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Solitary Wave Solution for Fractional-Order General Equal-Width Equation via Semi Analytical Technique

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Abstract: This paper presents an innovative approach to solve equal-width time-fractional-order equations that describe the behavior of waves in a certain physical system, using the Caputo operator to express the fractional derivative by improving the Taylor series expansion.Its convergence theorem is proven, and the error between the exact and approximate solutions is estimated; the resulting solutions are illustrated using graphs for different values of the fractional derivative order and time.The primary objective of this study is to demonstrate the effectiveness of the method in reducing computational effort for solving nonlinear fractional partial differential equations (NFPDEs).

Keywords: Taylor's series, The modified fractional Taylor expansion, Caputo operators, Riemann-Liouville fractional integral, time fractional-order general equal-width equations (FGEWE).

1 Introduction

During the past three decades or so, fractional evaluation equations have drawn the interest of many academics due to their broad use in a variety of modern research fields and industry. It has been demonstrated that time-fractional equations can be used to clarify a wide range of physical processes and address a number of problems. Applications for fractional calculus need to be more inventive [1, 2]. The fractional Caputo derivative was discovered by Ford and Simpson to be the most effective way for locating fractional problems because it constantly includes the preliminary conditions that are unavailable in various specific models [3, 4]. Since partial evaluation equations have a enormous range of applications in several technological and scientific fields, many academics have been working on them recently. The fractional equations have the potential to describe a multitude of intriguing phenomena within various fields, such as fluid and quantum mechanics, electrodynamics, material science, plasma physics, and waves and optical fiber, among others [5, 6]. One of the most important nonlinear partial differential equations is fractional equal width equation (FEWE), which express many intricate nonlinear phenomena in several fields, including science and engineering, for example plasma waves, chemical physics, fluids mechanics, material science, and other fields. The Equal-Width equations explained the behaviour of nonlinear waves in a number of nonlinear systems, such as shallow water waves, acoustic waves in enharmonic crystals, surface waves in compressible fluids, cold plasma, and others [7, 8]. The FGEWEs, obtained forlengthy waves that travel in the positive x direction takes the form [9, 10]:

$$\mathscr{D}_t^{\alpha}\varpi + a\varpi_x\varpi^p - \mu\varpi_{xxt} = 0,\ x \in [l,m],\ t \geqslant 0,\ 0 < \alpha \leqslant 1, \tag{1}$$

The equation you provided is a fractional-order partial differential equation that describes the behavior of waves in a certain physical system. In this equation, ϖ represents the wave amplitude, a and μ are physical parameters that affect the wave behavior, and p is a positive integer $p \in Z^+$ that determines the nonlinear behavior of the wave. The fractional-order derivative $\mathscr{D}_t^{\alpha}$ represents a fractional time derivative of order α, where $0 < \alpha \leqslant 1$. This means that the equation takes into account the memory effects of the system, where the behavior of the wave at a given time depends not only on

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the current time but also on its past behavior. The term $a\varpi_x\varpi^p$ represents the nonlinear behavior of the wave, where the wave amplitude affects its own propagation. The term $\mu\varpi_{xxt}$ represents the dispersion of the wave, where the wave behavior depends on its spatial curvature. The physical boundary conditions of the equation require that the wave amplitude approaches zero as x goes to infinity, which ensures that the wave does not have an infinite energy or amplitude in the physical system. Overall, this equation provides a mathematical model for understanding the behavior of waves in a nonlinear and dispersive medium with memory effects.

When we substitute $p = 1$ into the equation(1), we get an equation the time fractional-order equal-width equations (FEWE) and when we substitute $p = 2$, we get the fractional modified equal width wave (FMEWE) equation, for further details, see [11, 12].

Many different techniques have been used to solve nonlinear differential equations of fractional order by many researchers, including numerical and analytical ones, for example finite difference method [13]. In their research, K. M. Owolabi and A. Atangana utilized the Fourier pseudo-spectral method to solve the time-dependent fractional Schrödinger equation, both in its linear and nonlinear forms [14]. Similarly, Kh. K. Ali and M. Maneea employed the similarity method to solve the fractional Schrödinger equation in their own study [15]. Fractional differential transform method(FDTM), perturbation methods, differential transform method (DTM), Adomian decomposition technique, homotopy analysis method (HAM), New iterative technique, etc., see [16, 17] for more details.

Various techniques have been employed to obtain solutions for Fractional Equal-Width equations (FEWE). For example, in [18], the variational-iteration method was utilized by Youwei Zhang. The new iterative method (NIM) and homotopy perturbation method (HPM) were employed with the aid of Laplace transform and Caputo-Fabrizio operators in [19]. The homotopy-perturbation Sadik transform method (HPSTM) was utilized to solve equations containing Caputo-Prabhakar fractional derivatives in [20]. Also, the time-fractional-order non-linear equal width equations are solved utilizing the HPTM. By integrating the Yang transform with the homotopy perturbation method, the homotopy perturbation transform method (HPTM) is developed, as shown in [21]. The new auxiliary equation methodology (NAEM) was utilized in [22] to investigate various types of single wave solutions of the fractional modified equal-width wave equation (FMEWE). NFPDEs are a popular choice for modeling systems with long-range dependence and memory effects, due to their ability to capture complex nonlinear behavior. However, solving NFPDEs is computationally demanding, requiring the use of sophisticated methods. In this paper, we present an innovative technique for solving equal-width time-fractional-order equations using the Caputo operator. The method improves the Taylor series expansion to approximate the fractional derivative and proves its convergence and error bounds. By applying this method to solve NFPDEs, we aim to reduce the computational complexity and increase the accuracy of the solutions. Our main objective is to demonstrate the effectiveness of this method in solving NFPDEs and reducing computational effort. This technique include using fractional taylor series as a functional tool for solving the nonlinear partial differential equations, which has been advanced by [23, 24]. In another study, M. Sultana et al. employed a new analytic method to solve the fractional derivatives of Kortewege-DeVries Equations, as described in their research [25]. Similarly, Kh. K. Ali and M. Maneea utilized the same method to solve the Kudryashov Sinelshchikov equation in their study [26].

The article is structured as follows:

The second section of the paper provides a comprehensive overview of fractional derivatives and their properties. Section three details the current approach and its application to non-linear FPDEs, while section four focuses on discussing the convergence of the proposed method. In section five, examples are presented, demonstrating how the current technique can be applied to time fractional (EWEs). Section six introduces graphical representations of the solutions obtained in section five. Section seven presents a discussion about the findings and their implications. Finally, section eight concludes the research.

2 Basic Conceptes about Fractional -Calculus

This section introduces some fundamental concepts of fractional calculus [27, 28].

Definition 1. A real function $\mathscr{V}(\mathscr{T})$, where $\mathscr{T} > 0$, is considered to be in space $\mathbb{C}_\upsilon$, for $\upsilon \in \mathbb{R}$, if there exists a real number $\rho > \upsilon$ such that $\mathscr{V}(\mathscr{T}) = \mathscr{T}^\rho \mathscr{V}_1(\mathscr{T})$ where $\mathscr{V}_1(\mathscr{T}) \in \mathbb{C}(0,\infty)$. It is said to be in the space $\mathbb{C}_\upsilon^\aleph$, for $\aleph \in \mathbb{N}$.

Definition 2. The fractional integral of Riemann-Liouville of order α, $\alpha > 0$ of a function $\mathscr{V}(\mathscr{T})$ is defined as:

$$\begin{cases} J^\alpha \mathscr{V}(\mathscr{T}) = \dfrac{1}{\Gamma(\alpha)} \int_0^{\mathscr{T}} (\mathscr{T}-\zeta)^{(\alpha-1)} \mathscr{V}(\zeta)\, d\zeta, & \alpha > 0, \mathscr{T} > 0, \\ J^0 \mathscr{V}(\mathscr{T}) = \mathscr{V}(\mathscr{T}), \end{cases}$$

Some properties of J^α are : For $\gamma, \alpha \geqslant 0, \upsilon \geqslant -1$ given by

$$J^\alpha J^\gamma \mathscr{V}(\mathscr{T}) = J^{\alpha+\gamma}.$$

$$J^\alpha \mathscr{T}^\psi = \frac{\Gamma(\psi+1)}{\Gamma(\psi+\alpha+1)} \mathscr{T}^{\psi+\alpha}.$$

Definition 3. Caputo fractional derivative [29] is defined as:

$$\mathscr{D}^\alpha \mathscr{V}(\mathscr{T}) = J^{m-\alpha} \mathscr{D}^m \mathscr{V}(\mathscr{T}),$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{\mathscr{T}} (\mathscr{T}-\zeta)^{(m-\alpha-1)} \mathscr{V}^m(\zeta)\, d\zeta.$$

For $m-1<\alpha\leqslant m, m\in\mathbb{N}, \quad \mathscr{T}>0$.

Lemma 1. If $m-1<\alpha\leqslant m, m\in\mathbb{N}$, and $\mathscr{V}(\mathscr{T})\in\mathbb{C}^m_{-1}$.

$$\begin{cases} \mathscr{D}^\alpha J^\alpha \mathscr{V}(\mathscr{T}) = \mathscr{V}(\mathscr{T}), \\ J^\alpha \mathscr{D}^\alpha \mathscr{V}(\mathscr{T}) = \mathscr{V}(\mathscr{T}) - \sum_{k=0}^{m-1} \mathscr{V}^{(k)}(0)\frac{\mathscr{T}^k}{k!}. \end{cases}$$

For further details, see [30]. We utilize the Caputo fractional derivative because it enables the formulation of our work to include conventional initial and boundary conditions.

3 Methodology of the modified fractional Taylor expansion

Now, We discuss the main concepts for developing The current method for highly nonlinear fractional(PDE) in this section.

$$\mathscr{D}_t^{2\alpha}\varpi(x,t)=\mathscr{F}(\varpi,\varpi_x,,\mathscr{D}_t^\alpha\varpi,\mathscr{D}_x^\alpha\varpi,\ldots),\ 0<\alpha\leqslant 1, \tag{2}$$

with initial condition and

$$\varpi(x,0)=\psi_0(x),\ \mathscr{D}_t^\alpha\varpi(x,0)=\psi_1(x). \tag{3}$$

By applying the fractional integral to both sides of equation (2) from 0 to η, we obtain:

$$\begin{aligned}\mathscr{D}_t^\alpha\varpi(x,t)-\mathscr{D}_t^\alpha\varpi(x,0)&=\mathscr{I}_t^\alpha\ \mathscr{F}[\varpi],\\ \mathscr{D}_t^\alpha\varpi(x,t)&=\psi_1(x)+\mathscr{I}_t^\alpha\ \mathscr{F}[\varpi],.\end{aligned} \tag{4}$$

where $\mathscr{F}[\varpi]=\mathscr{F}(\varpi,\varpi_x,\varpi_{xx},\mathscr{D}_t^\alpha\varpi,\mathscr{D}_x^\alpha\varpi,D_x^{2\alpha}\varpi,\ldots)$. Then, When using integration again on both sides of equation (4) from 0 to t, we obtain,

$$\begin{aligned}\varpi(x,t)-\varpi(x,0)&=\psi_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}+\mathscr{I}_t^{2\alpha}\ \mathscr{F}[\varpi],\\ \varpi(x,t)&=\psi_0(x)+\psi_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}+\mathscr{I}_t^{2\alpha}\ \mathscr{F}[\varpi].\end{aligned} \tag{5}$$

The extended fractional Taylor series includes about $t=0$,

$$\mathscr{F}[\varpi]=\sum_{k=0}^{\infty}\frac{\mathscr{D}_t^{k\alpha}\mathscr{F}[\varpi_0]}{\Gamma(k\alpha+1)}t^{k\alpha},\quad \alpha>0.$$

$$\begin{aligned}\mathscr{F}[\varpi]=\mathscr{F}[\varpi_0]+\frac{\mathscr{D}_t^{\alpha}\mathscr{F}[\varpi_0]}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\mathscr{D}_t^{2\alpha}\mathscr{F}[\varpi_0]}{\Gamma(2\alpha+1)}t^{2\alpha}\\ +\frac{\mathscr{D}_t^{3\alpha}\mathscr{F}[\varpi_0]}{\Gamma(3\alpha+1)}t^{3\alpha}+\ldots+\frac{\mathscr{D}_t^{k\alpha}\mathscr{F}[\varpi_0]}{\Gamma(k\alpha+1)}t^{k\alpha}+\ldots\end{aligned} \tag{6}$$

substitute from (6) into (5), we obtain

$$\begin{aligned}\varpi(x,t)=\psi_0(x)+\psi_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}+\mathscr{I}_t^{2\alpha}\,[\mathscr{F}[\varpi_0]+\\ \frac{\mathscr{D}_t^{\alpha}\mathscr{F}[\varpi_0]}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\mathscr{D}_t^{2\alpha}\mathscr{F}[\varpi_0]}{\Gamma(2\alpha+1)}t^{2\alpha}\\ +\frac{\mathscr{D}_t^{3\alpha}\mathscr{F}[\varpi_0]}{\Gamma(3\alpha+1)}t^{3\rho}+\ldots],\end{aligned}$$

$$\begin{aligned}\varpi(x,t)=\psi_0(x)+\psi_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{\mathscr{F}[\varpi_0]}{\Gamma(2\alpha+1)}t^{2\alpha}+\\ \frac{\mathscr{D}_t^{\alpha}\mathscr{F}[\varpi_0]}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{\mathscr{D}_t^{2\alpha}\mathscr{F}[\varpi_0]}{\Gamma(4\alpha+1)}t^{4\rho}+\ldots\end{aligned} \tag{7}$$

This series can be expressed as follows:

$$\begin{aligned}\varpi(x,t)=a_0+a_1\frac{t^\alpha}{\Gamma(\alpha+1)}+a_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\\ a_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots+a_k\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}+\ldots,\end{aligned} \tag{8}$$

where

$$a_0=\psi_0(x)=\varpi(x,0),$$

$$a_1=\psi_1(x)=\mathscr{D}_t^\alpha\varpi x,0),$$

$$a_2=\mathscr{F}[\varpi_0]=\mathscr{D}_t^{2\alpha}\varpi(\alpha,0),$$

$$a_3=\mathscr{D}_t^{2\alpha}\mathscr{F}[\varpi_0]=\mathscr{D}_t^{3\alpha}\varpi(x,0),$$

$$a_4=\mathscr{D}_t^{3\alpha}\mathscr{F}[\varpi_0]=\mathscr{D}_t^{4\alpha}\varpi(x,0),\ldots\ and\ so\ on.$$

And so we can simply get the required approximate solution.

4 Convergence Analysis of The Current Technique

Consider as FPDE

$$\varpi(x,t)=\mathscr{A}(\varpi(x,t)), \tag{9}$$

In which $\mathscr{A}$ is nonlinear operator. The done result is congruent to the following sequence when using the techinque described

$$\vartheta_n=\sum_{i=0}^{n}\varpi_i=\sum_{i=0}^{n}\rho_i\frac{(\Delta t)^i}{i!}$$

Theorem 1. Let $\varpi(x,t)$ be the exact solution of Equation (9), and let $\varpi(x,t)\in\mathscr{H}$, $0\leqslant\rho<1$, where the Hilbert space show by$\mathscr{H}$. Then, the solution acquired $\sum_{i=0}^{n}\varpi_i$ will convergence ϖ if $\|\varpi_{i+1}\|\leqslant\rho\|\varpi_i\|\ \forall i\in\aleph\cup\{0\}$.

Proof

Our goal is to demonstrate that the Cauchy Sequence $\{\vartheta_n\}_{n=0}^{\infty}$ has a limit,

$$\begin{aligned}\|\vartheta_{n+1}-\vartheta_n\|=\|\varpi_{n+1}\|\leqslant\rho\,\|\varpi_n\|\leqslant\rho^2\,\|\varpi_{n-1}\|\\ \leqslant\ldots\leqslant\rho^n\,\|\varpi_1\|\leqslant\rho^{n+1}\,\|\varpi_0\|.\end{aligned} \tag{10}$$

$$\begin{aligned}||\vartheta_n-\vartheta_m|| &= ||(\vartheta_n-\vartheta_{n-1})+(\vartheta_{n-1}-\vartheta_{n-2})\\ &+\ldots+(\vartheta_{n-1}-\vartheta_{m+1})||\\ &\leqslant ||\vartheta_n-\vartheta_{n-1}||+||\vartheta_{n-1}-\vartheta_{n-2}||\\ &+\ldots+||\vartheta_{m+1}-\vartheta_m||\\ &\leqslant \rho^n\|\varpi_0\|+\rho^{n-1}\|\varpi_0\|+\ldots+\rho^{m+1}\|\varpi_0\|\\ &\leqslant (\rho^{m+1}+\rho^{m+1}+\ldots+\rho^n)\|\varpi_0\|\\ &= \rho^{m+1}\frac{1-\rho^{n-m}}{1-\rho}||\varpi_0||.\end{aligned} \tag{11}$$

Hence, $lim_{n,m\to\infty}\|\vartheta_n-\vartheta_m\|=0$, i.e.,$\{\vartheta_n\}_{n=0}^{\infty}$ is a Cauchy-Sequence in the Hilbert space $\mathscr{H}$. It means that,$\exists\vartheta.\,\vartheta\in\mathscr{H}, lim_{n\to\infty}\vartheta_n=\vartheta$, where $\vartheta=\varpi$.
In actuality, the theorem involves computing

$$\rho_n=\begin{cases}\dfrac{||\varpi_{n+1}||}{||\varpi_n||}, & ||\varpi_n||\neq 0\\ 0 & otherwise.\end{cases} \tag{12}$$

When $0\leqslant\rho_i<1$, $i=0,1,2,3,\ldots$, the series $\sum_{i=0}^{n}\varpi_i$ converges to the accurate solution ϖ for every $n\in\aleph\cup\{0\}$. For more information, see [31].

5 Applications

The (FGEW) equation, obtained for long waves that travel in the positive x direction takes the form

$$\mathscr{D}_t^{\alpha}\varpi+a\varpi_x\varpi^p-\mu\varpi_{xxt}=0,\quad 0<\alpha\leqslant 1. \tag{13}$$

The equation involves real parameters p, a, and μ, and it is stated in [17] that the initial condition is expressed as follows:

$$\varpi(x,0)=\left(\frac{(p+1)(p+2)c}{2a}\,sech^2\Big(\frac{p}{2\sqrt{\mu}}(x-x_0)\Big)\right)^{\frac{1}{p}}. \tag{14}$$

where x_0 and c are constants.

$$\varpi(x,0)=\left(\frac{(p+1)(p+2)c}{2a}\,sech^2\Big(\frac{p}{2\sqrt{\mu}}(x-x_0)\Big)\right)^{\frac{1}{p}}. \tag{15}$$

where c and x_0 are constants. The exact solution at $\alpha=1$ is

$$\varpi(x,t)=\left(\frac{(p+1)(p+2)c}{2a}\,sech^2\Big(\frac{p}{2\sqrt{\mu}}(x-ct-x_0)\Big)\right)^{\frac{1}{p}}. \tag{16}$$

To apply the current method, equation (13) can be written in the form

$$\mathscr{D}_t^{\alpha}\varpi=\mathscr{F}\Big[\varpi,\varpi_x,\varpi_{xxt}\Big] \tag{17}$$

The initial condition $\varpi(x,0)=\psi_0(x)$

$$a_0=\varpi(x,0)=\left(\frac{(p+1)(p+2)c}{2a}\,sech^2\Big(\frac{p}{2\sqrt{\mu}}(x-x_0)\Big)\right)^{\frac{1}{p}}.$$

$$\begin{aligned}a_1=\mathscr{F}\Big[\varpi_0\Big]&=\frac{1}{\sqrt{\mu}}\times a\;2^{-1/p}\Big(\frac{1}{a}\times\\ &(p+1)(p+2)c\,\mathrm{sech}^2\Big(\frac{p(x-x0)}{2\sqrt{\mu}}\Big)\Big)^{1/p}\times\ldots\end{aligned} \tag{18}$$

To find the coefficient a_2:

$$a_2=\mathscr{F}_{\varpi}(a_1)+\mathscr{F}_{\varpi_x}(a_1)_x+\mathscr{F}_{\varpi_{xxt}}(a_1)_{xxt}$$

By using mathematica, we can compute a_2 as follows:

$$\begin{aligned}a_2=&\frac{a^2 2^{-\frac{1}{p}-1}p\mathrm{sech}^2\Big(\frac{p(x-x0)}{2\sqrt{\mu}}\Big)\ldots}{\mu}+\ldots\\ &+\frac{ac2^{-1/p}p^2\mathrm{sech}^2\Big(\frac{p(x-x0)}{2\sqrt{\mu}}\Big)\ldots}{\mu}+\ldots\end{aligned} \tag{19}$$

To find the coefficient a_3:

$$\begin{aligned}a_3=&\mathscr{F}_{\varpi}(a_2)+\mathscr{F}_{\varpi_x}(a_2)_x+\mathscr{F}_{\varpi_{xxt}}(a_2)_{xxt}+\mathscr{F}_{\varpi\varpi}(a_1)^2\\ &+2\mathscr{F}_{\varpi\varpi_x}(a_1)(a_1)_x\mathscr{F}_{\varpi_x\varpi_x}((a_{1x})^2+2\mathscr{F}_{\varpi\varpi_{xxt}}(a_1)(a_1)_{xxt}\\ &+2\mathscr{F}_{\varpi_x\varpi_{xxt}}(a_{1x})(a_1)_{xxt}+\mathscr{F}_{\varpi_{xxt}\varpi_{xxt}}((a_1)_{xxt})^2.\end{aligned}$$

Hence a_3 becomes:

$$\begin{aligned}a_3=&\frac{3a^3 2^{-\frac{1}{p}-1}p^2\mathrm{sech}^2\Big(\frac{p(x-x0)}{2\sqrt{\mu}}\Big)\ldots}{\mu^{3/2}}+..\\ &+\frac{a^3 2^{1-\frac{1}{p}}p^2\left(\frac{(p+1)(p+2)c\,\mathrm{sech}^2\Big(\frac{p(x-x0)}{2\sqrt{\mu}}\Big)}{a}\right)^{1/p}\ldots}{\mu^{3/2}}+\ldots\end{aligned} \tag{20}$$

We can express the solution for equation (13) in the following form by substituting into equation (8):

$$\begin{aligned}\varpi(x,t)=&a_0+a_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+a_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\end{aligned} \tag{21}$$

where a_0, a_1, a_2, a_3, and so on, are constants.

$$\begin{aligned}\varpi(x,t)=&a_0+a_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+a_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\end{aligned} \tag{22}$$

We can calculate other coefficients, but we will stop at a_3 and compare the error results between exact solution and

the approximate solution obtained from this series.
Now, we can discuss three cases for the proposed equation. case I : If we take $p = 1$ at equation (13), we get the numerical results for FEWE. case II : If we take$p = 2$ at equation (13), we get the numerical results for FMEWE. case III : If we take $p = 3$ at equation (13), we get the numerical results for FGEWE, for further details, see [32].

5.1 Case I: $p = 1$

By choosing suitable values of x_0, μ, c and $a = p(p+1)$, we can display the exact and approximate results and the error estimation for various values of x, t, and α.
Table 1 presents the precise and estimated solutions, as well as the absolute error, for different x and α values at $t = 10$. These results are calculated at $a = 2$, $\mu = 1$, $c = 0.001$ and $x_0 = 0$.
Table 1 indicates that the estimated solution is in good agreement with the exact solution, as the maximum absolute error is only (2.98355 E-6), which is an acceptable value. Moreover, the outcomes are more accurate when the value of α is closer to an integer order.
By selecting suitable values for x_0, μ, c, and $a = p(p+1)$, the current method enables us to provide precise and approximate solutions, as well as error estimations, for different x, t, and α values.
Table 2 shows the exact and approximate solutions and the absolute error for $x = -4, x = 6$ at different values of t when $\alpha = 0.8$. From Table 2, we observe that the absolute error is still small when the time is increased up to $t = 20$, so we conclude that our method is very successful.

5.2 Case II: $p = 2$

We get the solution of the time-fractional modified equal width equation by substituting $p = 2$ in the equation(13).
At varying x, t, and α values, Table 3 displays the precise and estimated solutions, along with the error estimation, for Case II. We can see from Table 3 that the error estimation is acceptable because the maximum value of the absolute error is (1.74618 E-5). Moreover, when α is closer to the integer order, the results are better. Table 4 displays the precise and estimated outcomes of the FMEW equation (13) using the current method, as well as the absolute error, for different x and t values at $\alpha = 0.7$.
The Table 4 reveals that the absolute error remains small even when the time reaches $t = 20$, indicating that the current method is very effective.

5.3 Case III: $p = 3$

The solution of equation (13) with $p = 3$ gives us the time-fractional GEWE solution. Table 5 shows the comparison between the exact and approximated solutions and their absolute errors for various values of α at $t = 2$. As Table 5 shows, the approximate solution is close to the exact solution because the highest value of the absolute error is $(9.89236E-5)$, which is a reasonable value. Moreover, when α is closer to the integer order, the results are better. The exact and approximate results of the FGMEW equation (13) using the current technique are displayed in Table 6, together with the absolute error for $\alpha = 1$ at different values of x,t. The absolute error's maximum value is $(9.89236E-5)$, which shows that the approximate solution is close to the exact one. This value is acceptable Table 6. Furthermore, when α is nearer to the integer order, the results are better.

6 Graphical illustrations

The use of graphs is a powerful method for illustrating the relationship between the various parameters of a solution. Therefore, in this section, 2D and 3D graphs are utilized to visually demonstrate the solution $\varpi(x,t)$ for different values of α and t, and how they affect the solution. The initial two-dimensional graph displays the soliton wave solution, illustrating the trajectory of the wave at different α values by plotting $\varpi(x,t)$ against x.The findings indicate a convergence of the outcomes obtained via fractional-order analysis towards those attained through integer-order analysis. For different values of t, the wave solutions at $\alpha = 1$ are shown in the second 2D graph. The soliton's amplitude remains unchanged as it moves to the right. A 3D graph of the solution at $\alpha = 1$ is shown in the third plot.
The solution of equation (13) for the time-fractional EWE at $x_0 = 0, \mu = 4, c = 0.01, a = 2$ and $p = 1$ is shown in Figure 1. The soliton does not change its amplitude as it moves to the right. The third plot displays a 3D view of the solution at $\alpha = 1$. Figure 2 shows the solution of equation (13) for the FMEWE at $x_0 = 0, \mu = 6, c = 0.005, a = 6$ and $p = 2$. The solution of equation (13) for the FGEWE at $x_0 = 0, \mu = 4, c = 0.005, a = 12$ and $p = 3$ is shown in Figure 3.

7 Discussion

In this section, the results of applying the proposed solution method to the FGEW equation will be presented and discussed. The performance of the method will be evaluated by comparing the approximate solutions with the numerical results obtained using other existing methods. To verify the accuracy of the proposed method for obtaining solutions of the nonlinear fractional general equal width equation, three cases were considered. The approximate solutions obtained using the proposed

Table 1: The solutions for $p = 1$, including the exact and approx. solutions along with the absolute errors at $t = 10$

	$\alpha = 0.25$			$\alpha = 0.6$		
x	Exact	Approx.	Error	Exact	Approx	Error
-10	2.69665 E-7	2.72375 E-7	2.70964 E-9	2.69660 E-7	2.72374 E-7	2.70927 E-9
-7	5.40708 E-6	5.46120 E-6	5.41263 E-8	5.40708 E-6	5.46106 E-6	5.39770 E-8
-4	0.00010495	0.00010593	9.74801 E-7	0.00010495	0.00010588	9.21148 E-7
-1	0.00117421	0.00117719	2.98355 E-6	0.00117421	0.00117403	1.81199 E-6
5	0.00004028	0.00003989	3.89285 E-7	0.00004028	0.00003990	3.81436 E-7
8	2.03163 E-6	2.01144 E-6	2.01856 E-8	2.03163 E-6	2.0114 E-6	2.01650 E-8

Table 2: The solutions for $p = 1$, including the exact and approx. solutions along with the absolute errors at $\alpha = 0.8$

	$x = -4$			$x = 6$		
t	Exact	Approx.	Error	Exact	Approx.	Error
0	0.00010597	0.00010597	0.00000	0.000014799	0.000014799	0.00000
5	0.00010546	0.00010589	4.25568 E-7	0.000014872	0.000014800	7.21155 E-8
10	0.00010495	0.00010583	4.25568 E-7	0.000014947	0.000014802	7.21155 E-8
15	0.00010445	0.00010577	1.32059 E-6	0.000015021	0.000014803	2.18448 E-7
20	0.00010395	0.00010572	1.77195 E-6	0.000015096	0.000014804	2.92303 E-7

Table 3: The solutions for $p = 2$, including the exact and approx. solutions along with the absolute errors at $t = 15$

	$\alpha = 0.35$			$\alpha = 0.75$		
x	Exact	Approximate	Abs.Error	Exact	Approximate	Abs.Error
-10	2.82859 E-6	2.87134 E-6	4.27487 E-8	2.82859 E-6	2.87134 E-6	4.27487 E-8
-6	0.000154435	0.000156769	2.33389 E-6	0.000154435	0.000156769	2.33377 E-6
-4	0.001140770	0.001157970	1.72021 E-5	0.001140770	0.001157920	1.71518 E-5
4	0.001175480	0.001158020	1.74618 E-5	0.001175480	0.001158070	1.74116 E-5
6	0.000159138	0.000156769	2.36916 E-6	0.000159138	0.000156769	2.36904 E-6
10	2.91474 E-6	2.87134 E-6	4.33948 E-8	2.91474 E-6	2.87134 E-6	4.33948 E-8

technique were compared with the solutions obtained using VIM in these cases. The results of the comparison are presented in Tables (7-10). It was observed that the approximate solutions obtained using the proposed method were more accurate and closer to the exact solutions compared to those obtained using VIM. The absolute errors of the proposed method were found to be smaller than those of VIM, as shown in Tables (7-10). The calculation of absolute errors in these cases confirms the accuracy and capability of the proposed method in obtaining solutions of nonlinear fractional general equal width equations.
Table 7 compares the absolute errors obtained using the proposed method with those obtained using the variational-iteration method, as reported in reference [18], for case I. A comparison of the absolute error calculated by the variational-iteration method in [18] and the proposed technique at case II is shown in Table 8. The variational-iteration method is a way to find the solution $\varpi(x,t)$ using different values of α and t. The Table illustrates how the suggested technique performs better than the variational-iteration method. The absolute error calculated by the variational-iteration method in [18] and the proposed method for case III are compared in Table 9 In addition, Table 10 compares the absolute errors obtained using the homotopy perturbation transform method, as reported in [21], with those obtained using the proposed method for $t = 0.1$, $a = 2$, $\mu = 1$, $c = 1$, $\alpha = 0.8, 1$, $x_0 = 15$, and $p = 1$. Based on the results presented in Tables (7-9), we can conclude that the proposed method is more effective and accurate in obtaining solutions of FGEWEs compared to VIM. However, in Table 10, it is observed that the absolute error values obtained using the proposed method and the homotopy perturbation transform method are similar when $\alpha = 1$, but for values of α less than one, the proposed method performs better. We presented data in tables with different values for x and t, as well as various fractional derivative orders α, to show how our technique can solve nonlinear fractional partial differential equations with initial conditions. Through our analysis, we found that the technique consistently produced results with minimal error, even when these variables were changed. This suggests that the technique is highly efficient and can be easily applied to a variety of nonlinear fractional partial differential equations with initial conditions. It also provides a general framework that can be applied to a variety of physical systems. However, one potential demerit of the method is that it may not be suitable for all types of NFPDEs, and the accuracy of the results may depend on the specific system being modeled. Additionally, the method's limitations in

Table 4: The solutions for $p=2$, including the exact and approx. solutions along with the absolute errors at $\alpha=0.7$

	$x=-5$			$x=4$		
t	Exact	Approximate	Abs.Error	Exact	Approximate	Abs.Error
0	0.00115799	0.00115799	0.00000	0.000426126	0.000426126	0.00000
5	0.00115222	0.00115796	5.74007 E-6	0.000428262	0.000426127	2.13419 E-6
10	0.00114648	0.00115794	1.14633 E-5	0.000430408	0.000426128	4.27968 E-6
15	0.00114077	0.00115793	1.71608 E-5	0.000432565	0.000426129	6.43607 E-6
20	0.00113508	0.00115791	2.28316 E-5	0.000434733	0.000426130	8.60335 E-6

Table 5: The solutions for $p=3$, including the exact and approx. solutions along with the absolute errors at $t=2$

	$\alpha=0.5$			$\alpha=1$		
x	Exact	Approx.	Abs.Error	Exact	Approx.	Abs. Error
-5	0.00046709	0.00046718	9.34126 E-8	0.00046709	0.00046718	9.34238 E-8
-3	0.00345107	0.00345171	6.46521 E-7	0.00345107	0.00345175	6.79894 E-7
-1	0.02468990	0.02459100	9.89066 E-5	0.02468990	0.02467010	1.97560 E-5
1	0.02469880	0.02479770	9.89236 E-5	0.02469880	0.02471860	1.97731 E-5
3	0.00345245	0.00345180	6.46659 E-7	0.00345245	0.00345177	6.80032 E-7
5	0.00046727	0.00046718	9.34313 E-8	0.00046727	0.00046718	9.34425 E-8

Table 6: The exact and approx. , as well as the absolute errors, for $p=3$ and $\alpha=1$

	$x=-1$			$x=2$		
t	Exact	Approx.	Abs. Error	Exact	Approx.	Abs. Error
0.2	0.02469390	0.02469350	3.62322 E-7	0.00936834	0.00936817	1.68007 E-7
0.6	0.02469300	0.02469180	1.21248 E-6	0.00936871	0.00936821	5.01163 E-7
1.2	0.02469170	0.02468740	4.25558 E-6	0.00936927	0.00936831	9.60640 E-7
1.6	0.02469080	0.02468130	9.49864 E-6	0.00936964	0.00936845	1.19377 E-6
2.2	0.02468940	0.02466180	2.76133 E-5	0.00937020	0.00936889	1.31006 E-6
2.4	0.02468900	0.02465100	3.78197 E-5	0.00937039	0.00936910	1.25392 E-6

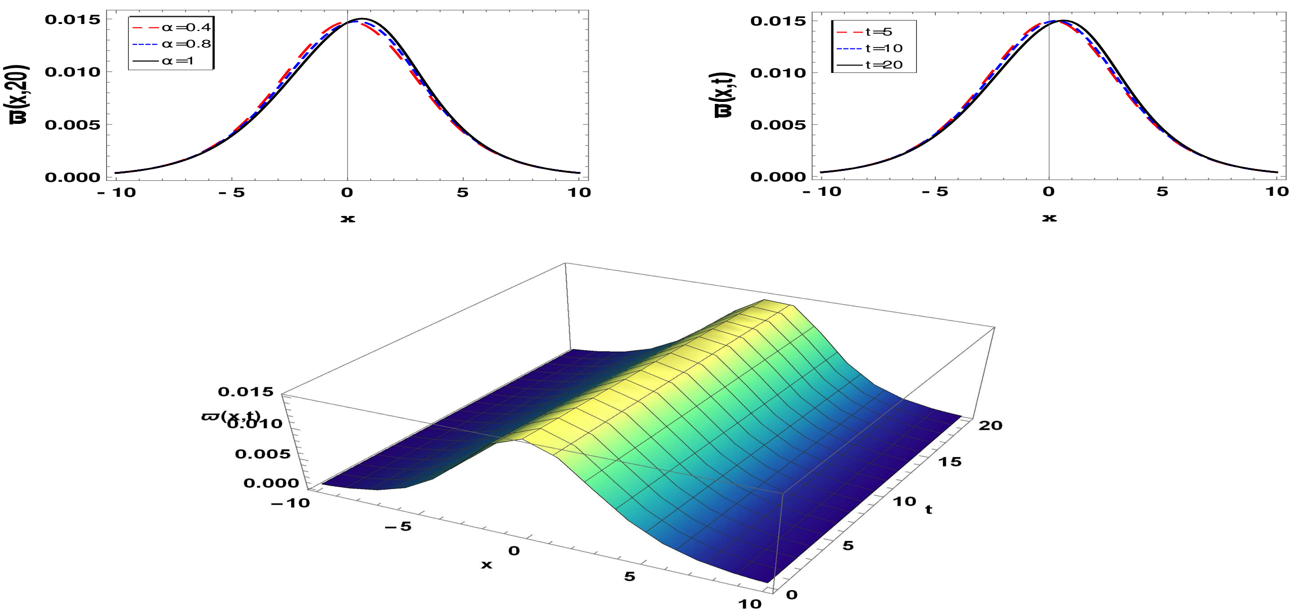

Fig. 1: The solution of time fractional-order equal-width equation at $x_0=0, \mu=4, c=0.01, p=1, a=2$

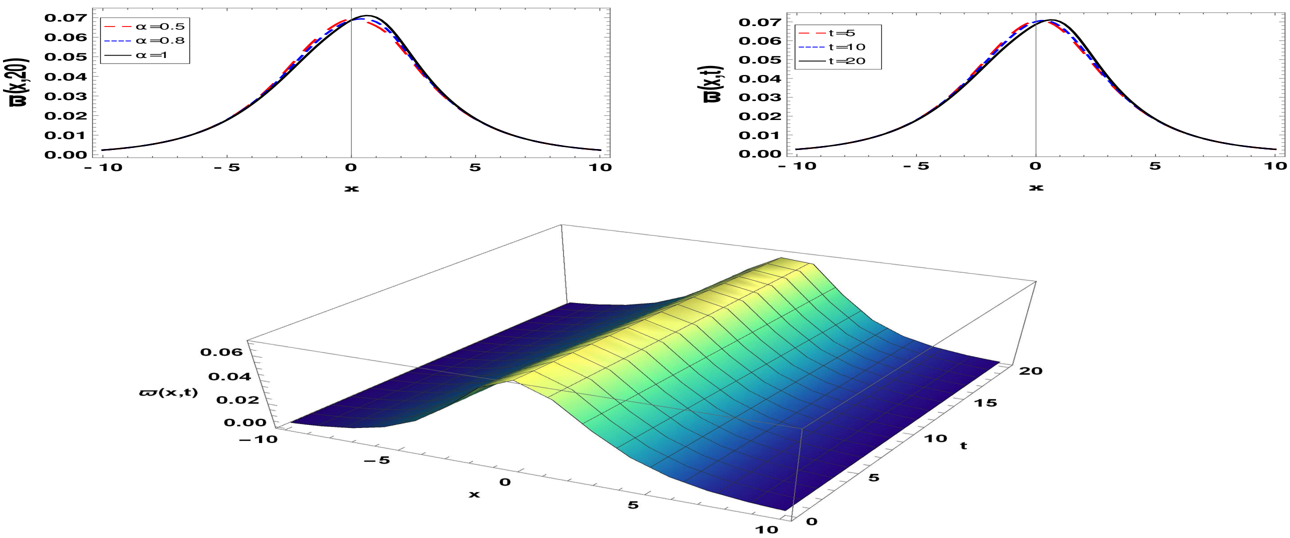

Fig. 2: The solution of time fractional-order modified equal-width equation at $x_0 = 0, \mu = 6, c = 0.005, p = 2, a = 6$

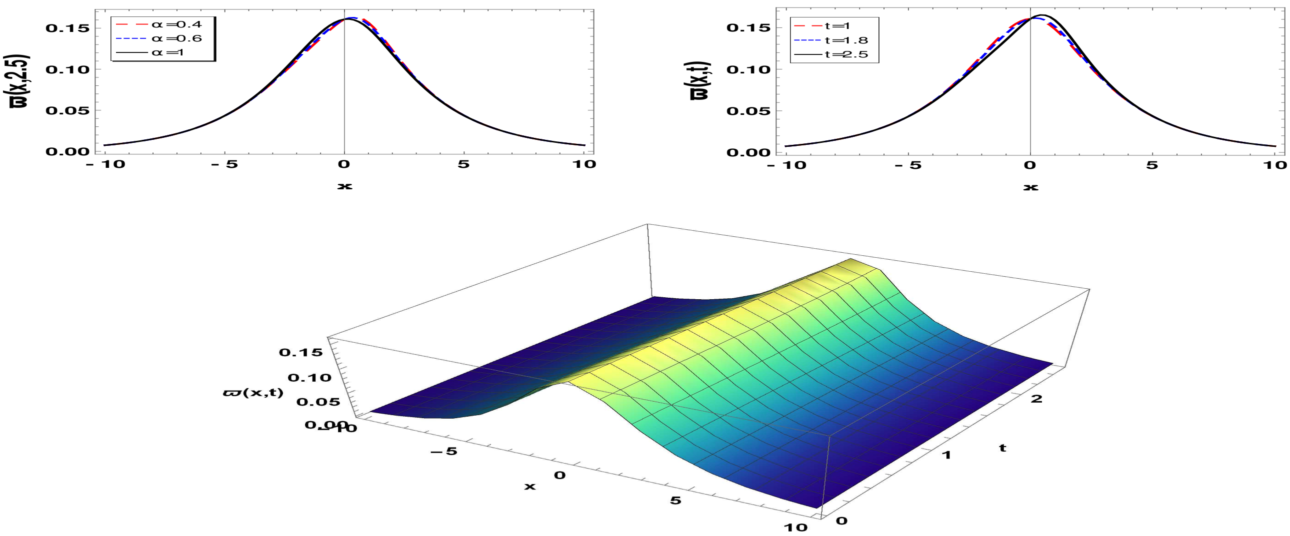

Fig. 3: The solution of time fractional-order General Equal-Width equation at $x_0 = 0, \mu = 4, c = 0.005, p = 3, a = 12$.

Table 7: Comparison between absolute errors of VIM [18] and the Current Method at $t = 1$

	$\alpha = 0.8$		$\alpha = 1$	
x	VIM error [18]	CM error	VIM error [18]	CM error
0	3.75 E-13	3.4022 E-12	3.75000 E -13	3.0000 E-12
0.1	5.95865 E-6	1.65090 E-9	5.95712 E-6	1.48649 E-9
0.2	2.31632 E-5	3.23706 E-9	2.31603 E-5	2.91354 E-9
0.3	4.97765 E-5	4.69162 E-9	4.97723 E-5	4.22000 E-9
0.4	8.30341 E-5	5.96028 E-9	8.30290 E-5	5.35619 E-9
0.5	1.19629 E-4	7.00129 E-9	1.19624 E-4	6.28402 E-9
0.6	1.56145 E-4	7.78754 E-9	1.56140 E-4	6.97887 E-9
0.7	1.89464 E-4	8.30728 E-9	1.89458 E-4	7.43026 E-9
0.8	2.17082 E-4	8.56344 E-9	2.17077 E-4	7.64125 E-9
0.9	2.37311 E-4	8.57191 E-9	2.37307 E-4	7.62679 E-9
1	2.49334 E-4	8.35898 E-9	2.49330 E-4	7.41133 E-9

Table 8: Comparison between absolute errors of VIM [18] and the Current Method at $t = 1$

	$\alpha = 0.5$		$\alpha = 1$	
x	VIM error [18]	CM error	VIM error [18]	CM error
0	7.07107 E-11	2.47487 E-9	7.07107 E-11	2.47487 E-9
0.1	1.04392 E-4	7.97582 E-7	1.04395 E-4	6.90625 E-7
0.2	3.73951 E-4	1.50510 E-6	3.73957 E-4	1.30254 E-6
0.3	7.03065 E-4	2.04674 E-6	7.03072 E-4	1.76893 E-6
0.4	9.77361 E-4	2.38205 E-6	9.77370 E-4	2.05447 E-6
0.5	1.12255 E-3	2.50674 E-6	1.12256 E-3	2.15565 E-6
0.6	1.12290 E-3	2.44666 E-6	1.12298 E-3	2.09550 E-6
0.7	1.00942 E-3	2.24632 E-6	1.00942 E-3	1.91339 E-6
0.8	8.32649 E-3	1.95646 E-6	8.32655 E-4	1.65410 E-6
0.9	6.40229 E-3	1.62423 E-6	6.40234 E-3	1.35912 E-6
1	4.64536 E-3	1.28735 E-6	4.64540 E-3	1.06155 E-6

Table 9: The exact solution, along with the approximate solutions and absolute errors

	$\alpha = 0.5$		$\alpha = 1$	
x	VIM error [18]	CM error	VIM error [18]	CM error
0	3.27593 E-10	3.24312 E-8	3.27593 E-10	3.24316 E-8
0.1	5.96967 E-3	1.59171 E-5	5.96968 E-3	7.21722 E-6
0.2	1.87328 E-2	2.84655 E-5	1.87329 E-2	1.28826 E-5
0.3	2.84744 E-2	3.55705 E-5	2.84744 E-2	1.65431 E-5
0.4	9.93070 E-2	3.70839 E-5	2.99308 E-2	1.65431 E-5
0.5	2.46717 E-2	3.42986 E-5	2.46717 E-2	1.50936 E-5
0.6	1.70609 E-2	2.90516 E-5	1.70609 E-2	1.25316 E-5
0.7	1.03465 E-2	2.29847 E-5	1.03465 E-2	9.62866 E-6
0.8	5.68225 E-3	1.72059 E-5	5.68225 E-3	6.90045 E-6
0.9	2.89539 E-3	1.22815 E-5	2.89539 E-3	4.60543 E-6
1	1.39460 E-3	8.38339 E-6	1.39460 E-3	2.81589 E-6

Table 10: Comparison between absolute errors of HPTM and the Current Method at $t = 0.1, a = 1, c = 1, \mu = 1$, and $x_0 = 15$.

	$\alpha = 0.8$		$\alpha = 1$	
x	HPTM error [21]	CM error	HPTM error [21]	CM error
0.5	6.81188E-4	1.09706 E-6	6.25000 E-8	1.09706 E-6
1	2.72475 E-3	1.80873 E-6	2.50000 E-7	1.80874 E-6
1.5	6,13069 E-3	2.98206 E-6	5.63000 E-7	2.98208 E-6
2	1.08990 E-2	4.91648 E-6	1.00000 E-6	4.91655 E-6
2.5	1.70297 E-2	8.10563 E-6	1.56300 E-6	8.10582 E-6
3	2.45227 E-2	1.33632 E-5	2.25000 E-6	1.33637 E-5
3.5	3.33782 E-2	2.20301 E-5	3.06000 E-6	2.20315 E-5
4	4.35960 E-2	3.63160 E-5	4.00000 E-6	3.63198 E-5
4.5	5.51762 E-2	5.98598 E-5	5.06000 E-6	5.98703 E-5
5	6.81188 E-2	9.86511 E-5	6.25000 E-6	9.86797 E-5

terms of the range of values for the fractional derivative order and time must be considered. Nevertheless, the presented approach is a valuable contribution to the field of NFPDEs and provides a promising avenue for future research.

8 Conclusion

In this paper, we proposed an innovative technique for finding approximate solutions of the fractional-order equal-width (FGEW) equation. The technique is based on the fractional Taylor's series and Caputo operator, which reduces the computation effort for solving nonlinear fractional partial differential equations (FPDEs). We demonstrated the effectiveness of the proposed technique by obtaining an analytical approximate solution for the FGEW equation in the form of a series. The results obtained were compared with numerical results obtained by the VIM, HPTM and good agreement was observed. We showed how our technique can solve nonlinear fractional partial differential equations with initial conditions by presenting data in tables with different values for x and t, and various orders of fractional derivative α. Our method can also be applied to other types of nonlinear fractional partial differential equations and more complex models, which could improve our knowledge of physical systems. Moreover, the technique can be extended to boundary-value problems, which have various applications.

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