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New Constructions of Strongly Partially Balanced Cross over Designs in Minimal Circular Periods of Three Different Sizes

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Abstract: Minimal strongly balanced cross over designs (CODs) are widely used to estimate direct effects and carry over effects independently. The situations where these minimal designs cannot be constructed, strongly partially balanced CODs (SPBCODs) are used. In this article, minimal circular SPBCODs are constructed in periods of three different sizes. Catalogues of these designs are also compiled.

Keywords: Residual effects; Carry over effects; RMDs; Minimal designs.
Mathematics Subject Classification (2010): 05B05; 62K10; 62K05.

1 Introduction

CODs are widely used to estimate direct effects and carry over effects independently, see [1]. In these designs, experimental subjects are either human or animal who can die or recover during their experimental period. It is, therefore, beneficial to use CODs in unequal period sizes. By the use of CODs which are also called repeated measurements designs (RMDs), there are chances of carry over or residual effects which can be neutralized through balanced or strongly balanced CODs (SBCODs). The experimenter always prefers minimal CODs. SBCODs are also important to estimate direct effects and carry over effects independently. The situations where minimal SBCODs cannot be constructed, strongly partially balanced CODs (SPBCODs) are used. Following are some important definitions.

- COD is minimal balanced if each treatment is immediately preceded exactly once by all others (excluding itself).
- COD is minimal strongly balanced if each treatment is immediately preceded exactly once by all others (including itself).
- COD is minimal strongly partially balanced if each treatment is immediately preceded by all others (including itself) either once or no time.
- If periods are divided in circular form where the treatments applied in the last period is considered the preceding value of first period then periods are considered as circular.

[2] introduced the idea of a CBRMDs. [3] presented non-circular BRMDs and SBRMDs in period of different sizes. [4] constructed non- circular minimal BRMDs for v odd. [5] constructed non-circular BRMDs. [6] constructed balanced cross over designs in circular periods using integer programming. [7] and [8] developed some generators to generate CBRMDs. [9] developed some generators for CBRMDs in periods of two different sizes. [10] presented MCPBRMDs in periods of two and three different sizes.

[11] presented some classes of CSBRMDs. [12] constructed some second order CBRMDs and CSBRMDs when $p < v$. [13] developed some series to generate MCSBRMDs in periods of three different sizes for p_1 even, $3 \leq p_2 \leq 10$ and

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$2 \leq p_2 \leq 9$, where $p_1 < p_2 < p_3$. [14] presented some generators for MCSBRMDs in two and three different period sizes. [15] presented some minimal circular strongly partially BRMDs (MCSPBRMDs) in periods of equal sizes using Rule I. [16] constructed some MCSPBRMDs in two different period sizes. Using Rule I, MCSPBCODs are not constructed in periods of three different sizes when p_1 is even, either of p_2 or p_3 is odd. In this article, therefore, minimal CSPBCODs are constructed and catalogues of these designs are also compiled for the following cases with $p_1 > p_2 > p_3$.

- $v \leq 60$, $p_1 = 8, 10, 12$, $p_2 = 6, 8, 10$ and $p_3 = 5$.
- $v \leq 60$, $p_1 = 8, 10, 12$, $p_2 = 7, 9, 11$ and $p_3 = 6$.
- $v \leq 60$, $p_1 = 10, 12$, $p_2 = 8, 10$ and $p_3 = 7$.
- $v \leq 60$, $p_1 = 10, 12$, $p_2 = 9, 11$ and $p_3 = 8$.
- $v \leq 60$, $p_1 = 12$, $p_2 = 10$ and $p_3 = 9$.
- $v \leq 60$, $p_1 = 12$, $p_2 = 11$ and $p_3 = 10$.

2 Method of Cyclic Shifts

Method of cyclic shifts introduced by [17] is used to construct several types of designs. Its Rule I is explained here to construct MCSBCODs and MCSPBCODs in three different period sizes.

- How a design is obtained from given set(s) of shifts using Rule I, is described here with the help of an example. A design is obtained through $S_1 = [2, 12, 3, 4, 5, 6, 7], S_2 = [9, 8, 11, 13, 19]$ and $S_3 = [15, 16, 17, 14]$ for $v = 20, p_1 = 8$, $p_2 = 6, p_3 = 5$.

Get arrays from $S_1 = [2, 12, 3, 4, 5, 6, 7]$ using v subjects. Consider $0, 1, ..., v-1$ as the respective elements of first period. Add 2 (mod 20) in these elements to obtain the elements of second period. Similarly add 12, 3, 4, 5, 6 and 7 (mod 20), see Table 1.

Table 1: Arrays obtained from S_1

Periods	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	1	2	3	4	5	6	7	8	9
2	2	3	4	5	6	7	8	9	10	11
3	14	15	16	17	18	19	0	1	2	3
4	17	18	19	0	1	2	3	4	5	6
5	1	2	3	4	5	6	7	8	9	10
6	6	7	8	9	10	11	12	13	14	15
7	12	13	14	15	16	17	18	19	0	1
8	19	0	1	2	3	4	5	6	7	8
Periods	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	10	11	12	13	14	15	16	17	18	19
2	12	13	14	15	16	17	18	19	0	1
3	4	5	6	7	8	9	10	11	12	13
4	7	8	9	10	11	12	13	14	15	16
5	11	12	13	14	15	16	17	18	19	0
6	16	17	18	19	0	1	2	3	4	5
7	2	3	4	5	6	7	8	9	10	11
8	9	10	11	12	13	14	15	16	17	18

Take v more subjects for $S_2 = [9, 8, 11, 13, 19]$ and get the design in the similar way as are taken through S_1, see Table 2.

Table 2: Arrays obtained from S_2

Periods	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
1	0	1	2	3	4	5	6	7	8	9
2	9	10	11	12	13	17	15	16	17	18
3	17	18	19	0	1	2	3	4	5	6
4	8	9	10	11	12	13	14	15	16	17
5	1	2	3	4	5	6	7	8	9	10
6	0	1	2	3	4	5	6	7	8	9
Periods	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
1	10	11	12	13	14	15	16	17	18	19
2	19	0	1	2	3	4	5	6	7	8
3	7	8	9	10	11	12	13	14	15	16
4	18	19	0	1	2	3	4	5	6	7
5	11	12	13	14	15	16	17	18	19	0
6	10	11	12	13	14	15	16	17	18	19

Take v more subjects for $S_3 = [15,16,17,14]$ and get the design in the similar way as are taken through S_1, see Table 3.

Table 3: Arrays obtained from S_3

Periods	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
1	0	1	2	3	4	5	6	7	8	9
2	15	16	17	18	19	0	1	2	3	4
3	11	12	13	14	15	16	17	18	19	20
4	8	9	10	11	12	13	14	15	16	17
5	2	3	4	5	6	7	8	9	10	11
Periods	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
1	10	11	12	13	14	15	16	17	18	19
2	5	6	7	8	9	10	11	12	13	14
3	1	2	3	4	5	6	7	8	9	10
4	18	19	0	1	2	3	4	5	6	7
5	12	13	14	15	16	17	18	19	0	1

- **Construction of minimal CSBCODs and minimal CSPBCODs through Rule I**

If $[q_{11},q_{12},...,q_{1(p_1-1)}]$, $[q_{21},q_{22},...,q_{2(p_2-2)}]$ and $[q_{31},q_{32},...,q_{3(p_3-3}]$ are the sets with $1 \leq q_{ij} \leq v-1$ and each of $1,2,...,v-1$ appears once in S^* then it will provide minimal CSBCOD in p_1, p_2 and p_3, where S^* contains (i) each element of all S_j, and (ii) v - [sum of all elements (mod v) in each S_j]. If most of $1,2,...,v-1$ appear once but a few appear no time then it will be minimal CSPBCOD.

Example 2.1: $S_1 = [2,12,3,4,5,6,7]$, $S_2 = [9,8,11,13,19]$ and $S_3 = [15,16,17,14]$ produce MCSPBCOD for $v = 20$ in $p_1 = 8, p_2 = 6, p_3 = 5$. Here $S^* = [2,12,3,4,5,6,7,1,9,8,11,13,19,0,15,16,17,14,18]$ contains each of $0,1,...,19$ exactly once except 10 which does not appear. Hence it is minimal CSPBCOD. Table 1, Table 2 and Table 3 jointly produce the minimal CSPBCOD for $v = 20$ in $p_1 = 8$, $p_2 = 6$ and $p_3 = 5$

3 Efficiency Measures

- [18] developed following efficiency of Separability (Es) for BRMDs. Design will be suitable to estimate direct effect and residual effects independently when value of Es is high.

$$Es = [1 - 1/(v\sqrt{v-1})] * 100\%$$

- Canonical efficiency factors are non-zero Eigen values of C* which provide efficiency of residual effects (Er), see [19] and [20]. Design will be suitable to estimate residual effects when value of Er is high.

4 Construction of minimal CSPBCODs in three different period sizes

Here, using Rule I MCSPBCODs are obtained for (i) p_1 even, p_2 even and p_3 odd, and (ii) p_1 even, p_2 odd and p_3 even.

4.1. Minimal CSPBCODs for $p_1 = 8, 10, 12$, $p_2 = 6, 8, 10$ and $p_3 = 5$

v	p_1	p_2	p_3	sets	Er	Es
20	8	6	5	[2,12,3,4,5,6,7]+[9,8,11,13,19]+[15,16,17,14]	0.95	0.78
28	8	6	5	[1,2,3,4,5,6,7]+[8,23,11,12,13,16,20]+[17,18,15,21,22]+ [24,25,26,27]	0.96	0.81
36	8	6	5	[8,2,3,4,5,6,7]+[24,11,10,12,13,14,15]+ [17,16,19,21,20,22,29]+[35,34,27,28,33]+[31,32,26,25]	0.97	0.82
44	8	6	5	[16,2,3,4,5,6,7]+[8,9,31,10,13,14,15]+[34,17,20,18,21,24,23]+ [25,26,27,28,29,30,11]+[33,35,36,37,38]+[39,40,42,43]	0.98	0.83
52	8	6	5	[24,1,2,3,5,6,7]+[8,9,10,12,25,14,15]+ [16,17,18,19,20,21,23]+[51,13,27,29,30,50,32]+ [34,35,36,37,38,39,41]+[42,43,44,45,46]+[48,49,31,33]	0.98	0.84
60	8	6	5	[32,2,3,4,5,6,7]+[16,23,11,20,13,14,15]+[54,17,19,12,21,22,55]+ [24,26,27,28,29,50,31]+[33,34,36,37,38,39,48]+ [41,42,43,44,46,52,47]+[49,51,18,53,9]+[10,56,58,59]	0.98	0.84
22	10	6	5	[1,9,4,3,7,6,5,8,21]+[18,12,15,14,16]+[17,10,19,20]	0.95	0.80
32	10	6	5	[19,1,2,3,4,5,6,7,8]+[10,11,12,13,15,14,17,30,18]+ [31,29,23,24,25]+[27,26,22,21]	0.97	0.83
42	10	6	5	[39,1,2,3,5,6,7,8,9]+[10,13,11,14,15,16,17,18,19]+ [36,34,23,24,25,27,26,28,29]+[31,32,22,33,20]+[37,38,12,41]	0.98	0.81
52	10	6	5	[51,30,3,4,5,41,7,8,9]+[11,21,13,14,15,16,17,18,19]+ [10,20,23,24,22,25,27,28,29]+[31,32,33,34,36,37,38,39,49]+ [40,42,43,45,46]+[47,48,6,1]	0.98	0.83
24	12	6	5	[1,2,3,4,5,6,7,8,10,9,17]+[13,14,15,16,18]+[19,21,22,23]	0.96	0.82
36	12	6	5	[1,2,3,4,5,6,7,30,8,31,11]+[14,15,12,17,16,19,21,20,22,23,24]+ [25,27,28,29,9]+[10,32,34,35]	0.97	0.85
48	12	6	5	[1,2,3,4,5,6,7,8,9,10,11]+[12,15,14,16,47,18,19,20,21,23,22]+ [26,25,27,28,31,29,32,33,34,35,36]+ [37,39,43,41,42]+[40,44,46,17]	0.98	0.87
60	12	6	5	[4,1,2,3,5,54,6,8,7,9,11]+[13,57,14,16,17,18,19,20,22,21,23]+ [24,25,27,26,28,29,33,31,32,36,34]+ [39,37,40,41,42,43,44,15,46,47,48]+ [49,50,51,53,45]+[55,56,58,59]	0.98	0.85
24	10	8	5	[1,2,3,4,5,6,7,9,11]+[19,8,15,13,16,17,18]+[10,21,22,23]	0.96	0.82
34	10	8	5	[12,33,2,3,4,6,5,7,21]+[10,11,14,15,13,16,19,18,20]+ [29,22,24,23,25,26,27]+[8,31,32,1]	0.97	0.85
44	10	8	5	[43,8,1,2,3,4,5,6,7]+[12,11,13,32,15,16,30,18,19]+ [21,24,23,25,26,27,28,29,17]+ [31,33,20,34,35,36,38]+[39,41,42,14]	0.98	0.89
54	10	8	5	[40,1,2,3,4,5,6,30,8]+[39,43,13,12,15,16,17,18,29]+ [22,34,23,24,26,25,28,39,29]+[31,33,32,35,21,36,37,38,7]+ [41,42,11,44,45,46,47]+[10,50,51,52]	0.96	0.87
26	12	8	5	[12,1,3,2,5,4,6,7,8,9,10]+[25,15,16,17,18,19,20]+ [21,22,23,14]	0.96	0.84
38	12	8	5	[1,2,3,4,5,6,7,8,9,10,21]+ [13,24,15,16,17,18,22,20,11,37,23]+ [25,27,26 ,28,29,30,32]+[34,35,36,14]	0.97	0.86
50	12	8	5	[1,2,3,4,5,6,7,8,9,44,11]+[13,14,30,41,17,18,19,20,21,23,22]+ [27,24,28,29,15,31,32,33,34,35,36]+[38,39,40,16,42,43,45]+ [46,47,48,49]	0.98	0.88
28	12	10	5	[1,18,2,3,8,5,6,7,4,9,11]+[25,13,16,15,17,19,21,20,22]+ [23,24,12,26]	0.96	0.85
40	12	10	5	[14,1,2,3,4,5,6,8,9,11,10]+[12,24,13,23,16,17,18,21,22,15,19]+ [39,27,28,30,29,31,32,33,34]+[35,36,26,38]	0.94	0.87
52	12	10	5	[38,1,3,2,4,5,6,7,8,9,10]+[46,44,51,20,18,17,19,15,21,22,23]+ [24,25,28,29,27,30,31,32,33,34,35]+ [37,39,40,41,42,43,13,45,12]+[47,48,49,14]	0.98	0.88

4.2. Minimal CSPBCODs for $p_1 = 8, 10, 12$, $p_2 = 7, 9, 11$ and $p_3 = 6$

v	p_1	p_2	p_3	sets	Er	Es
22	8	7	6	[16,2,1,3,4,6,7]+[8,9,12,13,14,15]+[10,18,19,20,21]	0.95	0.73
30	8	7	6	[1,2,3,4,5,6,9]+[8,29,17,11,14,12,16]+[27,18,19,20,22,23]+ [24,25,26,10,28]	0.97	0.82
38	8	7	6	[10,1,2,3,5,6,7]+[8,32,34,12,13,14,30]+[16,17,18,21,23,22,15]+ [25,26,27,28,29,24]+[33,11,35,36,37]	0.97	0.83
46	8	7	6	[18,1,2,3,4,5,6]+[8,9,10,12,13,14,15]+[36,43,45,19,22,21,24]+ [25,26,27,28,29,30,32]+[31,34,35,16,37,38]+[40,41,42,17,44]	0.98	0.84
54	8	7	6	[1,2,3,4,5,26,6]+[24,11,12,10,13,14,15]+ [16,17,18,20,21,22,29]+ [25,39,28,23,30,31,48]+[50,35,34,36,37,38,40]+ [41,42,43,44,45,47]+[32,49,33,51,52]	0.98	0.85
24	10	7	6	[1,2,3,4,5,6,10,9,8]+[7,13,14,15,16,20]+[19,17,21,22,23]	0.96	0.82
34	10	7	6	[1,2,3,4,5,6,7,8,9]+[33,11,13,18,14,15,16,19,10]+ [12,24,22,25,26,27]+[28,29,30,32,20]	0.97	0.84
44	10	7	6	[43,2,1,4,5,6,10,8,9]+[41,11,12,13,14,15,16,17,18]+ [21,24,23,25,26,27,28,29,30]+[3,33,34,35,36,37]+ [38,39,40,7,32]	0.98	0.86
54	10	7	6	[1,2,3,53,10,6,7,8,18]+[12,36,13,14,15,16,17,9,19]+ [42,21,22,24,23,26,25,28,30]+[31,33,34,35,5,37,38,39,40]+ [41,4,43,44,45,47]+[48,49,51,52,20]	0.98	0.87
26	12	7	6	[12,1,3,2,5,4,6,7,8,9,10]+[14,15,21,17,18,19]+[20,16,22,23,24]	0.96	0.84
38	12	7	6	[2,1,3,4,5,6,34,8,9,32,11]+[13,14,15,16,17,33,18,21,22,23,24]+ [25,20,27,28,30,31]+[10,26,7,35,36]	0.97	0.86
50	12	7	6	[34,1,2,3,4,5,6,8,9,10,11]+[12,14,13,15,16,17,20,18,30,22,23]+ [27,29,28,26,21,40,32,33,19,35,36]+[47,38,39,49,41,44]+ [43,45,46,48,31]	0.98	0.88
26	10	9	6	[1,2,3,4,5,6,14,8,9]+[10,7,12,20,16,17,18,19]+[15,21,22,23,24]	0.96	0.84
36	10	9	6	[27,2,3,5,4,6,7,8,9]+[12,28,13,14,16,17,30,19,20]+ [21,24,22,23,25,26,29,10]+[15,31,32,34,35]	0.97	0.85
46	10	9	6	[1,2,3,4,5,6,7,8,10]+[39,21,13,14,15,16,17,18,19]+ [45,22,202,24,25,26,27,28,29]+ [33,31,35,34,36,32,38,43]+[41,37,42,9,11]	0.95	0.86
56	10	9	6	[11,1,2,3,4,5,6,7,8]+[10,12,13,14,15,16,17,52,19]+ [34,23,22,24,25,26,27,28,29]+[32,21,33,35,50,37,38,39,20]+ [42,43,44,45,46,47,48,36]+[51,18,53,54,55]	0.97	0.87
28	12	9	6	[19,2,3,5,4,6,7,8,9,10,11]+[12,15,17,16,1,18,27,21]+ [22,23,24,25,26]	0.96	0.85
40	12	9	5	[1,2,3,4,6,5,7,8,9,10,25]+[14,13,16,17,18,19,15,21,22,23,24]+ [39,26,27,30,28,31,37,33]+[36,34,32,12,11]	0.97	0.87
52	12	9	6	[50,1,2,3,4,5,6,8,9,12,7]+[10,14,16,15,17,18,19,20,21,22,23]+ [25,24,27,29,28,30,32,33,34,35,36]+ [51,11,39,41,42,43,44,45]+[46,47,38,40,37]	0.98	0.88
30	12	11	6	[1,3,2,4,29,6,7,11,9,8,10]+[13,27,14,17,16,19,18,21,20,22]+ [24,25,26,12,28]	0.97	0.86
42	12	11	6	[1,2,3,4,5,6,25,8,10,9,11]+[13,14,30,16,18,17,19,20,23,24,22]+ [26,7,28,29,12,31,32,33,34,35]+[15,37,38,39,40]	0.98	0.88
54	12	11	6	[42,1,2,4,5,6,7,8,9,11]+[12,13,14,16,15,17,18,20,21,22,29]+ [45,26,25,28,23,30,31,32,33,34,35]+ [39,38,40,41,53,43,44,51,46,37]+[48,49,50,24,47]	0.97	0.85

4.3. Minimal CSPBCODs for $p_1 = 10, 12$, $p_2 = 8, 10$ and $p_3 = 7$

v	p_1	p_2	p_3	sets	Er	Es
26	10	8	7	[1,2,3,5,4,6,7,8,16]+[12,25,14,24,9,18,17]+ [19,20,21,22,23,15]	0.96	0.84
36	10	8	7	[1,2,3,4,5,6,7,35,9]+[11,26,12,13,14,15,16,34,19]+ [23,21,24,25,10,27,28]+[29,30,31,32,33,17]	0.97	0.85
46	10	8	7	[1,2,3,4,5,6,7,8,10]+[29,30,12,13,14,15,16,17,18]+ [22,19,24,25,26,45,28,9,11]+[31,32,33,34,35,37,38]+ [39,41,42,43,44,27]	0.98	0.86
56	10	8	7	[1,2,3,4,5,6,7,19,9]+[10,11,12,13,14,15,16,17,42]+ [21,22,23,24,26,25,29,30,27]+[54,32,33,34,35,51,36,39,40]+ [55,8,43,44,45,46,47]+[49,50,37,52,20,41]	0.98	0.87
28	12	8	7	[18,2,1,3,5,4,6,8,7,9,10]+[13,12,15,17,16,19,20]+ [21,22,23,24,25,26]	0.96	0.85
40	12	8	7	[14,1,3,4,5,6,8,7,9,11,10]+[13,12,16,15,17,18,19,22,23,24,21]+ [37,27,28,29,30,31,32]+[33,34,35,36,38,39]	0.97	0.87
52	12	8	7	[38,1,2,3,4,5,6,7,8,9,11]+[13,12,16,40,17,18,19,20,45,22,23]+ [25,24,30,27,29,31,32,33,34,35,36]+ [37,39,14,41,42,43,44]+[21,46,47,48,49,50]	0.98	0.88
30	12	10	7	[2,1,3,4,29,6,9,7,10,11,8]+[12,13,24,17,18,19,20,21,22]+ [23,16,25,27,26,5]	0.97	0.80
42	12	10	7	[1,2,3,4,5,8,6,9,25,10,11]+ [13,12,14,40,17,18,19,20,36,23,24]+ [41,38,39,28,30,29,31,32,33]+[34,22,27,37,15,7]	0.98	0.83
54	12	10	7	[42,2,1,3,4,5,6,7,8,9,10]+[12,13,17,15,16,18,19,20,21,22,29]+ [25,26,24,28,23,30,53,31,33,34,36]+ [37,38,39,40,41,43,44,45,51]+[47,48,50,46,52,32]	0.98	0.88
28	10	9	8	[11,1,2,3,4,5,7,8,9]+[10,20,13,16,15,17,18,19]+ [21,22,23,24,25,26,27]	0.95	0.92
38	10	9	8	[31,1,2,3,4,5,6,7,8]+[11,10,13,14,15,16,17,26,18]+ [21,22,32,24,25,20,27,28]+[30,23,33,34,35,36,37]	0.96	0.91
48	10	9	8	[1,2,3,4,5,6,7,8,12]+[10,11,13,15,16,18,17,21,14]+ [39,38,22,23,25,27,26,28,29]+[30,44,34,35,36,37,19,20]+ [40,41,42,43,32,45,46]	0.96	0.83
58	10	9	8	[13,1,3,4,5,6,7,8,9]+[12,10,42,14,15,16,17,18,19]+ [20,22,21,46,23,25,26,27,50]+[24,32,33,34,35,36,37,38,40]+ [41,43,44,45,31,47,48,49]+[28,51,52,53,54,55,57]	0.96	0.87

4.4. Minimal CSPBCODs for $p_1 = 12$, $p_2 = 9, 11$ and $p_3 = 8$

v	p_1	p_2	p_3	sets	Er	Es
30	12	9	8	[1,2,4,3,5,6,7,10,9,8,11]+[12,13,16,17,19,18,21,20]+ [22,23,25,26,27,28,29]	0.97	0.86
42	12	9	8	[18,1,3,2,4,5,6,7,8,9,11]+[13,12,14,15,16,32,20,19,22,24,23]+ [25,28,26,27,36,29,31,17]+[34,35,30,38,39,40,41]	0.98	0.88
54	12	9	8	[42,2,1,3,4,5,6,7,8,9,10]+[12,13,14,16,15,17,18,21,19,22,29]+ [26,24,28,25,23,48,31,32,34,33,35]+[37,38,36,40,41,43,44,45]+ [46,47,30,49,50,51,52]	0.98	0.88
32	12	11	8	[1,2,3,4,5,6,7,9,8,10,13]+[11,15,14,17,19,18,20,21,22,23]+ [24,25,26,27,29,30,31]	0.95	0.88
44	12	11	8	[2,1,3,4,5,6,7,8,9,19,43]+[13,12,15,16,17,18,20,10,21,41,23]+ [11,26,28,27,29,30,31,24,34,33]+[36,37,38,39,40,32,42]	0.98	0.88
56	12	11	8	[1,2,3,5,4,6,7,8,9,10,11]+[12,14,15,16,17,18,19,34,21,22,23]+ [24,25,26,27,30,31,33,20,55,36,29]+ [37,38,39,40,41,42,44,43,32,47]+[48,49,50,51,52,53,54]	0.98	0.89

4.5. Minimal CSPBCODs for $p_1 = 12$, $p_2 = 10$ and $p_3 = 9$

v	p_1	p_2	p_3	sets	Er	Es
32	12	10	9	[1,2,3,4,5,6,7,8,9,10,11]+[15,13,14,17,18,21,19,20,24]+ [22,23,25,27,26,28,29,12]	0.97	0.87
44	12	10	9	[2,1,3,4,5,6,7,8,34,9,10]+[13,14,15,16,17,18,19,21,24,28,23]+ [26,25,27,20,29,30,31,33,32]+[35,36,37,38,39,40,41,42]	0.98	0.88
56	12	10	9	[1,2,3,5,4,6,7,8,9,10,11]+[14,13,15,29,17,18,19,30,21,22,47]+ [24,25,26,27,16,20,31,32,33,34,12]+ [38,39,40,41,42,44,43,45,23]+[48,49,50,51,52,53,54,55]	0.98	0.89

4.6. Minimal CSPBCODs for $p_1 = 12$, $p_2 = 11$ and $p_3 = 10$

v	p_1	p_2	p_3	sets	Er	Es
34	12	11	10	[1,2,3,4,5,6,7,8,9,12,11]+[15,16,10,18,14,19,33,21,22,23]+ [24,25,26,28,27,29,30,31,32]	0.97	0.88
46	12	11	10	[26,1,2,3,5,4,6,7,8,9,11]+[14,12,15,16,17,37,19,20,22,21,24]+ [25,18,27,28,29,30,31,32,33,34]+[36,38,39,40,41,42,43,44,45]	0.98	0.89
58	12	11	10	[50,1,2,3,4,5,6,7,8,9,10]+[34,13,14,16,17,18,19,20,21,22,23]+ [24,25,26,27,28,39,33,31,12,35,36]+ [37,38,30,40,42,41,43,44,45,46]+[48,47,51,52,53,54,55,56,57]	0.98	0.89

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