Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170305

Legendre Polynomials' Second Derivative Tau Method for Solving Lane-Emden and Ricatti Equations

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Received: 30 Oct. 2022, Revised: 2 Mar. 2023, Accepted: 30 Mar. 2023
Published online: 1 May 2023

Abstract: This paper investigates a highly efficient method that depends on the Tau method for solving initial and boundary value problems. The second derivatives of Legendre polynomials (SDLPs) have been used as novel basis functions. A linearization relation for the presented basis functions has been introduced and proved to avoid any issues arising during tau's integration, especially for the nonlinear problems. Consequently, some essential integrations have been determined. Moreover, we used those relations to construct explicit forms for approximating the solutions of Lane-Emden and the Recatti equations. In addition, the presented strategy's converge and error analysis are discussed carefully and in-depth. Finally, the mentioned IVPs have been solved via the proposed method. The results have been compared with the others' methods, which showed our technique's accuracy, efficiency, and stability.

Keywords: Second derivative Legendre polynomials, Tau method, Error analysis, linearization relation, Explicit forms, Lane-Emden and Riccati Equations.

1 Introduction

Solving differential equations is highly important because used to simulate scientific and engineering phenomena [1–3]. That's why many authors are interested in finding the solution to these differential equations using different methods. However, in some cases, we can not find the exact solution by traditional methods. So, numerical methods such as finite element [4], finite difference [5], and spectral methods [6–8] have been used to approximate the solutions. Several authors prefer the spectral method because the approximate solution is more efficient and accurate than the other methods.

The main idea of the spectral methods is to represent the approximate solution as follows [9]:

$$u(\mathfrak{q}) \approx u_n(\mathfrak{q}) = \sum_{i=0}^{n} A_i p_i(\mathfrak{q}),$$

where A_i is unknown coefficients and $p_i(\mathfrak{q})$ is the basis functions.

The spectral methods will convert the differential equation into a system of algebraic equations. The unknown of that system is the coefficients $\{A_i\}_{i=0}^{i=n}$. Then, we can find these coefficients using different analytical or approximated methods. Consequently, the approximate solution of the differential equation can be found using these coefficients.

The spectral methods can be categorized into three primary types: Galerkin [10], Tau [11], and pseudo-spectral [12, 13]. In comparison, the authors [14–16] applied the pseudo-Galerkin method as one of the residual methods.

Choosing the basis functions is a critical part of the spectral methods. Polynomials such as Legendre polynomials (LPs) [17], Chebyshev polynomials [18], and Ultraspherical polynomials [19] have been utilized as basis functions. At the same time, the authors in [14, 15, 20–22] applied the first and second derivatives of Legendre and Chebyshev polynomials. This concept shows more accurate and efficient approximate solutions. The authors in [16] used the SDLPs as a new base function. Some relations of these basis functions, such as

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an operational matrix for derivatives, have been introduced. The second Legendre derivatives in the Pseudo-Galerkin method verified accuracy, efficiency, and stability.

This paper aims to extend the work in [16]. The tau method will be applied to approximate the solutions of the Lane-Emden and Riccati equations to get more accurate and efficient ones. As it is known, we will face complex integrations via the Tau method, especially in nonlinear problems. Therefore, a linearization relation for the polynomials will be derived.

This paper is planned as follows: The second section presents essential relations and theorems of PLs and SDLPs. Then, a novel Linearization relation of the presented basis functions has been investigated in the third section. In the fourth section, the technique of solution is discussed. In addition, the third section will be ended with an Algorithm of the introduced method. Consequently, error analysis of differential equations is studied in the fifth section. Then, we will be resolved examples for Lane-Emden and Riccati equations during the sixth section. Finally, the paper ended with a brief conclusion in the seventh section.

2 Preliminaries

During this section, we will present several essential concepts, theorems, and properties of the LPs, $\mathfrak{L}_n(\mathfrak{q})$, and SDLPs, ${}^{s\mathcal{D}}L_n(\mathfrak{q})$, of degree n, where $\mathfrak{q} \in [-1,1]$.

LPs can be presented via the relation [9]:

$$(n+1)\mathfrak{L}_{n+1}(\mathfrak{q}) = (2n+1)\mathfrak{q}\mathfrak{L}_n(\mathfrak{q}) - n\mathfrak{L}_{n-1}(\mathfrak{q}), \quad n = 1,2,\cdots, \tag{1}$$

where $\mathfrak{L}_0(\mathfrak{q}) = 1$, $\mathfrak{L}_1(\mathfrak{q}) = \mathfrak{q}$.

LPs and the first derivatives of LPs satisfy the following boundaries:

$$|\mathfrak{L}_n(\mathfrak{q})| \leq 1, \tag{2}$$

$$\mathfrak{L}_n(1) = 1, \tag{3}$$

$$\mathfrak{L}_n(-1) = (-1)^n, \tag{4}$$

$$\mathfrak{L}'_n(1) = \frac{1}{2}n(n+1), \tag{5}$$

$$\mathfrak{L}'_n(-1) = \frac{(-1)^{n-1}}{2}n(n+1). \tag{6}$$

$$\int_{-1}^{1} \mathfrak{L}'_i(\mathfrak{q})\mathfrak{L}'_j(\mathfrak{q})\,(1-\mathfrak{q}^2)\,d\mathfrak{q} = \begin{cases} 0 & i \neq j, \\ \frac{2i(i+1)}{2i+1} & i = j, \end{cases} \tag{7}$$

SDLPs ${}^{s\mathcal{D}}L_n(\mathfrak{q})$ of degree n can be defined as [16]:

$${}^{s\mathcal{D}}L_n(\mathfrak{q}) = \frac{d^2\mathfrak{L}_{n+2}(\mathfrak{q})}{d\mathfrak{q}^2}, \tag{8}$$

where n be any positive integer.

Also, SDLPs can be obtained using the recurrence relation:

$$(n+1)\,{}^{s\mathcal{D}}L_{n+1}(\mathfrak{q}) = (2n+5)\mathfrak{q}\,{}^{s\mathcal{D}}L_n(\mathfrak{q}) - (n+4)\,{}^{s\mathcal{D}}L_{n-1}(\mathfrak{q}), \quad n = 1,2,\cdots, \tag{9}$$

where ${}^{s\mathcal{D}}L_0(\mathfrak{q}) = 3$, ${}^{s\mathcal{D}}L_1(\mathfrak{q}) = 15\mathfrak{q}$.

The boundaries of SDLPs and their derivatives satisfy:

$${}^{s\mathcal{D}}L_n(\pm 1) = \frac{(\pm 1)^n}{8}(n+1)_4, \tag{10}$$

$${}^{s\mathcal{D}}L'_n(\pm 1) = \frac{(\pm 1)^{n-1}}{48}(n)_6, \tag{11}$$

$${}^{s\mathcal{D}}L''_n(\pm 1) = \frac{(\pm 1)^n}{384}(n-1)_8, \tag{12}$$

where $(m)_n = \frac{\Gamma(m+n)}{\Gamma(m)}$ is Pochhammer symbol.

The orthogonal relation for SDLPs is presented as:

$$\int_{-1}^{1} {}^{s\mathcal{D}}L_i(\mathfrak{q})\,{}^{s\mathcal{D}}L_j(\mathfrak{q})w(\mathfrak{q})d\mathfrak{q} = \begin{cases} 0 & i \neq j, \\ \frac{2(i+1)_4}{2i+5} & i = j, \end{cases} \tag{13}$$

where $w(\mathfrak{q}) = (1-\mathfrak{q}^2)^2$.

The SDLPs' moment formula is given by:

$$\mathfrak{q}^r\,{}^{s\mathcal{D}}L_n(\mathfrak{q}) = \sum_{j=1}^{r+1} F_{r,j}\,{}^{s\mathcal{D}}L_{b-2}(\mathfrak{q}), \tag{14}$$

where

$$F_{r,j} = \begin{cases} \alpha_n & r=1, j=1, \\ \gamma_n & r=1, j=2, \\ F_{r-1,1}\alpha_{n-r+1} & r \geq 2, j=1, \\ F_{r-1,j-1}\gamma_{b-3} + F_{r-1,j}\,\alpha_{b-1} & r \geq 2, 2 \leq j \leq r, \\ F_{r-1,r}\gamma_{n+r-1} & r \geq 2, j = r+1, \end{cases}$$

with $\alpha_n = \frac{n+4}{2n+5}$, $\gamma_n = \frac{n+1}{2n+5}$, and $b = n - r + 2j$.

The m^{th} derivatives of ${}^{s\mathcal{D}}L(\mathfrak{q})$ is:

$$D^m[{}^{s\mathcal{D}}L(\mathfrak{q})] = M^m.{}^{s\mathcal{D}}L(\mathfrak{q}), \tag{15}$$

where $M^m = (\Re_{ij}^{(m)})_{i,j=1}^{r+1}$, such that:

$$\Re_{ij}^{(m)} = \frac{1}{2^{m-1}(m-1)!}\begin{cases} G, & i > j, (i+m+j) even, \\ 0 & otherwise, \end{cases} \tag{16}$$

where $G = (2j+3)\prod_{f=0}^{m-2}[i+m-2f)(i+m-2f-1) - (j+1)(j+2)]$.

We will investigate and prove the SDLPs linearization relation in the next section. Then, some essential integrations will be determined. Those results will be used to construct explicit forms for the algebraic systems of the coefficients A_i of Lane-Emden, and the Recatti equations' approximated solutions via the Tau method.

3 Second Derivatives of Legendre Polynomials Linearization Formula

The section starts with the linearization formula of SDLPs as follows.

Lemma 1.*The product of two SDLPs can be presented as:*

$$ {}^{s\mathcal{D}}L_{\mathfrak{n}}(\mathfrak{q})\,{}^{s\mathcal{D}}L_{\mathfrak{m}}(\mathfrak{q}) = \frac{8}{3\pi}\sum_{i=0}^{min(n,m)} K_{n,m,i}\,{}^{s\mathcal{D}}L_{n+m-2i}(\mathfrak{q}), \tag{17} $$

where

$$ K_{n,m,i} = \frac{c(n-i+1)_{\frac{3}{2}}(m-i+1)_{\frac{3}{2}}(m+n-i+\frac{7}{2})_{\frac{3}{2}}}{(m+n-2i+1)_4}, $$

and $c = (2m+2n-4i+5)(i+1)_{\frac{3}{2}}$.

*Proof.*From [23], the product of two Ultraspherical polynomials is

$$ C_m^{(v)}(\mathfrak{q})C_n^{(v)}(\mathfrak{q}) = \sum_{i=0}^{min(n,m)} L_{m,n,s}C^{v}_{m+n-2i}(\mathfrak{q}), \tag{18} $$

where

$$ L_{m,n,s} = \frac{m+n+v-2i}{m+n+v-i}\frac{(v)_i(v)_{m-i}(v)_{n-i}}{i!(m-i)!(n-i)!}\frac{(2v)_{m+n-i}}{(v)_{m+n-i}}\frac{(m+n-2i)!}{(2v)_{m+n-2i}}, $$

Since

$$ \frac{dC_n^{(v)}(\mathfrak{q})}{d\mathfrak{q}} = 2vC_{n-1}^{(v+1)}(\mathfrak{q}), \tag{19} $$

$$ \frac{d^2C_{n+2}^{(v)}(\mathfrak{q})}{d\mathfrak{q}^2} = 4v(v+1)C_n^{(v+2)}(\mathfrak{q}). \tag{20} $$

and using the relation between Ultraspherical polynomials and Legendre polynomials as:

$$ C_n^{(\frac{1}{2})}(\mathfrak{q}) = \mathfrak{L}_n(\mathfrak{q}), \tag{21} $$

Thus:

$$ C_n^{(\frac{5}{2})}(\mathfrak{q}) = \frac{1}{3}\,{}^{s\mathcal{D}}L_n(\mathfrak{q}). \tag{22} $$

Using relation (18) and relation (22), the required relation will be obtained.

Lemma 2.*The integrations of SDLPs multiplied by* $\mathfrak{q}$, $\mathfrak{q}^2$ *and* $\mathfrak{q}^3$ *are determined by:*

$$ \int_{-1}^{1}\mathfrak{q}\,{}^{s\mathcal{D}}L_n(\mathfrak{q})d\mathfrak{q} = (1-(-1)^n)[\frac{(n+2)(n+3)}{2}-1], \tag{23} $$

$$ \int_{-1}^{1}\mathfrak{q}^2\,{}^{s\mathcal{D}}L_n(\mathfrak{q})d\mathfrak{q} = (1-(-1)^{n-1})[\frac{(n+2)(n+3)}{2}-2], \tag{24} $$

$$ \int_{-1}^{1}\mathfrak{q}^3\,{}^{s\mathcal{D}}L_n(\mathfrak{q})d\mathfrak{q} = (1-(-1)^{n-2})[\frac{(n+2)(n+3)}{2}-3]. \tag{25} $$

*Proof.*Use Eqs. (5), (6), and (14) to get the desired relations.

Lemma 3.*The integration of the product of two SDLPs given by:*

$$ \int_{-1}^{1}{}^{s\mathcal{D}}L_n(\mathfrak{q})\,{}^{s\mathcal{D}}L_m(\mathfrak{q})d\mathfrak{q} = \frac{8}{3\pi}\sum_{i=0}^{min(n,m)} K_{n,m,i}\,\frac{1}{2}(r+2)(r+3)[1-(-1)^{r+1}], \tag{26} $$

where $r = n+m-2i$, *and* $K_{n,m,i}$ *as in Eq. (17).*

*Proof.*From Eq. (17):

$$ \int_{-1}^{1}{}^{s\mathcal{D}}L_n(\mathfrak{q})\,{}^{s\mathcal{D}}L_m(\mathfrak{q})d\mathfrak{q} = \frac{8}{3\pi}\sum_{i=0}^{min(n,m)} K_{n,m,i}\int_{-1}^{1}{}^{s\mathcal{D}}L_{n+m-2i}(\mathfrak{q})d\mathfrak{q}, \tag{27} $$

Using the definition of the SDLPs (8) and the boundaries (5) and (6):

$$ \begin{aligned} \int_{-1}^{1}{}^{s\mathcal{D}}L_{n+m-2i}(\mathfrak{q})d\mathfrak{q} &= [\mathfrak{L}'(\mathfrak{q})_{n+m-2i+2}]_{-1}^{1} \\ &= \frac{1}{2}(n+m-2i+2)(n+m-2i+3) - \\ &\quad \frac{(-1)^{n+m-2i+1}}{2}(n+m-2i+2)(n+m-2i+3) \\ &= \frac{1}{2}(n+m-2i+2)(n+m-2i+3) \\ &\quad [1-(-1)^{n+m-2i+1}], \end{aligned} \tag{28} $$

Thus the proof is completed.

Theorem 1.*The integration of two SDLPs multiplied by* $\mathfrak{q}^2-\mathfrak{q}$ *can be presented as:*

$$ \int_{-1}^{1}{}^{s\mathcal{D}}L_n(\mathfrak{q})\,{}^{s\mathcal{D}}L_m(\mathfrak{q})w^*(\mathfrak{q})d\mathfrak{q} = \frac{8}{3\pi}\sum_{i=0}^{min(n,m)} K_{n,m,i}\left[(-1)^l[(l+2)(l+3)-3]-1\right], \tag{29} $$

where $l = n+m-2i$, $K_{n,m,i}$ *as in Eq. (17), and* $w^*(\mathfrak{q}) = \mathfrak{q}^2 - \mathfrak{q}$.

*Proof.*From Eq. (17):

$$\int_{-1}^{1} {}^{s\mathcal{D}}L_n(\mathfrak{q})\, {}^{s\mathcal{D}}L_m(\mathfrak{q}) w^*(\mathfrak{q}) d\mathfrak{q} = \frac{8}{3\pi} \sum_{i=0}^{min(n,m)} K_{n,m,i} \int_{-1}^{1} {}^{s\mathcal{D}}L_{n+m-2i}(\mathfrak{q}) w^*(\mathfrak{q}) d\mathfrak{q}. \tag{30}$$

The proof will be completed using Eqs. (23) and (24),

Now, the mathematical relationships and integrations are ready for us to apply the Tau method easily.

4 Second Legendre derivatives Tau Method (SDLP-TM)

This section will introduce the technique of finding the approximate solution of IVP and BVP using SDLPs via the Tau method.

Consider the linear/nonlinear ordinary differential equation:

$$f\left(\mu_m(\mathfrak{q})u^{(m)}(\mathfrak{q}), \mu_{m-1}(\mathfrak{q})u^{(m-1)}(\mathfrak{q}), \cdots, \mu_1(\mathfrak{q})u^{(1)}(\mathfrak{q}), \mu_0(\mathfrak{q})u(\mathfrak{q}), \mu(\mathfrak{q})\right) = 0, \tag{31}$$

where $-1 \le \mathfrak{q} \le 1$, subject to the initial and boundary conditions:

$$\begin{array}{ll} u(-1) = \alpha_0, & u(1) = \beta_0, \\ u'(-1) = \alpha_1, & u'(1) = \beta_1, \\ \vdots & \vdots \\ u^{(s)}(-1) = \alpha_s, & u^{(r)}(1) = \beta_r, \end{array} \tag{32}$$

such that $\{\alpha_i\}_0^s$ and $\{\beta_s\}_0^r$ are constants, $\mu(\mathfrak{q})$, $\mu_i(\mathfrak{q})$ are real valued functions. The required solution will be approximated by:

$$u(\mathfrak{q}) \approx u_n(\mathfrak{q}) = \sum_{i=0}^{n} A_i\, {}^{s\mathcal{D}}L_i(\mathfrak{q}), \tag{33}$$

where A_i are constant.

Eqs. (15),(16) can be used to represent the derivatives of the unknown function $u(\mathfrak{q})$ as follows:

$$\frac{d^m u_n(\mathfrak{q})}{d\mathfrak{q}^m} = \sum_{i=0}^{n} \sum_{k=0}^{i-m} A_i \Re_{ik}^{(m)}\, {}^{s\mathcal{D}}L_k(\mathfrak{q}). \tag{34}$$

At $m = 0$: the equation (34) will be equivalent to equation (33). Substituting from Eq. (34) into the BVP (31-32) to get the residual:

$$R_n = f\left(\mu_m(\mathfrak{q}) \sum_{i=0}^{n} \sum_{k=0}^{i-m} A_i \Re_{ik}^{(m)}\, {}^{s\mathcal{D}}L_k(\mathfrak{q}),\ \mu_{m-1}(\mathfrak{q}) \sum_{i=0}^{n} \sum_{k=0}^{i-m+1} A_i \Re_{ik}^{(m-1)}\, {}^{s\mathcal{D}}L_k(\mathfrak{q}), \cdots,\ \mu_0(\mathfrak{q}) \sum_{i=0}^{n} A_i\, {}^{s\mathcal{D}}L_i(\mathfrak{q}), \mu(\mathfrak{q})\right) = 0, \quad -1 \le \mathfrak{q} \le 1, \tag{35}$$

with initial/boundary condition:

$$\begin{array}{l} \sum_{j=0}^{n} \frac{(-1)^i}{8} A_i (i+1)_4 = \alpha_0, \\ \sum_{i=0}^{n} \frac{1}{8} A_i (i+1)_4 = \beta_0, \\ \sum_{i=0}^{n} \sum_{k=0}^{i-1} \frac{(-1)^k}{8} A_i \Re_{ik}^{(1)} (k+1)_4 = \alpha_1, \\ \sum_{i=0}^{n} \sum_{k=0}^{i-1} \frac{1}{8} A_i \Re_{ik}^{(1)} (k+1)_4 = \beta_1, \\ \vdots \\ \sum_{i=0}^{n} \sum_{k=0}^{i-s} \frac{(-1)^k}{8} A_i \Re_{ik}^{(s)} (k+1)_4 = \alpha_s, \\ \sum_{i=0}^{n} \sum_{k=0}^{i-r} \frac{1}{8} A_i \Re_{ik}^{(r)} (k+1)_4 = \beta_r. \end{array} \tag{36}$$

Then, by applying the Tau method to get the integration:

$$\int_{-1}^{1} R_n\, {}^{s\mathcal{D}}L_k(\mathfrak{q}) w^*(\mathfrak{q}) d\mathfrak{q}, \quad k = 0, 1, 2, \ldots, n-m, \tag{37}$$

where $w^*(\mathfrak{q}) = \mathfrak{q}^2 - \mathfrak{q}$ is the appreciate weight function. Eqs. (36) with Eqs. (37) generate an algebraic equations system for unknown coefficients A_n. Consequently, we will solve this system to get the approximate solution of ODEs.

The steps of the solution will be summarized in Algorithm (1). This Algorithm helps the reader to code the approximation methods easily with any suitable software.

The following two subsections will explicitly express the Lane-Emden and Riccati Equations' algebraic systems.

4.1 Riccati Equation via SDLP-TM

Theorem 2.*Consider the following Riccati equation:*

$$u'(\mathfrak{q}) = P(\mathfrak{q}) + Y(\mathfrak{q})u(\mathfrak{q}) + Z(\mathfrak{q})u^2(\mathfrak{q}), \quad 0 \le \mathfrak{q} \le 1. \tag{38}$$

Algorithm 1 Steps for solving ODE via SDLP-TM

Step 1: Input $n \in \mathbb{N}$.
Step 2: Expand the dependent variable of the BVP using the spectral expansion (33).
Step 3: Shift the independent variable to [-1,1].
Step 4: Construct the operational matrix (Eq. (15)).
Step 5: Expand the BVP as depicted in Eqs. (35) and (36).
Step 6: Apply Tau method's integration Eq. (37) to get the algebraic system of the unknowns A_i.
Step 7: Solve the previous system to obtain A_i.
Step 8: Substitute into the spectral expansion, step (2), by A_i, step (7), to get the approximated solution of the BVP.

So, the unknown coefficient A_i for the Riccati equation's algebraic system:

$$\frac{16}{3\pi}\sum_{i=0}^{n}\sum_{r=0}^{i-1}\sum_{s_1=0}^{min(r,k)} A_i \Re_{ir}^{(1)} K_{r,k,s_1} Q_{r,k,s_1} - \sum_{e_1=0}^{t_1}\sum_{o_1=1}^{e_1+1} p_{e_1} F_{e_1,o_1} Q_{2o_1-e_1,k,1} - \frac{8}{3\pi}\sum_{i=0}^{n}\sum_{e_2=0}^{t_2}\sum_{o_2=1}^{e_2+1}\sum_{s_2=0}^{min(h_1,k)} A_i y_{e_2} F_{e_2,o_2} K_{h_1,k,s_2} Q_{h_1,k,s_2} - \frac{64}{9\pi^2}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{e_3=0}^{t_3}\sum_{o_3=1}^{e_3+1}\sum_{s_3=0}^{min(j,k)}\sum_{s_4=0}^{min(h_2,h_3)} A_i A_j z_{e_3} F_{e_3,o_3} K_{j,k,s_3} K_{h_2,h_3,s_4} Q_{h_2,h_3,s_4}, \tag{39}$$

where
$h_1 = i + 2o_2 - e_2 - 2,$
$h_2 = i - e_3 + 2o_3 - 2,$
$h_3 = j + k - 2s_3,$
$F_{n,m}$ *as in Eq. (14),*
$K_{n,m,i}$ *as in Eq. (17),*
$Q_{n,m,s} = ((-1)^{n+m-2s}[(n+m-2s+2)(n+m-2s+3)-3]-1),$
$P(\mathfrak{q}) = \sum_{e_1=0}^{t_1} p_{e_1}\mathfrak{q}^{e_1},$
$Y(\mathfrak{q}) = \sum_{e_2=0}^{t_2} y_{e_2}\mathfrak{q}^{e_2},$
$Z(\mathfrak{q}) = \sum_{e_3=0}^{t_3} z_{e_3}\mathfrak{q}^{e_3},$
p_{e_1}, y_{e_2} *and* z_{e_3} *are constants.*

Proof. Using Eqs. (33), (34), we obtain the residual:

$$R_n = 2\sum_{i=0}^{n}\sum_{r=0}^{i-1} A_i \Re_{ir}^{(1)}\,{}^{s\mathcal{D}}L_r(\mathfrak{q}) - \sum_{e_1=0}^{t_1} p_{e_1}\mathfrak{q}^{e_1} - \sum_{i=0}^{n}\sum_{e_2=0}^{t_2} y_{e_2}\mathfrak{q}^{e_2} A_i\,{}^{s\mathcal{D}}L_i(\mathfrak{q}) - \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{e_3=0}^{t_3} A_i A_j z_{e_3}\mathfrak{q}^{e_3}\,{}^{s\mathcal{D}}L_i(\mathfrak{q})\,{}^{s\mathcal{D}}L_j(\mathfrak{q}), \quad -1 \le \mathfrak{q} \le 1. \tag{40}$$

By applying relation (37) with the aid of relations (14), (17), and (29), the proof will be completed.

4.2 Lane-Emden Equation via SDLP-TM

Similar to the previous subsection, an algebraic system for the spectral's coefficients of the Lane-Emden equation will be introduced.

Theorem 3. *Consider the following Lane-Emden equation:*

$$\mathfrak{q}u''(\mathfrak{q}) + \alpha u'(\mathfrak{q}) + \beta\mathfrak{q}u(\mathfrak{q}) = \gamma U(\mathfrak{q}, u(\mathfrak{q})), \tag{41}$$

where $0 \le \mathfrak{q} \le X$.

Then, the Lane-Emden equation's algebraic system for the unknown coefficient A_i will be:

$$\frac{32}{3\pi}\sum_{i=0}^{n}\sum_{r=0}^{i-2}\sum_{j_1=1}^{2}\sum_{s_1=0}^{min(t,k)} A_i \Re_{ir}^{(2)} F_{1,j_1} K_{t,k,s_1} Q_{t,k,s_1} + \frac{16}{3\pi}X\alpha\sum_{i=0}^{n}\sum_{r=0}^{i-1}\sum_{s_2=0}^{min(r,k)} A_i \Re_{ir}^{(1)} K_{r,k,s_2} Q_{r,k,s_2} + \frac{8}{3\pi}X^2\beta\sum_{i=0}^{n}\sum_{j_2=0}^{2}\sum_{s_3=0}^{min(p,k)} A_i F_{1,j_2} K_{p,k,s_3} Q_{p,k,s_3} - \gamma W, \tag{42}$$

where
α, β, *and* γ *are constants,*
$t = r + 2j_1 - 3,$
$p = i + 2j_2 - 3,$
$U(\mathfrak{q}, u(\mathfrak{q}))$ *is a real-valued function,*
$W = \int_{-1}^{1} X^2\,{}^{s\mathcal{D}}L_k(\mathfrak{q}) U(\mathfrak{q}, u(\mathfrak{q})) w^*(q)dq,$
$F_{n,m}$ *as in Eq. (14),*
$K_{n,m,i}$ *as in Eq. (17),*
$Q_{n,m,s} = ((-1)^{n+m-2s}[(n+m-2s+2)(n+m-2s+3)-3]-1).$

Proof. Using Eqs. (33) and (34), the residual of the shifted Lane-Emden equation will be:

$$\begin{aligned} R_n =&4\mathfrak{q}\sum_{i=0}^{n}\sum_{r=0}^{i-2} A_i \Re_{ir}^{(2)}\, {}^{s\mathcal{D}}L_r(\mathfrak{q}) \\ &+2X\alpha\sum_{i=0}^{n}\sum_{r=0}^{i-1} A_i \Re_{ir}^{(1)}\, {}^{s\mathcal{D}}L_r(\mathfrak{q}) \\ &+X^2\beta\,\mathfrak{q}\sum_{i=0}^{n} A_i\, {}^{s\mathcal{D}}L_i(\mathfrak{q}) \\ &-X^2\gamma U(\mathfrak{q},u(\mathfrak{q})), \end{aligned} \tag{43}$$

where $-1 \le \mathfrak{q} \le 1$.

By applying relation (37) and using relations (14), (17), and (29), the required result will be proved.

In the next section, the error analysis and convergence will study the proposed method's accuracy. Moreover, we will investigate the global errors of the Lane-Emden and the Recatti equations.

5 Error Analysis

The following definition and theorems are necessary to discuss error analysis and convergence.

Definition 1. ***Lipschitz Condition:*** *[24] Let $u(\mathfrak{q})$ be piecewise continuous in $\mathfrak{q}$, and defined on $[a,b]$, then $u(\mathfrak{q})$ is said to satisfy a Lipschitz condition on $[a,b]$ if there exists a constant $\zeta > 0$ such that:*
$||u(\mathfrak{q}) - u(\hat{\mathfrak{q}})|| \le \zeta||\mathfrak{q} - \hat{\mathfrak{q}}||, \forall\hat{\mathfrak{q}} \in [a,b]$,
where ζ is called the Lipschitz constant.

Theorem 4. *[16] If $|u^{(p)}(\mathfrak{q})| \le M$, that can be spectrally expanded, as presented in Eq. (33), in terms of SDLPs. Then:*

$$|A_i| < \frac{2^{p+2}}{i^{p+1}}M, \quad \forall\, i > 2. \tag{44}$$

Theorem 5. *[16] Let $u(\mathfrak{q})$ be any continuous function that satisfies the previous theorem, then :*

$$|u(\mathfrak{q}) - u_n(\mathfrak{q})| \lesssim O\left(\frac{1}{n}\right)^{p-4}. \tag{45}$$

Corollary 1. *Let $u(\mathfrak{q})$ be any continuous function that satisfies Theorems (4), (5), then:*

$$(i)|u'(\mathfrak{q}) - u_n'(\mathfrak{q})| \lesssim O\left(\frac{1}{n^{p-6}}\right), \tag{46}$$

$$(ii)|u''(\mathfrak{q}) - u_n''(\mathfrak{q})| \lesssim O\left(\frac{1}{n^{p-8}}\right), \tag{47}$$

$$(iii)\,|u^2(\mathfrak{q}) - u_n^2(\mathfrak{q})| \lesssim O\left(\frac{1}{n^{p-4}}\right). \tag{48}$$

Theorem 6. *Consider the function $u(\mathfrak{q})$ that satisfies Theorems (4), (5) and let $U(\mathfrak{q};u)$ satisfies Lipschitz condition (1) for the variable u with constant ζ, then the global error of:*

i)The Lane-Emden equation
$\mathfrak{q}\,u''(\mathfrak{q}) + \alpha\,u'(\mathfrak{q}) + \beta\,\mathfrak{q}\,u(\mathfrak{q}) = \gamma\,U(\mathfrak{q};u)$; $0 \le \mathfrak{q} \le X$,
will be $O\left(\frac{1}{n^{p-8}}\right)$,

ii)The Ricatti equation
$u'(\mathfrak{q}) = P(\mathfrak{q}) + Y(\mathfrak{q})\,u(\mathfrak{q}) + Z(\mathfrak{q})\,u^2(\mathfrak{q}), ;\, 0 \le \mathfrak{q} \le 1,$
will be $O\left(\frac{1}{n^{p-6}}\right)$.

Proof. Lane-Emden equation and its residual in the shifted domain $-1 \le \mathfrak{q} \le 1$ will be:

$$4\mathfrak{q}u'' + 2X\alpha u' + X^2\beta\mathfrak{q}\,u - X^2\gamma U(\mathfrak{q};u) = 0. \tag{49}$$

$$4\mathfrak{q}u_n'' + 2X\alpha u_n' + X^2\beta\mathfrak{q}\,u_n - X^2\gamma U(\mathfrak{q};u_n) \approx 0. \tag{50}$$

So the global error takes the form:

$$\begin{aligned} |e_n| = &|4\mathfrak{q}\,(u''(\mathfrak{q}) - u_n''(\mathfrak{q})) \\ &+ 2X\alpha\,(u'(\mathfrak{q}) - u_n'(\mathfrak{q})) \\ &+ X^2\beta\,\mathfrak{q}\,\,(u(\mathfrak{q}) - u_n(\mathfrak{q})) \\ &-X^2\gamma\,\,(U(\mathfrak{q};u_n) - U(\mathfrak{q};u))|\,. \end{aligned} \tag{51}$$

Since U satisfies Lipschitz condition, $|\mathfrak{q}| \le 1$, and by corollary (1):

$$\begin{aligned} |e_n| \lesssim &4\,O\left(\frac{1}{n^{p-8}}\right) + 2X\,\alpha\,O\left(\frac{1}{n^{p-6}}\right) \\ &+ X^2(\gamma\,\zeta + \beta)\,O\left(\frac{1}{n^{p-4}}\right) \\ &\lesssim O\left(\frac{1}{n^{p-8}}\right). \end{aligned} \tag{52}$$

Similar procedures can prove the global error of the Ricatti equation.

$$2u'(\mathfrak{q}) - P(\mathfrak{q}) - Y(\mathfrak{q})\,u(\mathfrak{q}) - Z(\mathfrak{q})\,u^2(\mathfrak{q}) = 0. \tag{53}$$

$$2u_n'(\mathfrak{q}) - P(\mathfrak{q}) - Y(\mathfrak{q})u_n(\mathfrak{q}) - Z(\mathfrak{q})\,u_n^2(\mathfrak{q}) \approx 0. \tag{54}$$

Then:

$$\begin{aligned} |e_n| = &|2\,\,(u'(\mathfrak{q}) - u_n'(\mathfrak{q})) - Y(\mathfrak{q})\,(u(\mathfrak{q}) - u_n(\mathfrak{q})) \\ &-Z(\mathfrak{q})\,\,(u^2(\mathfrak{q}) - u_n^2(\mathfrak{q}))|\,. \end{aligned} \tag{55}$$

Consequently:

$$\begin{aligned} |e_n| \lesssim &2\,O\left(\frac{1}{n^{p-6}}\right) - \zeta O\left(\frac{1}{n^{p-4}}\right) \\ &- \zeta O\left(\frac{1}{n^{p-4}}\right) \\ &\lesssim O\left(\frac{1}{n^{p-6}}\right). \end{aligned} \tag{56}$$

In the next section, the introduced method SDLP-TM will be applied. Several tables and figures will be presented to prove the accuracy and efficiency of the method compared with the other methods.

6 Numerical Examples

In the present section, we will find approximate solutions for the Lane–Emden and Riccati equations by applying SDLP-TM. Consequently, the SDLP-TM's accuracy, efficiency, and stability have been shown.

*Example 1.*Consider the standard nonlinear Lane-Emden equation [16, 25–28]:

$$u''(\mathfrak{q}) + \frac{2}{\mathfrak{q}}u'(\mathfrak{q}) + u^{\mathfrak{t}}(\mathfrak{q}) = 0,\ 0 \le \mathfrak{t} \le 5, \qquad (57)$$

where initial conditions are $u(0) = 1$ and $u'(0) = 0$. The Lane-Emden equation has physical importance for the value of $\mathfrak{q}$ with $u(\mathfrak{q}) = 0$ and for $\mathfrak{t} = 0, 1$.

The first case at $\mathfrak{t} = 0$, $\mathfrak{q} \in (0, 2.5)$ with the exact solution is $u(\mathfrak{q}) = 1 - \frac{\mathfrak{q}^2}{3!}$. Shifting $\mathfrak{q}$ from (0,2.5) to the domain [-1,1], and applying the explicit form of the Lane-Emden equation using SDLP-TM (42):

$$\frac{32}{3\pi}\sum_{i=0}^{2}\sum_{r=0}^{i-2}\sum_{j_1=1}^{2}\sum_{s_1=0}^{min(t,k)} A_i\Re_{ir}^{(2)}F_{1,j_1}K_{t,k,s_1}Q_{t,k,s_1}$$
$$+\frac{80}{3\pi}\sum_{i=0}^{2}\sum_{r=0}^{i-1}\sum_{s_2=0}^{min(r,k)} A_i\Re_{ir}^{(1)}K_{r,k,s_2}Q_{r,k,s_2}$$
$$+\frac{50}{3\pi}\sum_{i=0}^{2}\sum_{j_2=0}^{2}\sum_{s_3=0}^{min(p,k)} A_iF_{1,j_2}K_{p,k,s_3}Q_{p,k,s_3} = 0.$$

Consequently, we will get the system:

$$\begin{aligned} &3A_0 - 15A_1 + 45A_2 = 1,\\ &15A_1 - 105A_2 = 0,\\ &-240A_1 + 672A_2 = 5. \end{aligned} \qquad (58)$$

Then: $A_0 = \frac{59}{252}$, $A_1 = \frac{-5}{144}$, and $A_2 = \frac{-5}{1008}$. Finally, $u_2(\mathfrak{q}) = \frac{71}{96} - \frac{25}{48}\mathfrak{q} - \frac{25}{96}\mathfrak{q}^2$, which equals the exact solution for $\mathfrak{q} \in (-1, 1)$.

The second case at $\mathfrak{t} = 1$, and the exact solution is $u(\mathfrak{q}) = \frac{sin\mathfrak{q}}{\mathfrak{q}}$. Table (1) shows the point-absolute error (point-AE) compared with the methods in [16, 27, 28].

*Example 2.*Consider the following nonlinear Riccati equation:

$$u'(\mathfrak{q}) - u^2(\mathfrak{q}) = 1\,, \quad 0 \le \mathfrak{q} \le 1,$$

with the initial condition $u(0) = 0$, and the exact smooth solution:

$$u(\mathfrak{q}) = \tan\mathfrak{q}.$$

The explicit form of the Riccati Eq. (39) will be used. Table (2) shows the efficiency of the presented method compared with [29, 30]. While Fig. (1) illustrates the stability of the SDLP-TM.

Table 1: The point-AE of Example 1 for $\mathfrak{t} = 1$.

$\mathfrak{q}$	[27]	[28]	[16]	SDLP-TM	
	$n=16$	$n=10$	$n=12$	$n=10$	$n=12$
0.1	6.2e-13	8.8e-13	3.3e-16	3.3e-16	1.1e-16
0.2	-	2.8e-12	1.1e-16	7.8e-16	1.1e-16
0.3	-	7.9e-13	2.2e-16	1.6e-15	0
0.4	-	5.1e-12	2.2e-16	5.4e-15	0
0.5	5.8e-13	1.0e-11	3.3e-16	1.7e-14	1.1e-16
0.6	-	9.6e-12	2.2e-16	2.5e-14	1.1e-16
0.7	-	3.4e-12	3.3e-16	2.1e-14	0
0.8	-	5.3e-12	3.3e-16	1.4e-14	0
0.9	4.6e-13	1.2e-11	3.3e-16	7.4e-15	1.1e-16

Table 2: The point-AE for Example 2.

$\mathfrak{q}$	[29]	[30]	SDLP-TM	
	$n=14$	$n=15$	$n=15$	$n=25$
0	0	0	1.6e-16	3.5e-17
0.1	6.8e-10	8.2e-10	7.8e-11	4.2e-17
0.2	6.0e-10	1.1e-07	5.2e-12	1.7e-16
0.3	1.3e-09	1.9e-06	5.2e-10	8.3e-16
0.4	1.3e-10	1.6e-05	2.0e-09	2.0e-15
0.5	7.4e-10	8.1e-05	3.7e-09	8.1e-15
0.6	2.7e-09	3.3e-04	3.9e-09	5.6e-15
0.7	2.6e-09	1.1e-03	5.6-09	9.9e-15
0.8	2.6e-09	3.5e-03	9.0e-09	1.3e-14
0.9	8.5e-10	1.1e-03	1.2-08	3.9-14
1.0	1.2e-09	2.8e-03	1.4e-08	1.5e-13

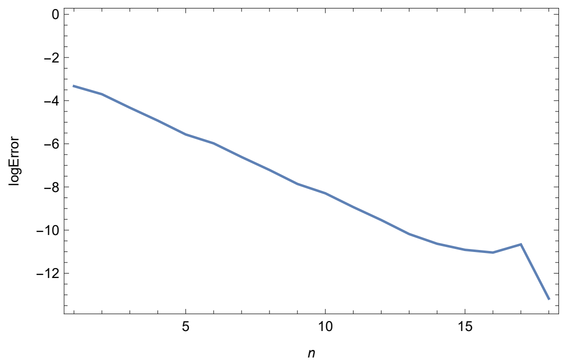

Fig. 1: Log-error for Example 2.

*Example 3.*Consider the following nonlinear Riccati equation [29, 31]:

$$u'(\mathfrak{q}) + u^2(\mathfrak{q}) = 1, \quad 0 \le \mathfrak{q} \le 1, \qquad (59)$$

with the initial condition $u(0) = 0$. The exact solution of that equation is $u(\mathfrak{q}) = \tanh\mathfrak{q}$. The efficiency and accuracy of the proposed method have been shown in Table (3). Fig. (2) presents the stability of the approximate solution.

Table 3: The point-AE Example 3.

q	[31]	[29]	[32]	[33]	SDLP-TM
0.2	3.4e-11	5.0e-11	2.0e-11	1.7e-05	1.4e-16
0.4	2.9e-11	5.3e-11	5.0e-11	1.4e-05	1.7e-16
0.6	2.4e-11	1.0e-11	1.4e-10	1.3e-05	0
0.8	1.8e-11	3.0e-11	3.0e-11	8.9e-06	2.2e-16
1	1.5e-11	1.0e-11	2.0e-11	7.8e-06	1.1e-16

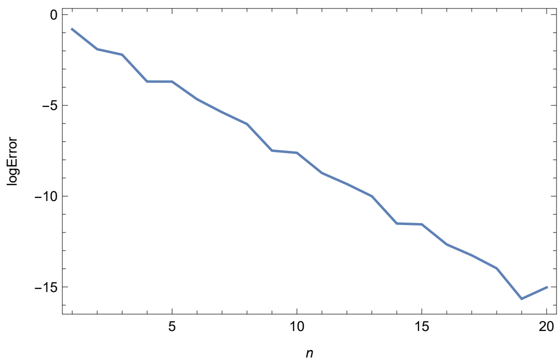

Fig. 2: Log-error for Example 3.

*Example 4.*Consider the following nonlinear Riccati equation [34]:

$$u'(\mathfrak{q}) + (u(\mathfrak{q}))^2 - 2\,u(\mathfrak{q}) = 1\,, \quad u(0) = 0 \quad 0 \le \mathfrak{q} \le 1, \tag{60}$$

with the exact solution: $u(\mathfrak{q}) = 1 + \sqrt{2}\tanh(\frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}) + \sqrt{2}\mathfrak{q})$. Table (4) shows the efficiency of the presented method compared with other methods.

7 Conclusions

This study presented a highly effective strategy that employs the second derivative of Legendre (SDLPs) as basis functions. Those novel basis functions have been used via the Tau method (SDLP-TM) to approximate solutions for linear and nonlinear ordinary differential equations. A linearization relation for the SDLPs has been investigated and proved. Additionally, some important integrations have been introduced. The established relations and integration have been used to utilize the Tau method's integration. These procedures are used to construct explicit algebraic systems for the Land-Emden and Riccati equations. Then, an algorithm for the presented method is created. Also, the converge and error analysis of the proposed technique are well covered, and the global upper bounds of the errors for the discussed problems have been determined. Finally, the SDLP-TM's accuracy, efficiency, and stability have been demonstrated numerically.

Acknowledgment

The authors would like to thank the anonymous reviewers for carefully reading the article and for their constructive and valuable comments, which have improved the paper's present form. The authors also sincerely thank the Helwan School of Numerical Analysis in Egypt (HSNAE) members for their valuable effort and support.

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Table 4: The point-AE for Example (4).

q	SDLP-TM			[34]	[35]		[16]
	$n = 10$	$n = 14$	$n = 22$	$n = 10$	$n = 10$	$n = 14$	$n = 22$
0	2.2e-16	2.2e-16	0	-	0	0	8.0e-17
0.1	3.8e-09	1.8e-11	5.6e-16	4.6e-09	1.0e-06	8.5e-08	9.3e-13
0.2	7.0e-09	1.6e-11	2.2e-16	9.7e-10	1.1e-06	8.8e-08	1.1e-12
0.3	1.8e-09	2.0e-10	8.9e-16	3.7e-09	1.2e-06	1.0e-07	1.3e-12
0.4	7.6e-08	4.9e-10	1.8e-15	1.3e-09	1.4e-06	1.1e-07	1.4e-12
0.5	2.1e-07	1.7e-09	2.3e-14	1.9e-09	1.5e-06	1.2e-07	1.5e-12
0.6	6.7e-07	5.4e-10	1.7e-14	2.7e-09	1.6e-06	1.3e-07	1.5e-12
0.7	6.4e-07	1.9e-11	1.3e-14	4.3e-09	1.5e-06	1.2e-07	1.5e-12
0.8	6.4e-07	4.0e-10	1.1e-14	2.4e-09	1.5e-06	1.2e-07	1.5e-12
0.9	5.8e-07	4.6e-10	1.3e-14	3.6e-10	1.4e-06	1.2e-07	1.3e-12
1	3.9e-09	4.9e-10	1.9e-14	7.0e-09	6.6e-07	4.3e-08	1.6e-12

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