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Fuzzy Time Series Inference for Stationary Linear Processes: Features and Algorithms With Simulation

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Abstract: The primary objective of this article is to estimate the unknown parameters of stationary linear processes based on a fuzzy time series approach to observations that follow AR (1) processes. Predicted observations are obtained using fuzzy time series. Both actual and forecasted observations are utilized to study various classic method's estimators for the autoregressive parameter. The comparisons between actual and forecasted observations in all estimating processes are discussed based on the mean squared errors. Furthermore, to investigate the extent to which fuzzy time series can enhance estimates produced by traditional estimating techniques. Based on these comparisons, it is possible to explore how fuzzy time series contribute to the improvement of classical methods' estimations.

Keywords: Time series; AR(1) model; Simulation; Fuzzy inference; Forecasting; Statistics and numerical data.

1 Introduction

Since Zadeh proposed the fuzzy set/group in 1965, significant progress has been noted in both theory and applications. Fuzzy sets are the sets whose elements have degree of membership. Let R be the universe of discourse, $R = r_1, r_2, ..., r_n$ and let D be a fuzzy set in the universe of discourse R defined as follows:

$$D = f_D(r_1)/(r_1)+f_D(r_2)/(r_2)+...+f_D(r_n)/(r_n), \quad (1)$$

where f_D is the membership function of D, $f_D : R \rightarrow [0,1], f_D(r_i)$ indicates the grade of membership of r_i in the fuzzy set D, $f_D(r_i) \in [0,1]$, and $1 \leq i \leq n$,. If there exists a fuzzy relationship $K(t-1,t)$, such that $F(t) = F(t-1) \times K(t-1,t)$, where $\times$ represents an operator, then $F(t)$ is said to be caused by $F(t-1)$. The relationship between $F(t)$ and $F(t-1)$ is denoted by $F(t-1) \rightarrow F(t)$. If $F(t-1) = D_i$ and $F(t) = D_j$, the logical relationship between $F(t)$ and $F(t-1)$ is denoted by $D_i \rightarrow D_j$, where D_i is called the left hand side and D_j the right hand side of the fuzzy relation. One such application in the field of time series entitled fuzzy time series (FTS). Let $Y_t(t = ..., 0, 1, 2, ...)$, a subset of real numbers, be the universe of discourse on which fuzzy sets $f_i(t)(i = 1, 2, ...)$ are defined. If $F(t)$ is a collection of $f_i(t)(i = 1, 2, ...)$, then $F(t)$ is called a FTS on $Y_t(t = ..., 0, 1, 2, ...)$. Based on FTS, statisticians can predict time series values that contain linguistic features, which is not the case in the case of classical time series. The FTS technique based on fuzzy set theory was first put forth by [1], [2] and [3]. [4] and [5] used fuzzy group relation in determination of fuzzy relations stage. For modelling time series with a trend component, [6] developed a novel FTS modelling approach. [7] and [8] presented a novel method that optimize the length of

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intervals in FTS. [9] suggested using FTS to estimate enrollments with a greater forecast accuracy rate. [10] suggested a new FTS forecasting technique. Using fuzzy logical relationships and similarity metrics, [11] proposed a strategy for forecasting FTS data. [12] put out a brand-new forecasting technique built on FTS, which can manage point, interval, and distribution forecasts while utilizing stochastic and fuzzy patterns in the data. The combined resilient FTS technique for time series forecasting was shown in [13]. [14] advised conducting research on the most effective machine learning methods for high order forecasting. [15] unveiled a brand-new long-short-term memory-based deep intuitionistic fuzzy time series forecasting technique. [16] proposed using fuzzy time series algorithms to forecast (IN) direct short-term solar electricity. [17] explored the effects of fuzzy approaches on spectral analysis estimators. [18] investigated how FTS affected the INAR(1) process' smoothing estimates.

This paper's primary goal is to apply FTS as a novel way to enhance the conventional methods' estimations of the unknown parameters of stationary processes. This procedure is done on stationary AR(1) model. So, the method suggested in [19] is utilized to transform the time series that follow AR(1) model into FTS whose observations are fuzzy observations. The fuzzy observations are converted using Chen method also to numerical observations and called forecasted observations. These two categories of observations are substituted in the classic method's estimators "maximum likelihood (MLE), moments, and ordinary least squares (OLS)" for the autoregressive parameter. The comparisons between the actual and forecasted observations in all estimating processes based on mean squared error (MSE). In all of the aforementioned estimating approaches, the MSE is used to compare the estimations arising from the actual observations with the forecasted observations.

The structure of this paper is as follows: In Section 2, the fundamentals of fuzziness and FTS are presented. The standard techniques, such as MLE, OLS, and moments, are used in Section 3 to estimate the unknown parameter of the AR(1) model. In Section 4, a brief illustration of how to transform the AR(1) series into a FTS is provided. Further, in this section, the simulation experiments using a range of sample sizes and various values for the AR(1) parameter as the default parameter values is constructed. The paper's conclusion is provided in Section 5.

2 AR(1) Parameter: Different Estimation techniques

Let Y_t is a stationary AR(1) process where

$$Y_t = \phi_1 Y_{t-1} + a_t, \tag{2}$$

where Y_t observation at time t, and $|\phi_1| < 1$ autoregressive parameter and a_t stochastic error (white noise) follows a normal distribution which $a_t \sim N(0, \sigma_a^2))$, then $Y_t \sim N(0, \frac{\sigma_a^2}{1-\phi_1^2})$. The autocovariance function of Y_t can be formulated as

$$\gamma_k = \phi_1 \gamma_{k-1}, k \geq 1. \tag{3}$$

The autocorrelation function (ACF) can listed as

$$\rho_k = \frac{\gamma_k}{\gamma_0} = {\phi_1}^k, \tag{4}$$

where $\gamma_0 = Var(Y_t)$. The partial autocorrelation function (PACF) can be expressed as

$$\begin{cases} \Phi_{11} = \rho_1 = \phi_1, \\ \Phi_{kk} = 0, k > 1. \end{cases} \tag{5}$$

Now, based on a sample size n, various estimation approaches on stationary AR(1) can be derived and discussed.

2.1 MLE technique: Theory and implementation

The most important step to study the MLE method is to evaluate the sample joint distribution which are also called the likelihood function. Because the dependence between all observations $Y_2, Y_3, ..., Y_n$ and the first observation Y_1, the likelihood function cannot written as multiplication of marginal probability density function (PDF), but make transformation as following

$$\begin{aligned} L &= f(Y_1, Y_2, ..., Y_n) \\ &= f(Y_2, Y_3, ..., Y_n|Y_1) f(Y_1) = f(Y_1) f(a_2, a_3, ..., a_n)|J|, \end{aligned} \tag{6}$$

where

$$|J| = \begin{vmatrix} \frac{\partial a_2}{\partial Y_2} & \frac{\partial a_2}{\partial Y_3} & \cdots & \frac{\partial a_2}{\partial Y_n} \\ \frac{\partial a_3}{\partial Y_2} & \frac{\partial a_3}{\partial Y_3} & \cdots & \frac{\partial a_3}{\partial Y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial Y_2} & \frac{\partial a_n}{\partial Y_3} & \cdots & \frac{\partial a_n}{\partial Y_n}. \end{vmatrix}.$$

Since $Y_t \sim N(0, \frac{\sigma_a^2}{1-\phi_1^2})$ then,

$$f(Y_1) = \frac{1}{\sqrt{\frac{2\pi\sigma_a^2}{1-\phi_1^2}}} \exp\left[\frac{-(1-\phi_1^2)Y_1^2}{2\sigma_a^2}\right]. \tag{7}$$

Since $a_t \sim N(0, \sigma_a^2))$, then the joint PDF for stochastic error can be formulated as

$$\begin{aligned} f(a_1, a_2, \ldots, a_n) &= \prod_{t=1}^{n} f(a_t) = (2\pi\sigma_a^2)^{\frac{-n}{2}} \exp\left[\frac{-\sum_{t=1}^{n} a_t^2}{2\sigma_a^2}\right] \\ &= (2\pi\sigma_a^2)^{\frac{-(n-1)}{2}} \exp\left[\frac{-\sum_{t=2}^{n} a_t^2}{2\sigma_a^2}\right]. \end{aligned} \tag{8}$$

Since

$$\begin{aligned}
Y_1 &= \phi_1 Y_0 + a_1 \rightarrow \quad \text{Let}' \text{ take } Y_0 = 0,\\
Y_2 &= \phi_1 Y_1 + a_2 \rightarrow \quad a_2 = Y_2 - \phi_1 Y_1\\
Y_3 &= \phi_1 Y_2 + a_3 \rightarrow \quad a_3 = Y_3 - \phi_1 Y_2\\
&\vdots\\
Y_n &= \phi_1 Y_{n-1} + a_n \rightarrow \quad a_n = Y_n - \phi_1 Y_{n-1},
\end{aligned}$$

the Jacobian can be listed as

$$|J| = \begin{vmatrix} \frac{\partial a_2}{\partial Y_2} & \frac{\partial a_2}{\partial Y_3} & \cdots & \frac{\partial a_2}{\partial Y_n} \\ \frac{\partial a_3}{\partial Y_2} & \frac{\partial a_3}{\partial Y_3} & \cdots & \frac{\partial a_3}{\partial Y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial Y_2} & \frac{\partial a_n}{\partial Y_3} & \cdots & \frac{\partial a_n}{\partial Y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 \ldots 0 \\ -\phi_1 & 1 \ldots 0 \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \ldots 1 \end{vmatrix} = 1,$$

substituting by (7), (8), and Jacobian in (6), the MLE function for series $\{Y_t\}$ can presented as

$$L = (2\pi\sigma_a^2)^{\frac{-n}{2}}(1-\phi_1^2)^{\frac{1}{2}} \exp\left[\frac{-[(1-\phi_1^2)Y_1^2 + \sum_{t=n}^{n} a_t^2]}{2\sigma_a^2}\right]. \tag{9}$$

By taking the logarithm of (9), and the differential with respect to ϕ_1, then

$$\frac{-\hat{\phi}_1}{1-\hat{\phi}_1^2}\sigma_a^2 + \sum_{t=2}^{n} Y_t Y_{t-1} - \hat{\phi}_1 \sum_{t=3}^{n} Y_{t-1}^2 = 0. \tag{10}$$

After simplistic, the MLE estimator for ϕ_1 can be listed as

$$\hat{\phi}_1 = \frac{(n-2)\sum_{t=2}^{n} Y_t Y_{t-1}}{(n-1)\sum_{t=3}^{n} Y_{t-1}^2}. \tag{11}$$

In this part, a technique for Chen [19] which turning regular time series into fuzzy time series and providing forecasting observations for the subsequent series is provided. The six steps that make up this approach are as follows: Define the discourse universe (a collection of observations made using the model chosen for this paper) and divide it into equally long intervals; determine how many observations are contained in each interval, which will result in a redistribution for each interval based on the number of observations in each interval; Depending on the newly separated intervals, define the linguistic values that the fuzzy set represents; the real observations be fuzzed; We next employ a series of rules to assess if the trend of the forecasting is upward or downward, which means we defuzzify the fuzzy output into forecasted output. discover and build fuzzy logical linkages based on the fuzzified observations.

To properly describe this strategy, which is both lengthy and challenging, see [19] for additional information. In the first phase of this procedure, Chen divided 20 observations into seven intervals of equal length. Here, we used the number of intervals $K = 1 + 3.322 \, log \, n$, where n is the actual number of observations produced by the model discussed in this study, and this produced better results.

2.2 Moments approach: Theory and implementation

This method, also referred to as Yule-Walker estimating, produces accurate estimators for AR models but less efficient ones for MA or ARMA processes. This method's fundamental idea is to solve a series of equations that give the necessary estimators by equating the population moments with the sample moments. For instance, the sample counterpart of the first population moment, $\mu = E(Y_t)$, is $m_1 = \bar{Y}_t$. This yields $\hat{\mu} = \bar{Y}_t$ right away. To estimate the autoregressive parameters of a model AR(1) in (11). It is known from (4) that $\rho_1 = \phi_1$. Now, take $\rho_1 = (\hat{\rho_1})$, where $(\hat{\rho_1})$ is the sample ACF (see [20]). Then, $\hat{\phi_1}$ can be reported as

$$\hat{\phi_1} = \hat{\rho_1} = \frac{n}{n-1}\frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=1}^{n} Y_t^2}. \tag{12}$$

2.3 OLS approach: Theory and implementation

This method relies on minimizing the sum of the square of errors for the observed values with regard to ϕ_1. From (11), the sum of the square of random errors $S(\phi_1)$ can be formulated as

$$S(\phi_1) = \sum_{t=2}^{n} a_t^2 = \sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})^2. \tag{13}$$

For the purpose of minimizing the error, the first derivative of the $S(\phi_1)$ is derive to get

$$\frac{\partial S(\phi_1)}{\partial \phi_1} = 2\sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})(-Y_{t-1}) = 0, \tag{14}$$

$$\phi_1 \sum_{t=2}^{n} Y_{t-1}^2 = \sum_{t=2}^{n} Y_t Y_{t-1}, \tag{15}$$

hence, the OLS estimator for ϕ_1 can be listed as

$$\hat{\phi_1} = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2}. \tag{16}$$

3 Practical Side: Analyzing and discussing

In order to demonstrate how Chen's method which is proposed in this work is applied on AR(1) process, an example is discussed based on 50 observations only.

3.1 Numerical implementation under algorithm diagram

Let us take for example the default values as $n = 50$, $\phi_1 = 0.4$, $\sigma_a = 2$ and $Y_0 = 0.3$. Now, generate the random error where $a_t \sim N(0, \sigma_a^2)$, and therefore generate the observations of AR(1) such that $Y_t = \phi_1 Y_{t-1} + a_t$, the generated observations can be listed in Table 4.

The actual observations will now be converted to fuzzy observations using Chen's method, and forecasted observations will then be obtained. Chen's approach consists of six steps as following:

1. Define the discourse universe and divide it into equally long intervals. Take the minimum and the maximum value from the actual observations and symbolized them by symbols X_{min} and X_{max}, respectively. After that, the universe of discourse R id defined as $R = [X_{min}, X_{max}]$. From Table 4, $X_{min} = -13.9850$ and $X_{max} = 16.7953$ then $R = [-13.9850, 16.7953]$. Chen take number of intervals is seven, but length R is $L = 16.7953 - (-13.9850) = 30.7803$, then length of each interval is $l = 30.7803/7 = 4.3972$, that is mean dividing R into seven evenly lengthy. The results can be listed in Table 1.

2. Get the observations number in each interval. In Table 2 , a summary of the distributions of the observations across various intervals is presented.

Divide the interval with the most observations into four equal-length sub-intervals. Take the interval with the second-highest number of observations and divide it into three equal-sized sub-intervals. Take the period with the third-highest number of observations and divide it into two intervals of equal length. The interval with the fourth largest number of observations is chosen, the interval length is left unchanged, and the intervals with no observations are discarded as shown in Table 3

3. Define each fuzzy set D_i based on the re-divided intervals and fuzzify the observations which in Table 4, where fuzzy set D_i denotes a linguistic value of the observations represented by a fuzzy set, and $1 \leq i \leq 16$. To express the fuzzy sets, expressions or vocabularies are used, for example, D_1 =very very very very small, D_2 =very very very very small, D_3 = very very very small, D_4 = very very small, D_5 = very small, D_6 = small, D_7 = moderate, D_8 = after moderate, D_9 = , D_{10} = much much, D_{11} = much much much, D_{12} = too much, D_{13} = too much much, D_{14} = too much much much, D_{15} = too much much much much, D_{16} = too much much much much much, defined as follows

$$
\begin{aligned}
D_1 =& 1/r_1 + 0.5/r_{2,1} + 0/r_{2,2} + ... + 0/r_{7,2},\\
D_2 =& 0.5/r_1 + 1/r_{2,1} + 0.5/r_{2,2} + 0/r_{3,1} + ... + 0/r_{7,2},\\
D_3 =& 0/r_1 + 0.5/r_{2,1} + 1/r_{2,2} + 0.5/r_{3,1} + 0/r_{3,2} + ...\\
&+ 0/r_{7,2},\\
D_4 =& 0/r_1 + 0/r_{2,1} + 0.5/r_{2,2} + 1/r_{3,1} + 0.5/r_{3,2} + 0/r_{3,3}\\
&+ ... + 0/r_{7,2},\\
D_5 =& 0/r_1 + 0/r_{2,1} + 0/r_{2,2} + 0.5/r_{3,1} + 1/r_{3,2} + 0.5/r_{3,3}\\
&+ 0/r_{4,1} + ... + 0/r_{7,2},\\
D_6 =& 0/r_1 + ... + 0/r_{3,1} + 0.5/r_{3,2} + 1/r_{3,3} + 0.5/r_{4,1}\\
&+ 0/r_{4,2} + 0/r_{4,3} + ... + 0/r_{7,2},\\
D_7 =& 0/r_1 + ... + 0/r_{3,2} + 0.5/r_{3,3} + 1/r_{4,1} + 0.5/r_{4,2}\\
&+ 0/r_{4,3} + ... + 0/r_{7,2},\\
D_8 =& 0/r_1 + ... + 0/r_{3,3} + 0.5/r_{4,1} + 1/r_{4,2} + 0.5/r_{4,3}\\
&+ 0/r_{4,4} + ... + 0/r_{7,2},\\
D_9 =& 0/r_1 + ... + 0/r_{4,1} + 0.5/r_{4,2} + 1/r_{4,3} + 0.5/r_{4,4}\\
&+ 0/r_{5,1} + ... + 0/r_{7,2},\\
D_{10} =& 0/r_1 + ... + 0.5/r_{4,3} + 1/r_{4,4} + 0.5/r_{5,1} + 0/r_{5,2} + ...\\
&+ 0/r_{7,2},\\
D_{11} =& 0/r_1 + ... + 0.5/r_{4,4} + 1/r_{5,1} + 0.5/r_{5,2} + 0/r_{5,3} + ...\\
&+ 0/r_{7,2},\\
D_{12} =& 0/r_1 + ... + 0/r_{4,4} + 0.5/r_{5,1} + 1/r_{5,2} + 0.5/r_{5,3}\\
&+ 0/r_6 + 0/r_{7,1} + 0/r_{7,2},\\
D_{13} =& 0/r_1 + ... + 0/r_{5,1} + 0.5/r_{5,2} + 1/r_{5,3} + 0.5/r_6\\
&+ 0/r_{7,1} + 0/r_{7,2},\\
D_{14} =& 0/r_1 + ... + 0/r_{5,2} + 0.5/r_{5,3} + 1/r_6 + 0.5/r_{7,1}\\
&+ 0/r_{7,2},\\
D_{15} =& 0/r_1 + ... + 0/r_{5,3} + 0.5/r_6 + 1/r_{7,1} + 0.5/r_{7,2},\\
D_{16} =& 0/r_1 + ... + 0/r_6 + 0.5/r_{7,1} + 1/r_{7,2},
\end{aligned}
$$

In order to keep things simple, the membership values of fuzzy set D_i are either $0, 0.5$ or 1 where $1 \leq i \leq 16$ are assumed. The results can be listed in Table 4.

4. Fuzzify the actual observations. Finding connections between the actual observations and the fuzzy sets is the process of "fuzzification". Every observation that is actualize fuzzified based on its highest membership. If the fuzzy set D_k has the maximum degree of belongingness for the created observations time variable, $F(t-1)$, then $F(t-1)$ is fuzzified as D_k. Utilizing this principle. The results can be listed in Table 5.

5. Identify and establish fuzzy logical relationships. If the actual observation $F(t-1)$ is fuzzified as D_k and $F(t)$ as D_m , then D_k is related to D_m . This relationship as $D_k \rightarrow D_m$ is denoted, where D_k the current state of observation and D_m the next state of observation. For example, in Table 5, the observation 1 is fuzzified as D_9 and the observation 2 is fuzzified as D_{14}, which provides the following relationship $D_9 \rightarrow D_{14}$. All fuzzy logical relationships can be obtained by applying this reasoning to all observations as shown in Table 5. For the same relationships which may appear more than once, this repetition is ignored.

6. Defuzzify the fuzzy output into forecasted output. Divide each interval obtained in step two into four

equal-length sub intervals, and use the $0.25-$point and $0.75-$point of each interval as the forecasting's downward and upward pivot points. By applying the following rules, it may be established whether the trend of the forecasting is upward or downward when predicting the fuzzified observations. Assume that the fuzzy logical relationship is $D_i \rightarrow D_j$ where D_i denotes the fuzzified observation number $n-1$ and D_j denotes the fuzzified observation number n, then

I.If $j > i$ and $((Y_{n-1} - Y_{n-2}) - (Y_{n-2} - Y_{n-3}))$ are
 (a)Positive, then the trend of the forecasting will go up, and to forecast the fuzzified observations rule two is utilized.
 (b)Negative, then the trend of the forecasting will go down, and to forecast the fuzzified observations rule three is applied.

II.If $j < i$ and $((Y_{n-1} - Y_{n-2}) - (Y_{n-2} - Y_{n-3}))$ are
 (a)Positive, then the trend of the forecasting will go up, and to forecast the fuzzified observations rule two is applied.
 (b)Negative, then the trend of the forecasting will go down, and to forecast the fuzzified observations rule three is used.

III.If $j = i$ and $((Y_{n-1} - Y_{n-2}) - (Y_{n-2} - Y_{n-3}))$ are
 (a)Positive, then the trend of the forecasting will go up, and to forecast the fuzzified observations rule two is utilized.
 (b)Negative, then the trend of the forecasting will go down, and to forecast the fuzzified observations rule three is used.

As for rule No. 1, apply with the first, second and third observations. For rules No. 1, 2, and 3, see [19]. Using these rules, the defuzzifcation of fuzzy observations to forecasted observations can be listed in 7 where this table lists the AR(1) process-actual observations as well as the forecasted observations that were predicted using the FTS method.

Figure 1 illustrates the simulated series of $\{Y_t, t = 1, 2, ..., 300\}$ from AR(1) model at $n = 300, \phi_1 = 0.4, \sigma_a = 2$ and $Y_0 = 0.3$. Figure 2 illustrate the simulated series of the forecasted AR(1)'s observations based on FTS. From Figures 1 and 2, we can conclude that the process is stationary, so the plot satisfy the definition of the AR(1) model. Also, it can be concluded that the FTS does not affect the behavior of the process, but maintains its properties and behavior.

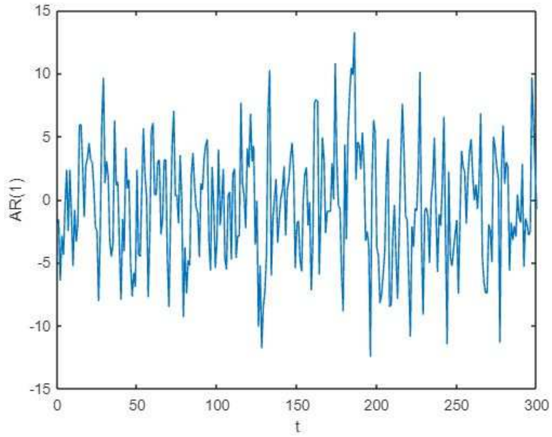

Fig. 1: Simulated series of AR(1) model at $n = 300$, $\phi_1 = 0.4$, $\sigma_a = 2$ and $Y_0 = 0.3$.

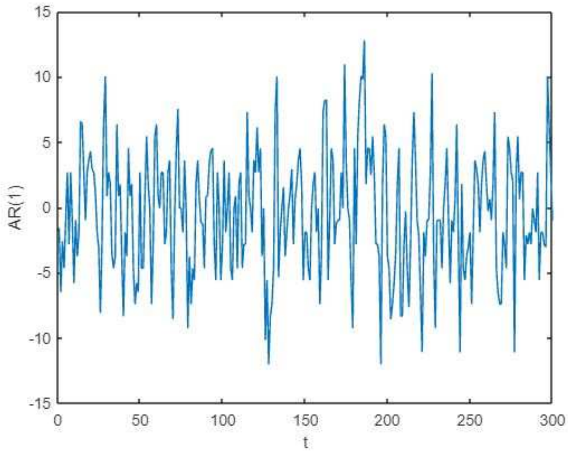

Fig. 2: Simulated series of forecasted AR(1)'s observations based on fuzzy time series.

3.2 Stages of constructing the simulation

Seven stages make up the simulation experiments for this part which was coded in the program (see Appendix).

1.Stage of selecting the sample size n and the default value for the AR(1) parameter as the default parameter values.
2.Stage of generate the observations: Generate the observations by generating the random errors according to normal distribution and therefore the observations $(Y_1, Y_2, ..., Y_n)$ can be obtained.
3.Take the actual observations AO in stage II and transform them to fuzzy observations and then to forecasted observations FO as in the previous example.
4.Both types of observations can be substituted in the estimator produced from all the estimation methods.
5.Repeat this experiment 1000 once, and thus produces 1000 value for $\hat{\phi}_1$ in case of actual observations and forecasted observations for each one of the different estimation methods.
6.Using MSEs to compare between the $\hat{\phi}_1$'s which mentioned in the previous step.

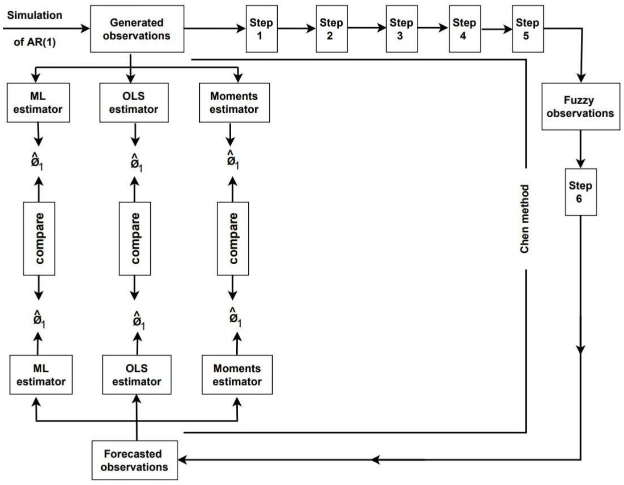

Fig. 3: Diagram for Chen method and some of stages of constructing the simulation.

7.Repeat this comparison numerous times using various sample sizes and various default values.

Figure 3 is a diagram of the mechanism for converting the actual observations from the AR(1) model into fuzzy observations and then into forecasted observations using Chen method, and also depicts the comparison between the estimators of the three classical methods in both cases.

3.3 Performance of estimation techniques

In this segment, the MSE of the MLE, moments, and OLS approaches is reported for the AR(1) parameter based on actual and forecasted observations. According to Table 8, some significant results can be lised as follows: Whenever the sample size increases, the MSE decrease; the MSE for (ϕ_1) results in case of forecasted observations are less than the MSE for (ϕ_1) results in case of actual observations along the table, which demonstrates that the three estimate methods' estimator values were enhanced using the FTS technique; and when looking at the results of the three estimation methods, can be noted:

–The MSEs of MLE are less than the MSEs of OLS, while the MSEs of OLS are less than the MSEs of Moments, which gives the preference to the MLE, OLS and Moments respectively.
–When the sample size increases, the MSEs for the three estimation methods may be coincides.

4 Conclusions

In light of the effort to identify the best parameter estimators, a novel method for doing so, called fuzzy logic is proposed. In this study, fuzzy time series (FTS) were employed to improve estimate of the stationary AR(1) process's unknown parameter using traditional estimation methods "MLE, OLS and moments". The estimators of these techniques were studied and discussed for both the actual observations generated by the model and the observations estimated by the FTS. Utilizing the mean squared error, these comparisons were made. The results showed a preference in the estimated observations over the actual observations in all three estimation approaches. This shows that employing FTS as opposed to conventional approaches can improve the estimation of unknown parameters in stationary time series models.

Table 1: Equally lengthy intervals.

Interval No.	Interval range
r_1	[-13.985,-9.5878]
r_2	[-9.5878,-5.1906]
r_3	[-5.1906,-0.7934]
r_4	[-0.7934,3.6038]
r_5	[3.6038,8.0010]
r_6	[8.0010,12.398]
r_7	[12.398,16.795]

Table 2: The distribution of the observations in intervals.

Interval No.	Interval range	No. of obs in each interval
r_1	[-13.985,-9.5878]	1
r_2	[-9.5878,-5.1906]	3
r_3	[-5.1906,-0.7934]	12
r_4	[-0.7934,3.6038]	17
r_5	[3.6038,8.0010]	12
r_6	[8.0010,12.3981]	2
r_7	[12.398,16.795]	3

Table 3: Redivided intervals based on number of observations.

Interval No.	Interval range
r_1	[-13.98,-9.587]
$r_{2,1}$	[-9.587,-7.382]
$r_{2,2}$	[-7.382,-5.190]
$r_{3,1}$	[-5.190,-3.721]
$r_{3,2}$	[-3.721,-2.254]
$r_{3,3}$	[-2.254,-0.793]
$r_{4,1}$	[-0.793,0.305]
$r_{4,2}$	[0.305,1.405]
$r_{4,3}$	[1.405,2.504]
$r_{4,4}$	[2.504,3.603]
$r_{5,1}$	[3.603,5.061]
$r_{5,2}$	[5.061,6.5321]
$r_{5,3}$	[6.5321,8.001]
r_6	[8.001,12.39]
$r_{7,1}$	[12.39,14.59]
$r_{7,2}$	[14.59,16.79]

Table 4: The actual observations.

Observation No.	Observation value
Y_1	2.2707
Y_2	8.2438
Y_3	-5.7379
Y_4	1.1535
Y_5	1.7365
Y_6	-4.5362
Y_7	-3.5488
Y_8	-0.049
Y_9	14.294
Y_{10}	16.7953
Y_{11}	1.3186
Y_{12}	12.6671
Y_{13}	7.9685
Y_{14}	2.9352
Y_{15}	4.033
Y_{16}	0.7934
Y_{17}	-0.1792
Y_{18}	5.8871
Y_{19}	7.991
Y_{20}	8.8652
Y_{21}	6.2321
Y_{22}	-2.3371
Y_{23}	1.9341
Y_{24}	7.2946
Y_{25}	4.8734
Y_{26}	6.0881
Y_{27}	5.3428
Y_{28}	0.9234
Y_{29}	1.5448
Y_{30}	-2.5312
Y_{31}	2.5411
Y_{32}	-3.5718
Y_{33}	-5.7042
Y_{34}	-5.5197
Y_{35}	-13.985
Y_{36}	0.1595
Y_{37}	1.3646
Y_{38}	-2.4739
Y_{39}	4.4916
Y_{40}	-5.0494
Y_{41}	-2.4287
Y_{42}	-1.9373
Y_{43}	0.5019
Y_{44}	1.4522
Y_{45}	-2.8786
Y_{46}	-1.2717
Y_{47}	-1.1682
Y_{48}	2.0436
Y_{49}	5.1905
Y_{50}	6.5133

Appendix

The next pseudo code consists of five parts
First part: Choosing default values: n, ϕ_1, σ_a and Y_0.
Second part :Generating the random error a = normrnd$(0, \sigma_a^2, n, 1)$ and therefore we can get the actual observations $(Y_1, Y_2, \ldots, Y_n)$.

Table 5: Fuzzy logical relationships.

$D_9 \to D_{14}$	$D_{14} \to D_3$	$D_3 \to D_8$	$D_8 \to D_9$
$D_9 \to D_4$	$D_4 \to D_5$	$D_5 \to D_7$	$D_7 \to D_{15}$
$D_{15} \to D_{16}$	$D_{16} \to D_8$	$D_8 \to D_{15}$	$D_{15} \to D_{13}$
$D_{13} \to D_{10}$	$D_{10} \to D_{11}$	$D_{11} \to D_8$	$D_8 \to D_7$
$D_7 \to D_{12}$	$D_{12} \to D_{13}$	$D_{13} \to D_{14}$	$D_{14} \to D_{12}$
$D_{12} \to D_5$	$D_5 \to D_8$	$D_8 \to D_{13}$	$D_{13} \to D_{11}$
$D_{11} \to D_{12}$	$D_{12} \to D_{12}$	$D_{12} \to D_8$	$D_9 \to D_5$
$D_5 \to D_{10}$	$D_{10} \to D_5$	$D_5 \to D_3$	$D_3 \to D_3$
$D_3 \to D_1$	$D_1 \to D_7$	$D_7 \to D_8$	$D_8 \to D_5$
$D_5 \to D_{11}$	$D_{11} \to D_4$	$D_5 \to D_6$	$D_6 \to D_8$
$D_6 \to D_6$	$D_6 \to D_9$	$D_9 \to D_{12}$	

Third part: Fuzzification the observations by the method mentioned in this paper.

1.Find the minimum of values $Y_1, Y_2, \ldots, Y_n$ and the maximum of values $Y_1, Y_2, \ldots, Y_n$ and take number of intervals $k = 7$.
2.Find the length of each interval L=(max(Y)-min(Y))/7 and get all intervals from 1 to 7.
for i from 1 to n and for j from 1 to k
if Y(i) belong to interval(j)
then A(j,1,i)=Y(i);
3.Determine how much an element within each interval by using the following function
for i from 1 to n
le(i)=length(nonzeros(A(i,:,:)));
4.Sort the intervals r=[sort(interval(:,1)) sort(out(:,2))];
5.Find the middle value and downward, upward and length for the new intervals r which need to them when applying the rules.

Fourth part:
for i from 1 to length(r)
if Y(r) belong to r(i)
for j from 1 to length(r)
if Y(r+1) belong to r(j)
if (j>i)
if (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))>0)→ apply Rule 2 ;
elseif (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))<0 → apply Rule 3;
elseif (j<i)
if (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))>0)→ apply Rule 2;
elseif (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))<0)→ apply Rule 3;
else if (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))>0)→apply Rule 2;
elseif (((Y(r)-Y(r-1))-(Y(r-1)-Y(r-2)))<0)→ apply Rule 3.
apply Rule 1 with the first and the second and the third observations.
Substitute by both the actual observations (which in second part) and the forecasted observations separately in the method's estimators and compare between the results.
Fifth part: Showing results.

Table 6: Actual and fuzzified observations.

Observation No.	Actual observations	Interval	Fuzzified observations
Y_1	2.2707	[1.40516,2.50446]	D_9
Y_2	8.2438	[8.0010,12.3981]	D_{14}
Y_3	-5.7379	[-7.3892,-5.1906]	D_3
Y_4	1.1535	[0.30586,1.40516]	D_8
Y_5	1.7365	[1.40516,2.50446]	D_9
Y_6	-4.5362	[-5.1906,-3.7248]	D_4
Y_7	-3.5488	[-3.7248,-2.2591]	D_5
Y_8	-0.049	[-0.7934,0.30586]	D_7
Y_9	14.294	[12.3981,14.5967]	D_{15}
Y_{10}	16.7953	[14.5967,16.7953]	D_{16}
Y_{11}	1.3186	[0.30586,1.40516]	D_8
Y_{12}	12.6671	[12.3981,14.5967]	D_{15}
Y_{13}	7.9685	[6.5352,8.0010]	D_{13}
Y_{14}	2.9352	[2.50446,3.6038]	D_{10}
Y_{15}	4.033	[3.6038,5.0694]	D_{11}
Y_{16}	0.7934	[0.30586,1.40516]	D_8
Y_{17}	-0.1792	[-0.7934,0.30586]	D_7
Y_{18}	5.8871	[5.0694,6.5352]	D_{12}
Y_{19}	7.991	[6.5352,8.0010]	D_{13}
Y_{20}	8.8652	[8.0010,12.3981]	D_{14}
Y_{21}	6.2321	[5.0694,6.5352]	D_{12}
Y_{22}	-2.3371	[-3.7248,-2.2591]	D_5
Y_{23}	1.9341	[0.30586,1.40516]	D_8
Y_{24}	7.2946	[6.5352,8.0010]	D_{13}
Y_{25}	4.8734	[3.6038,5.0694]	D_{11}
Y_{26}	6.0881	[5.0694,6.5352]	D_{12}
Y_{27}	5.3428	[5.0694,6.5352]	D_{12}
Y_{28}	0.9234	[0.30586,1.40516]	D_8
Y_{29}	1.5448	[1.40516,2.50446]	D_9
Y_{30}	-2.5312	[-3.7248,-2.2591]	D_5
Y_{31}	2.5411	[2.50446,3.6038]	D_{10}
Y_{32}	-3.5718	[-3.7248,-2.2591]	D_5
Y_{33}	-5.7042	[-7.3892,-5.1906]	D_3
Y_{34}	-5.5197	[-7.3892,-5.1906]	D_3
Y_{35}	-13.985	[-13.9850,-9.5878]	D_1
Y_{36}	0.1595	[-0.7934,0.30586]	D_7
Y_{37}	1.3646	[0.30586,1.40516]	D_8
Y_{38}	-2.4739	[-3.7248,-2.2591]	D_5
Y_{39}	4.4916	[3.6038,5.0694]	D_{11}
Y_{40}	-5.0494	[-5.1906,-3.7248]	D_4
Y_{41}	-2.4287	[-3.7248,-2.2591]	D_5
Y_{42}	-1.9373	[-2.2591,-0.7934]	D_6
Y_{43}	0.5019	[0.30586,1.40516]	D_8
Y_{44}	1.4522	[1.40516,2.50446]	D_9
Y_{45}	-2.8786	[-3.7248,-2.2591]	D_5
Y_{46}	-1.2717	[-2.2591,-0.7934]	D_6
Y_{47}	-1.1682	[-2.2591,-0.7934]	D_6
Y_{48}	2.0436	[1.40516,2.50446]	D_{19}
Y_{49}	5.1905	[5.0694,6.5352]	D_{12}
Y_{50}	6.5133	[5.0694,6.5352]	D_{12}

Table 7: Actual and forecasted observations.

Observation No.	Actual observations	Trend the forecasting	Forecasted observations
Y_1	2.2707		
Y_2	8.2438	Middle value	10.1995
Y_3	-5.7379	Upward; 0.75-point	-5.7403
Y_4	1.1535	Middle value	0.8555
Y_5	1.7365	Middle value	1.9548
Y_6	-4.5362	Middle value	-4.4578
Y_7	-3.5488	Middle value	-2.9920
Y_8	-0.049	Downward; 0.25-point	-0.5186
Y_9	14.294	Middle value	13.4974
Y_{10}	16.7953	Middle value	15.6960
Y_{11}	1.3186	Middle value	0.8555
Y_{12}	12.6671	Middle value	13.4974
Y_{13}	7.9685	Middle value	7.2680
Y_{14}	2.9352	Middle value	3.0541
Y_{15}	4.033	Upward; 0.75-point	4.7031
Y_{16}	0.7934	Downward; 0.25-point	0.5807
Y_{17}	-0.1792	Middle value	-0.2438
Y_{18}	5.8871	Middle value	5.8024
Y_{19}	7.991	Middle value	7.2681
Y_{20}	8.8652	Downward; 0.25-point	9.1003
Y_{21}	6.2321	Upward; 0.75-point	6.1688
Y_{22}	-2.3371	Middle value	-2.9920
Y_{23}	1.9341	Middle value	1.9548
⋮	⋮	⋮	⋮
Y_{34}	-5.5197	Middle value	-6.2899
Y_{35}	-13.985	Upward; 0.75-point	-10.6871
Y_{36}	0.1595	Middle value	-0.2438
Y_{37}	1.3646	Middle value	0.8555
Y_{38}	-2.4739	Middle value	-2.9920
Y_{39}	4.4916	Middle value	4.3366
Y_{40}	-5.0494	Middle value	-4.4578
Y_{41}	-2.4287	Middle value	-2.9920
Y_{42}	-1.9373	Middle value	-1.5263
Y_{43}	0.5019	Middle value	0.8555
Y_{44}	1.4522	Downward; 0.25-point	1.67998
Y_{45}	-2.8786	Middle value	-2.9920
Y_{46}	-1.2717	Middle value	-1.5263
Y_{47}	-1.1682	Middle value	-1.5263
Y_{48}	2.0436	Upward; 0.75-point	2.2296
Y_{49}	5.1905	Middle value	5.8023
Y_{50}	6.5133	Downward; 0.25-point	5.4359

Table 8: Showing the MSEs for the three methods for the AR(1) parameter in the two cases of actual observations and forecasted observations.

n	ϕ_1	σ	Y_0	MSE					
				MLE		Moments		OLS	
				AO	FO	AO	FO	AO	FO
100	-0.3	4.5	2.5	0.0092661	0.00921863	0.0094827	0.0094534	0.0093573	0.0093177
	0.1	3	0.8	0.0094529	0.00942243	0.0098318	0.0097003	0.0096316	0.0096003
	-0.4	1.6	1	0.008625	0.00852349	0.0087275	0.0087189	0.008646	0.0086007
	0.7	2.8	3.9	0.0060178	0.00600065	0.0062388	0.0062099	0.0061484	0.006127
	0.2	2.3	4.8	0.0089998	0.00893687	0.0092826	0.0092207	0.0091095	0.0090464
	-0.8	3.8	0.7	0.0047336	0.00465002	0.005264	0.004999	0.0047454	0.0047111
	0.5	7	1.7	0.0080631	0.00804791	0.008083	0.0080777	0.0080676	0.0080535
	-0.9	5.5	4.2	0.0032079	0.0032022	0.0034497	0.0033877	0.0032304	0.0032188
200	-0.3	4.5	2.5	0.0041677	0.00414024	0.0042358	0.00422	0.0041971	0.0041688
	0.1	3	0.8	0.0050984	0.00499012	0.0051891	0.00512	0.0051419	0.0050921
	-0.4	1.6	1	0.0042642	0.00426013	0.0043157	0.0042968	0.0042903	0.0042785
	0.7	2.8	3.9	0.0027698	0.00273933	0.0027848	0.002779	0.0027792	0.0027512
	0.2	2.3	4.8	0.0046311	0.00460839	0.0046994	0.004675	0.0046706	0.0046464
	-0.8	3.8	0.7	0.0019592	0.00193296	0.0020882	0.001999	0.0019647	0.0019512
	0.5	7	1.7	0.0038805	0.003801	0.0039363	0.0039112	0.0038842	0.0038777
	-0.9	5.5	4.2	0.0010452	0.00103209	0.0012033	0.0012007	0.0011826	0.0011598
300	-0.3	4.5	2.5	0.0031185	0.00311298	0.0031398	0.0031281	0.0031333	0.0031267
	0.1	3	0.8	0.0032582	0.00323003	0.003304	0.0032991	0.0032777	0.0032644
	-0.4	1.6	1	0.0029265	0.0028999	0.0029371	0.0029007	0.0029405	0.0029276
	0.7	2.8	3.9	0.0017379	0.00168829	0.0017475	0.001713	0.0017503	0.0017333
	0.2	2.3	4.8	0.0032457	0.00317653	0.0032539	0.00319	0.0032727	0.0032119
	-0.8	3.8	0.7	0.0012664	0.00120939	0.0012668	0.0012498	0.0013213	0.0012809
	0.5	7	1.7	0.0025966	0.0025629	0.0025892	0.002577	0.0025944	0.0025901
	-0.9	5.5	4.2	0.000822	0.00081199	0.0008239	0.00082	0.0008482	0.0008411
400	-0.3	4.5	2.5	0.0023702	0.00232981	0.0023735	0.002353	0.0023782	0.0023599
	0.1	3	0.8	0.0023081	0.00229098	0.0023143	0.0022976	0.002321	0.0023025
	-0.4	1.6	1	0.0020885	0.00199837	0.0020891	0.0020377	0.0020919	0.002067
	0.7	2.8	3.9	0.0020919	0.00203465	0.0022416	0.0021674	0.0022194	0.0021804
	0.2	2.3	4.8	0.0024723	0.00245747	0.002483	0.0024795	0.002493	0.0024848
	-0.8	3.8	0.7	0.0009938	0.00096309	0.0009991	0.0009865	0.0010297	0.0009955
	0.5	7	1.7	0.0020134	0.00200323	0.0020217	0.002012	0.0020228	0.0020194
	-0.9	5.5	4.2	0.0005525	0.00055094	0.0005558	0.0005539	0.0005587	0.0005561
500	-0.3	4.5	2.5	0.00174	0.00171646	0.0017423	0.0017394	0.0017465	0.001742
	0.1	3	0.8	0.0020765	0.00207579	0.0020852	0.0020844	0.002094	0.0020932
	-0.4	1.6	1	0.0017073	0.00168736	0.001708	0.0016903	0.001717	0.0017023
	0.7	2.8	3.9	0.000962	0.00096003	0.0009624	0.000962	0.0009639	0.0009627
	0.2	2.3	4.8	0.0019338	0.0019119	0.0019377	0.001915	0.0019424	0.0019192
	-0.8	3.8	0.7	0.0007273	0.00070763	0.0007302	0.0007217	0.0007517	0.0007376
	0.5	7	1.7	0.0013328	0.0013022	0.0013361	0.0013102	0.0013328	0.0013177
	-0.9	5.5	4.2	0.0004126	0.0004119	0.0004144	0.000413	0.0004186	0.0004167

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References

[1] Q. Song and B. S. Chissom. Fuzzy time series and its models. *Fuzzy sets and systems*, **54(3)**, 269– 277 (1993).
[2] Q. Song and B. S. Chissom. Forecasting enrollments with fuzzy time series—part i, *Fuzzy sets and systems*, **54(1)**, 1–9 (1993).
[3] Q. Song and B. S. Chissom. Forecasting enrollments with fuzzy time series-part ii, *Fuzzy sets and systems*, **62(1)**, 1–8 (1994).
[4] K. Huarng. Effective lengths of intervals to improve forecasting in fuzzy time series. *Fuzzy sets and systems*, **123(3)**, 387–394 (2001).
[5] H. Yu. A refined fuzzy time-series model for forecasting, *Physica A: Statistical Mechanics and its Applications*, **346(3-4)**, 657–681 (2005).
[6] K. Huarng and T. H. Yu. Ratio-based lengths of intervals to improve fuzzy time series forecasting, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, **36(2)**, 328–340 (2006).
[7] E. Egrioglu, C. Aladag, U. Yolcu, V. R. Uslu, and M. Basaran. Finding an optimal interval length in high order fuzzy time series, *Expert Systems with Applications*, **37(7)**, 5052–5055 (2010).
[8] E. Egrioglu, C. H. Aladag, M. A. Basaran, U. Yolcu, and V. R. Uslu. A new approach based on the optimization of the length of intervals in fuzzy time series, *Journal of Intelligent & Fuzzy Systems*, **22(1)**, 15–19 (2011).
[9] P. Saxena, K. Sharma, and S. Easo. Forecasting enrollments based on fuzzy time series with higher forecast accuracy rate, *Int. J. Computer Technology & Applications*, **3(3)**, 957–961 (2012).
[10] Q. Cai, D. Zhang, W. Zheng, and S. C. Leung. A new fuzzy time series forecasting model combined with ant colony optimization and auto-regression, *Knowledge-Based Systems*, **74**, 61– 68 (2015).
[11] S. Cheng, S. Chen, and W. Jian. Fuzzy time series forecasting based on fuzzy logical relationships and similarity measures, *Information Sciences*, **327**, 272–287 (2016).
[12] P. C. de Lima Silva, H. J. Sadaei, R. Ballini, and F. G. Guimar˜aes. Probabilistic forecasting with fuzzy time series, *IEEE Transactions on Fuzzy Systems*, **28(8)**, 1771–1784 (2019).
[13] O. C. Yolcu and H. Lam. A combined robust fuzzy time series method for prediction of time series, *Neurocomputing*, **247**, 87–101 (2017).
[14] S. Panigrahi and H. Sekhar. Behera. A study on leading machine learning techniques for high order fuzzy time series forecasting, *Engineering Applications of Artificial Intelligence*, **87**, 103245, (2020).
[15] C. Kocak, E. Egrioglu, and E. Bas. A new deep intuitionistic fuzzy time series forecasting method based on long short-term memory, *The Journal of Supercomputing*, **77**, 6178–6196 (2021).
[16] V. M. Serrano, J. N. Maciel, J. J. G. Ledesma, and O. H. Ando. Fuzzy time series methods applied to (in) direct short-term photovoltaic power forecasting, *Energies*, **15(3)**, 845 (2022).
[17] A. A. M. Teamah, H. M. Faied, and M. H. El-Menshawy. Effect of fuzzy time series technique on estimators of spectral analysis, *Recent Advances in Mathematical Research and Computer Science*, **6**, 29–38 (2022).
[18] M. El-Morshedy, M. M.A. El-Menshawy, M. H.and Almazah, R. M. El-Sagheer, and M. S. Eliwa. Effect of fuzzy time series on smoothing estimation of the inar (1) process, *Axioms*, **11(9)**, 423 (2022).
[19] S. Chen, C. Hsu, et al. A new method to forecast enrollments using fuzzy time series, *International Journal of Applied Science and Engineering*, **2(3)**, 234–244 (2004).
[20] W. W. S. Wei, *Time series analysis*, Univariate and Multivariate Methods. (1990).

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