}f}{dt^{\alpha_2}}\right)\\&=\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\frac{d}{dt^{\alpha_1}}\left(\frac{df}{dt^{\alpha_2}}\right).\end{aligned} \tag{11}$$

$$x^{(\alpha)_2} = f^{(\alpha)_2} = D^{\alpha,\alpha} f := D^{\alpha}\left(D^{\alpha} f\right) = \frac{d^{\alpha}}{dt^{\alpha}}\left(\frac{d^{\alpha} f}{dt^{\alpha}}\right) = \left(\Gamma(\alpha+1)\right)^2 \frac{d}{dt^{\alpha}}\left(\frac{df}{dt^{\alpha}}\right), \tag{12}$$

$$x^{(2)} = f^{(2)} = D^2 f := x^{(1)_2}, \tag{13}$$

$$x^{(\beta)} = f^{(\beta)} = D^{\beta} f := x^{(\alpha,1)} = D^{\alpha,1} = D^{\alpha}\left(Df\right), \quad 1 < \beta = \alpha+1 \leq 2. \tag{14}$$

Second extension integrals

$$x^{(-\alpha_1,-\alpha_2)} = f^{(-\alpha_1,-\alpha_2)} = I^{\alpha_1,\alpha_2} f := I^{\alpha_1}\left(I^{\alpha_2} f\right) = \frac{1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \int \left\{ \int f(t)\, dt^{\alpha_2} \right\} dt^{\alpha_1}, \tag{15}$$

$$x^{(-\alpha)_2} = f^{(-\alpha)_2} = I^{\alpha,\alpha} f := I^{\alpha}\left(I^{\alpha} f\right) = \frac{1}{\left(\Gamma(\alpha+1)\right)^2} \int \left\{ \int f(t)\, dt^{\alpha} \right\} dt^{\alpha}, \tag{16}$$

$$x^{(-2)} = f^{(-2)} = I^2 f := I\left(If\right) = \int \left\{ \int f(t)\, dt \right\} dt, \tag{17}$$

$$x^{(-\beta)} = f^{(-\beta)} = I^{\beta} f := x^{(-1,-\alpha)} = I^{1,\alpha} = I\left(I^{\alpha} f\right), \quad 1 < \beta = \alpha+1 \leq 2. \tag{18}$$

Remark 6*Note that*

1. $D^{\beta}\left(t^{\beta}\right) = \Gamma(\beta+1), \qquad 1 < \beta \leq 2$
2. $D^{\alpha_1+\alpha_2} f \neq D^{\alpha_1,\alpha_2} f \neq D^{\alpha_2,\alpha_1} f$
3. $D^{\alpha_1,\alpha_2} I^{\alpha_2,\alpha_1} f(t) = f(t)$.

8 $\alpha-$ Operators of higher extension

We can extend the above notations to higher extension, for examples
Higher extension derivatives

$$x^{(\alpha_1,\alpha_2,\alpha_3)} = D^{\alpha_1,\alpha_2,\alpha_3} f := D^{\alpha_1}\left(D^{\alpha_2,\alpha_3} f\right) = D^{\alpha_1,\alpha_2}\left(D^{\alpha_3} f\right), \tag{19}$$

$$x^{(\alpha)_3} = f^{(\alpha)_3} := D^{\alpha,\alpha,\alpha} f, \tag{20}$$

$$x^{(n)} = f^{(n)} = D^n f := x^{(1)_n}, \tag{21}$$

$$x^{(\beta)} = f^{(\beta)} = D^{\beta} f := D^{\alpha}\left(D^n f\right), \quad n < \beta = \alpha+1 \leq n+1. \tag{22}$$

Higher extension integrals We have the followings

$$x^{(-\alpha_1,-\alpha_2,-\alpha_3)} = f^{(-\alpha_1,-\alpha_2,-\alpha_3)} = I^{\alpha_1,\alpha_2,\alpha_3} f := I^{\alpha_1,\alpha_2}\left(I^{\alpha_3} f\right) = I^{\alpha_1}\left(I^{\alpha_2,\alpha_3} f\right), \tag{23}$$

$$x^{(-\alpha)_3} = f^{(-\alpha)_3} := I^{\alpha,\alpha,\alpha} f, \tag{24}$$

$$x^{(-n)} = f^{(-n)} = I^n f := x^{(-1)_n}, \tag{25}$$

$$\begin{gathered} x^{(-\beta)} = f^{(-\beta)} := I^n \left(I^\alpha f\right), \\ n < \beta = \alpha + n \leq n + 1. \end{gathered} \tag{26}$$

From theorem 11,for $n < \beta = \alpha + n \leq n + 1,$ we can define $\underline{I}^\beta$ and $\overline{I}^\beta$ respectively by

$$\begin{aligned} \underline{I}^\beta f := & \frac{1}{\Gamma(\alpha+1)\,n!} \int_{t_0}^{t} (t-\tau)^n f(\tau) d\tau^\alpha \\ = & \frac{1}{\Gamma(\alpha)\,n!} \int_{t_0}^{t} (t-\tau)^n \tau^{\alpha-1} f(\tau) d\tau, \end{aligned} \tag{27}$$

$$\begin{aligned} \overline{I}^\beta f := & \frac{1}{\Gamma(\alpha+1)\,n!} \int_{t}^{t_f} (\tau-t)^n f(\tau) d\tau^\alpha \\ = & \frac{1}{\Gamma(\alpha)\,n!} \int_{t}^{t_f} (\tau-t)^n \tau^{\alpha-1} f(\tau) d\tau, \end{aligned} \tag{28}$$

9 Conclusion

In this paper, we have overcome the problem of multiple previous definitions of fractional calculus by putting an accurate definition of the $\alpha-$ calculus using the normal way. We concluded from this definition that:

1. the fractional calculus is a complex valued function and that it does not depend on gamma function, but the gamma is set in the definition just for normalization.
2. the definition here depends on the principal root of the real number t (positive or negative) but the definition ([15]) is given for a positive real number only.
3. here we have overcome the deficiency in the definition of Riemann -Liouville(see table 3 and table 4).

References

[1] K. B. Oldham and J. Spanier, *The fractional calculus*, London: Academic Press, 1974.
[2] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Vol. **204**, 2006.
[3] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.
[4] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, theory and application. Gordon and Breach, Yverdon, 1993.
[5] J. Cresson, Fractional embedding of differential operators and Lagrangian systems. *J. Math. Phys.* **48**, 033504 (2007).
[6] G. Jumarie, Fractional Hamilton-Jacobi equation for the optimal control of nonrandom fractional dynamics with fractional cost function. *J. Appl. Math. Comput.* **23**, 215–228 (2007).
[7] M. Klimek, Lagrangean and Hamiltonian fractional sequential mechanics. *Czech J. Phys.* **52**, 1247–1253 (2002).
[8] K. M. Kolwankar and A. D. Gangal, Holder exponents of irregular signals and local fractional derivatives. *Pramana J. Phys.* **48**, 49–68 (1997).
[9] A. Carpinteri and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer, Vienna, 1997.
[10] R. Hilfer, *Applications of fractional calculus in physics*, World Sci. Publishing, River Edge. 2000.
[11] G. Jumarie, On the representation of fractional Brownian motion as an integral with respect to (dt). *Appl. Math. Lett.* **18**, 739–748 (2005).
[12] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann–Liouville derivatives of order (α,β). *Math. Meth. Appl. Sci.* **30**, 1931–1939 (2007).
[13] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory. *Nonlin. Dyn.* **53**, 215–222 (2008).
[14] G. Jumarie, Fractional Hamilton-Jacobi equation for the optimal control of nonrandom fractional dynamics with fractional cost function. *J. Appl. Math. Comput.* **23**, 215–228 (2007).
[15] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative. *J. Comput. Appl. Math.* **264**, 65-70 (2014).

Table 3: Comparison between the definition of α derivative with the concept of Riemann-Liouville and our concept

Comparisons	Our Concept	Riemann-Liouville Concept
Derivative operator D^{α}, $0<\alpha\leq 1$	One definition for all $t\in\Re$ $(D^{\alpha}f(t)):=\Gamma(\alpha+1).\lim_{\tau\to t}\frac{f(\tau)-f(t)}{\tau^{\alpha}-t^{\alpha}}$. If f is differentiable, then $(D^{\alpha}f(t)):=\Gamma(\alpha).t^{1-\alpha}f'(t)$	Two definitions (left and right) for all $t\in[t_0,t_f]$ $(\underline{D}^{\alpha}f(t)):=\frac{1}{\Gamma(\alpha)}.\frac{d}{dt}\left(\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau\right)$ (left), $\left(\overline{D}^{\alpha}f(t)\right):=\frac{1}{\Gamma(\alpha)}.\frac{d}{dt}\left(\int_{t}^{t_f}(\tau-t)^{\alpha-1}f(\tau)d\tau\right)$ (right)
$\mathbf{D}^{\alpha}$**(constant)**	$D^{\alpha}(constant)=0$	$D^{\alpha}(constant)\neq 0$
Product Rule	$D^{\alpha}(u.v)=D^{\alpha}(u).v+u.D^{\alpha}(v)$	$D^{\alpha}(u.v)\neq D^{\alpha}(u).v+u.D^{\alpha}(v)$
$\mathbf{D}^{\alpha}$ **function**	It is a complex valued function	He did not explain the result of $D^{\alpha}f(t)$ when t is negative or what mean by $(t-\tau)^{\alpha-1}$ when $t-\tau<0$
When $\alpha=1$	It is completely identical to classical calculus when $\alpha=1$.	It does not coincide with classical calculus when $\alpha=1$.
Higher derivative operator $\mathbf{D}^{\beta}$, $\beta=\alpha+\mathbf{n}$	One definition for all $t\in\Re$ $\left(D^{\beta}f(t)\right):=D^{\alpha}\left(\frac{d^n f(t)}{dt^n}\right)$	Two definitions (left and right) for all $t\in[t_0,t_f]$ $\left(\underline{D}^{\beta}f(t)\right):=\frac{1}{\Gamma(\alpha)}.\frac{d^{n+1}}{dt^{n+1}}\left(\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau\right)$, $\left(\overline{D}^{\alpha}f(t)\right):=\frac{1}{\Gamma(\alpha)}.\frac{d^{n+1}}{dt^{n+1}}\left(\int_{t}^{t_f}(\tau-t)^{\alpha-1}f(\tau)d\tau\right)$

Table 4: Comparison between the definition of α integral with the concept of Riemann-Liouville and our concept

Comparisons	**Our Concept**	**Riemann-Liouville Concept**
Integral operator $I^{\alpha},\ 0<\alpha\leq 1$	For all $t\in[t_0,t_f]$ $(I^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha+1)} \cdot \lim_{n\to\infty} \lim_{\lvert P\rvert\to 0} \sum_{i=1}^{n} f(t_i^*)\left(t_{i-1}^{\alpha}-t_i^{\alpha}\right),$ Where $P=\{t_0,t_1,\ldots,t_n=t_f\}$ is any partition of $[t_0,t_f]$. For a continuous function f on $[t_0,t_f]$, we define lower and upper integral operator for $t\in[t_0,t_f]$. by: $(\underline{I}^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(\tau)^{\alpha-1} f(\tau)d\tau\right),$ $\left(\overline{I}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t}^{t_f}(\tau)^{\alpha-1} f(\tau)d\tau\right)$	Two definitions (left and right) for all $t\in[t_0,t_f]$ $(\underline{I}^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-\tau)^{\alpha-1} f(\tau)d\tau\right),$ $\left(\overline{I}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t}^{t_f}(\tau-t)^{\alpha-1} f(\tau)d\tau\right)$
$\mathbf{I}^{\alpha}$ **function**	It is a complex valued function	He did not explain the result of $I^{\alpha} f(t)$ when t is negative or what mean by $(t-\tau)^{\alpha-1}$ when $t-\tau<0$
Higher integral operator $\mathbf{I}^{\beta}$, $\beta=\alpha+\mathbf{n}$	For a continuous function f on $[t_0,t_f]$, we define lower and upper integral operator for $t\in[t_0,t_f]$, by: $\left(\underline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-\tau)^{n-1}(\tau)^{\alpha} f(\tau)d\tau\right),$ $\left(\overline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t}^{t_f}(\tau-t)^{n-1}(\tau)^{\alpha} f(\tau)d\tau\right)$	Two definitions (left and right) for all $t\in[t_0,t_f]$ $\left(\underline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-\tau)^{n+\alpha-1} f(\tau)d\tau\right),$ $\left(\overline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)}\left(\int_{t}^{t_f}(\tau-t)^{n+\alpha-1} f(\tau)d\tau\right)$