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Fractional Dynamic Inequalities of Hardy's Type on Time Scales

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Abstract: In this paper, I obtain some new fractional dynamic inequalities of Hardy and reversed Hardy via time scales. The main outcomes achieved through utilizing inequality of Hölder and reversed Hölder on fractional time scales. The inequalities obtained will lead to the classical inequalities which are established earlier.

Keywords: Conformable fractional calculus, Hardy inequality, Hölder inequality, Reversed Hardy inequality, Time scales

1 Introduction

Hardy in [1], established inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}h(i)\right)^{\beta}\leq\left(\frac{\beta}{\beta-1}\right)^{\beta}\sum_{\beta=1}^{\infty}h^{\beta}(n), \tag{1}$$

where $\beta>1$ and $h(n)\geq 0$ for $n\geq 1$. Hardy in [2], demonstrated continuous inequality

$$\int_0^{\infty}\left(\frac{1}{\eta}\int_0^{\eta}\lambda(\eta)d\eta\right)^{\beta}d\eta\leq\left(\frac{\beta}{\beta-1}\right)^{\beta}\int_0^{\infty}\lambda^{\beta}(\eta)d\eta, \tag{2}$$

for $\beta>1$, $\lambda\geq 0$, integrable on any interval $(0,\eta)$, λ^{β} is integrable and convergent over $(0,\infty)$. The constant $(\beta/(\beta-1))^{\beta}$ in (1) and (2) is the best possible.

Hardy in [3], to generalize inequality (1), he deduced that if $\beta>1$ and λ be non-negative for $\eta\geq 0$, then

$$\int_0^{\infty}\eta^{-\gamma}\left(\int_0^{\eta}\lambda(s)ds\right)^{\beta}d\eta\leq\left(\frac{\beta}{\gamma-1}\right)^{\beta}\int_0^{\infty}\eta^{\beta-\gamma}\lambda^{\beta}(\eta)d\eta,\ \text{for}\ \gamma>1, \tag{3}$$

and

$$\int_0^{\infty}\eta^{-\gamma}\left(\int_{\eta}^{\infty}\lambda(s)ds\right)^{\beta}d\eta\leq\left(\frac{\beta}{1-\gamma}\right)^{\beta}\int_0^{\infty}\eta^{\beta-\gamma}\lambda^{\beta}(\eta)d\eta,\ \text{for}\ \gamma<1. \tag{4}$$

The constants $(\beta/(\gamma-1))^{\beta}$ and $(\beta/(1-\gamma))^{\beta}$ are the best possible.

Sulaiman in [4], proved that if $\lambda\geq 0$, $\mu>0$, $\eta/\mu(\eta)$ be a non-increasing function, $\beta>1$ and $0<\omega<1$, then

$$\int_0^{\infty}\left(\frac{\int_0^{\eta}\lambda(\eta)d\eta}{\mu(\eta)}\right)^{\beta}d\eta\leq\frac{1}{\omega(1-\omega)^{\beta-1}(\beta-1)}\int_0^{\infty}\left(\frac{\eta\lambda(\eta)}{\mu(\eta)}\right)^{\beta}d\eta, \tag{5}$$

and also if $\lambda\geq 0$, $\mu>0$, $\eta/\mu(\eta)$ be a non-decreasing function, $0<\beta\leq 1$ and $\omega>0$, then

$$\int_0^{\infty}\left(\frac{\int_0^{\eta}\lambda(\eta)d\eta}{\mu(\eta)}\right)^{\beta}d\eta\geq\frac{1}{\omega(1+\omega)^{\beta-1}(1-\beta)}\int_0^{\infty}\left(\frac{\eta\lambda(\eta)}{\mu(\eta)}\right)^{\beta}d\eta. \tag{6}$$

Sroysang in [5], proved that if $\lambda\geq 0$, $\mu>0$, $\eta/\mu(\eta)$ be a non-increasing function, $0<\omega<1$, $\beta>1$ and $q>\beta-\omega(\beta-1)$, we get

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$$\int_0^\infty \frac{\left(\int_0^\eta \lambda(\eta)\,d\eta\right)^\beta}{\mu^q(\eta)} d\eta \leq \frac{1}{(1-\omega)^{\beta-1}((\omega-1)(\beta-1)+q-1)} \int_0^\infty \frac{(\eta\lambda(\eta))^\beta}{\mu^q(\eta)} d\eta, \tag{7}$$

and also if $\lambda \geq 0$, $\mu > 0$, $\eta/\mu(\eta)$ be a non-decreasing function, $0 < \beta < 1$, $\omega > 0$ and $q > \beta + \omega(\beta-1)$, then

$$\int_0^\infty \frac{\left(\int_0^\eta \lambda(\eta)\,d\eta\right)^\beta}{\mu^q(\eta)} d\eta \geq \frac{1}{(1+\omega)^{\beta-1}((\omega+1)(1-\beta)+q-1)} \int_0^\infty \frac{(\eta\lambda(\eta))^\beta}{\mu^q(\eta)} d\eta. \tag{8}$$

Mehrez in [6], proved that if $\lambda \geq 0$, $\mu > 0$, $\eta/\mu(\eta)$ be a non-increasing function, $\beta > 1$, $0 < \omega < 1$, $q > \frac{\beta-\omega(\beta-1)}{2}$ and $\Lambda(\eta) = \int_0^\eta \mu(\eta)\,d\eta$, then

$$\int_0^\infty \frac{\left(\int_0^\eta \lambda(\eta)\,d\eta\right)^\beta}{\Lambda^q(\eta)} d\eta \leq \frac{1}{(1-\omega)^{\beta-1}((\beta-1)(\omega-1)+2q-1)} \int_0^\infty \frac{(\eta\lambda(\eta))^\beta}{\Lambda^q(\eta)} d\eta, \tag{9}$$

and also if $\lambda \geq 0$, $\mu > 0$, $\eta/\mu(\eta)$ be a non-decreasing function, $0 < \beta < 1$, $\omega > 0$, $q > \frac{\beta+\omega(\beta-1)}{2}$ and $\Lambda(\eta) = \int_0^\eta \mu(\eta)\,d\eta$, then

$$\int_0^\infty \frac{\left(\int_0^\eta \lambda(\eta)\,d\eta\right)^\beta}{\Lambda^q(\eta)} d\eta \geq \frac{1}{(1+\omega)^{\beta-1}((\omega+1)(1-\beta)+2q-1)} \int_0^\infty \frac{(\eta\lambda(\eta))^\beta}{\Lambda^q(\eta)} d\eta. \tag{10}$$

Hilger established time scales theory, this notion was introduced in his thseis of Ph.D. to combine the discrete and continuous theorems into a single theorem [8]. Řehák in [7], employed a technique introduced by Elliott [9] and established the Hardy inequality on a time scale (2). In specifically, he was demonstrated that if the integral $\int_\theta^\infty (\xi(t))^\beta \Delta t$, exists (finite number), $\beta > 1$ and ξ be a non-negative *rd*-continuous, then

$$\int_\theta^\infty \left(\frac{1}{\sigma(\eta)-\theta}\int_\theta^{\sigma(\eta)} \xi(t)\Delta t\right)^\beta \Delta\eta \leq \left(\frac{\beta}{\beta-1}\right)^\beta \int_\theta^\infty (\xi(t))^\beta \Delta t. \tag{11}$$

Additionally, if $\xi(\eta)/\eta \to 0$ as $\eta \to \infty$, the constant is the best one.

lately, a variety inequalities that extend and generalize Hardy inequality have been established on a time scale $\mathbb{T}$. For extra details about Hardy dynamic inequalities via time scales; see [10]. For extra details about time scale analysis; see ([11], [12]) that organize and summarize most time scale calculus.

Theory of fractional calculus has a significant impact in mathematical analysis and applications. The research field of fractional calculus back to Riemann, Liouville and Abel; see [13] .

In [14] and [15], authors enlarged fractional order calculus into conformable calculus and presented new derivative definition.

Saker et al. in [16], gave conformable fractional version of Hardy inequality and shown, if a funtion ξ, is a non-negative over $(0,\infty)$, $0 < \alpha \leq 1$, $\beta > 1$ and $\eta^{\alpha-1}\xi(\eta)$ is continuous on $[0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{\eta}\int_0^\eta \xi(s)d_\alpha s\right)^p d_\alpha\eta \leq \left(\frac{\beta}{\beta-\alpha}\right)^\beta \int_0^\infty (\eta^{\alpha-1}\xi(\eta))^\beta d_\alpha\eta.$$

In [17] and [18], authors formed fractional calculus via time scales by combining a time scales calculus with a conformable fractional calculus.

The important question currently: Is it achievable to demonstrate new fractional generalization of Hardy and reversed Hardy inequalities via time scales? The paper goal is providing affirmative response to the above question. The following indicates how the paper is structured. The second section includes the basics of time scales calculus also some preliminaries about calculus of the conformable fractional on time scales. The third section we derive some extensions of inequalities Hardy and reversed Hardy via fractional time scales, in addition to conclusion section.

2 Preliminaries and Basic Lemmas

A time scale $\mathbb{T}$ defines as a non-empty closed subset of $\mathbb{R}$ (real line). The operators of backward and forward jump are presented as:

$$\rho(\gamma) := \sup\{\delta \in \mathbb{T} : \delta < \gamma\}$$

and

$$\sigma(\gamma) := \inf\{\delta \in \mathbb{T} : \delta > \gamma\}$$

respectively, with $\inf\mathbb{T} = \sup\emptyset$. $\gamma \in \mathbb{T}$, is known as right-dense when $\sigma(\gamma) = \gamma$ and $\gamma < \sup\mathbb{T}$, left-dense when $\rho(\gamma) = \gamma$ and $\gamma > \inf\mathbb{T}$, right-scattered when $\sigma(\gamma) > \gamma$ and left-scattered when $\rho(\gamma) < \gamma$.

$\mu : \mathbb{T} \to \mathbb{R}$ named right-dense continuous function (*rd*-continuous) whene μ continuous at points of left-dense and right-dense in $\mathbb{T}$, limits of left hand are finite and exist. $C_{rd}(\mathbb{T})$ indicated to all rd-continuous functions set.

The graininess function μ on $\mathbb{T}$ defined as:

$$\mu(\gamma) := \sigma(\gamma) - \gamma \geq 0,$$

each $\lambda : \mathbb{T} \to \mathbb{R}$, $\lambda^{\sigma}(\gamma)$ indicates to $\lambda(\sigma(\gamma))$.Suppose $\sup \mathbb{T} = \infty$, and define $[\eta,\tau]_{\mathbb{T}}$ as $[\eta,\tau]_{\mathbb{T}} := [\eta,\tau] \cap \mathbb{T}$. Define $\lambda^{\Delta}(\gamma)$ that number (if exists) with for anygiven $\varepsilon > 0$ there exist a neighborhood N to γ where

$$|[\lambda(\sigma(\gamma)) - \lambda(\delta)] - \lambda^{\Delta}(\gamma)[\sigma(\gamma) - \delta]| \leq \varepsilon|\sigma(\gamma) - \delta|, \quad \forall \delta \in N,$$

and we indicate to $\lambda^{\Delta}(\gamma)$ as Δ-derivative to λ at γ and λ be Δ-differentiable at γ. For Δ-differentiable functions λ and μ, (where $\mu\mu^{\sigma} \neq 0$ and μ^{σ} is the $\mu \circ \sigma$), we have

$$(\lambda\mu)^{\Delta} = \lambda^{\Delta}\mu + \lambda^{\sigma}\mu^{\Delta} = \lambda\mu^{\Delta} + \lambda^{\Delta}\mu^{\sigma}.$$

$$\left(\frac{\lambda}{\mu}\right)^{\Delta} = \frac{\lambda^{\Delta}\mu - \lambda\mu^{\Delta}}{\mu\mu^{\sigma}}.$$

Through the paper, Δ-integral is specified as, if $\theta^{\Delta}(\gamma) = \phi(\gamma)$, the Cauchy integral of ϕ is defined as

$$\int_{h}^{\gamma} \phi(\delta)\Delta\delta := \theta(\gamma) - \theta(h).$$

If $\phi \in C_{rd}(\mathbb{T})$,then the Cauchy integral $\theta(\gamma) := \int_{\gamma_0}^{\gamma} \phi(\delta)\Delta\delta$ exists, $\gamma_0 \in \mathbb{T}$ and satisfies $\theta^{\Delta}(\gamma) = \phi(\gamma)$, $\gamma \in \mathbb{T}$; see [11].

The chain rule over $\mathbb{T}$; (see [11, Theorem 1.87]), is presented b

$$(\mu \circ \zeta)^{\Delta}(\gamma) = \mu'(\zeta(r))\,\zeta^{\Delta}(\gamma), \text{ where } r \in [\gamma, \sigma(\gamma)],$$

where $\zeta : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable and $\mu : \mathbb{R} \to \mathbb{R}$ be continuously differentiable.

The Keller chain rule; (see [11, Theorem 1.90]), is presented by

$$(u^{\tau}(\gamma))^{\Delta} = \tau \int_{0}^{1} [hu^{\sigma}(\gamma) + (1-h)u(\gamma)]^{\tau-1}\, dhu^{\Delta}(\gamma).$$

The integration by parts form over $\mathbb{T}$ to θ, $\phi \in \mathbb{T}$, is presented by

$$\int_{h}^{k} \theta(\gamma)\phi^{\Delta}(\gamma)\Delta\gamma = \theta(\gamma)\phi(\gamma)|_{h}^{k} - \int_{h}^{k} \theta^{\Delta}(\gamma)\phi^{\sigma}(\gamma)\Delta\gamma. \tag{12}$$

Fubini theorem over $\mathbb{T}$ was presented in coming lemma; (see [19]).

Lemma 1. *Suppose that (H,M,θ_{Δ}), (G,L,η_{Δ}) are finite dimensional measured spaces over time scales, if $\chi : H \times G \to \mathbb{R}$ is Δ-integrable and define functions*

$$\varphi(\delta) = \int_{r} \chi(r,\delta)d\theta_{\Delta}(r), \ \text{for} \ \delta \in G,$$

and

$$\psi(r) = \int_{\delta} \chi(r,\delta)d\eta_{\Delta}(\delta), \ \text{for} \ r \in H,$$

then ψ is Δ-integrable over H, φ is Δ-integrable over G and

$$\int_{H} d\theta_{\Delta}(r) \int_{G} \chi(r,\delta)d\eta_{\Delta}(\delta) = \int_{G} d\eta_{\Delta}(\delta) \int_{H} \chi(r,\delta)d\theta_{\Delta}(r).$$

Definition 1. *Suppose a function $\lambda : \mathbb{T} \to \mathbb{R}$ be a real valued over $\mathbb{T}$ and $\alpha \in (0,1]$, $\gamma \in \mathbb{T}^{k}$. Then, for $\gamma > 0$, we define $T_{\alpha}^{\Delta}\lambda(\gamma)$ be the number (provided exist) with for any given $\varepsilon > 0$,there exist a neighborhood N of γ where*

$$\left|[\lambda^{\sigma}(\gamma) - \lambda(\delta)]\gamma^{1-\alpha} - T_{\alpha}^{\Delta}\lambda(\gamma)(\sigma(\gamma) - \delta)\right| \leq \varepsilon\,|\sigma(\gamma) - \delta)|, \ \forall \gamma \in N.$$

$T_{\alpha}^{\Delta}\lambda(\gamma)$ is named conformable fractional derivative to λ of order α at γ over $\mathbb{T}$ and $T_{\alpha}^{\Delta}\lambda(0) = \lim_{\gamma\to 0} T_{\alpha}^{\Delta}\lambda(\gamma)$.

Theorem 1.*Suppose that $\alpha \in (0,1]$ and Ω, $\Phi : \mathbb{T} \to \mathbb{R}$ are α-fractional differentiable. Then*

(i)The sum $\Omega + \Phi : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable and

$$T_{\alpha}^{\Delta}(\Omega + \Phi) = T_{\alpha}^{\Delta}(\Omega) + T_{\alpha}^{\Delta}(\Phi).$$

(ii)For $n \in \mathbb{R}$, $n\Omega : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable and

$$T_{\alpha}^{\Delta}(n\Omega) = nT_{\alpha}^{\Delta}(\Omega).$$

(iii)If Ω and Φ are rd-continuous we get $\Omega\Phi : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable and

$$T_{\alpha}^{\Delta}(\Omega\Phi) = T_{\alpha}^{\Delta}(\Omega)\Phi + (\Omega \circ \sigma)T_{\alpha}^{\Delta}(\Phi) = T_{\alpha}^{\Delta}(\Omega)(\Phi \circ \sigma) + \Omega T_{\alpha}^{\Delta}(\Phi).$$

(iv)If Ω is rd-continuous, then $1/\Omega$ be α-fractional differentiable and

$$T_{\alpha}^{\Delta}\left(\frac{1}{\Omega}\right) = -\frac{T_{\alpha}^{\Delta}(\Omega)}{\Omega(\Omega \circ \sigma)}.$$

(v)If Ω and Φ are rd-continuous, then Ω/Φ is α-fractional differentiable and

$$T_{\alpha}^{\Delta}(\Omega/\Phi) = \frac{T_{\alpha}^{\Delta}(\Omega)\Phi - \Omega T_{\alpha}^{\Delta}(\Phi)}{\Phi(\Phi \circ \sigma)},$$

valid $\forall \gamma \in \mathbb{T}^{k}$ for $\Phi(\gamma)(\Phi(\sigma(\gamma) \neq 0$.

Lemma 2. *Suppose that $\alpha \in (0,1]$, $\Theta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\Phi : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable. we get $(\Theta \circ \Phi) : \mathbb{T} \to \mathbb{R}$ also α-fractional differentiable with*

$$T_{\alpha}^{\Delta}(\Theta \circ \Phi)(\delta) = \left[\int_{0}^{1} \Theta'\left(\Phi(\delta) + h\Phi(\delta)\delta^{\alpha-1}T_{\alpha}^{\Delta}(\Phi)(\delta)\right)dh\right] T_{\alpha}^{\Delta}(\Phi)(\delta).$$

Lemma 3. *Let $\alpha \in (0,1]$, $\Phi : \mathbb{T} \to \mathbb{R}$ be continuous and α-fractional differentiable at $\delta \in \mathbb{T}^k$, and $\Theta : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, we get $d \in [\delta, \sigma(\delta)]$,with*

$$T_\alpha^\Delta(\Theta \circ \Phi)(\delta) = \Theta'(\Phi(d)) T_\alpha^\Delta(\Phi)(\delta).$$

Definition 2. *Let $0 < \alpha \leq 1$ and function $\lambda : \mathbb{T} \to \mathbb{R}$ be a regulated. Then α-fractional integral of λ, is defined as*

$$\int \lambda(\delta)\Delta^\alpha \delta = \int \lambda(\delta)\delta^{\alpha-1}\Delta\delta.$$

Theorem 2.*Let $h,\ k,\ c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $\lambda,\ \mu : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions. Then*

(i) $\int_h^k [\lambda(\delta)+\mu(\delta)]\Delta^\alpha\delta = \int_h^k \lambda(\delta)\Delta^\alpha\delta + \int_h^k \mu(\delta)\Delta^\alpha\delta.$
(ii) $\int_h^k \beta\lambda(\delta)\Delta^\alpha\delta = \beta\int_h^k \lambda(\delta)\Delta^\alpha\delta.$
(iii) $\int_h^k \lambda(\delta)\Delta^\alpha\delta = -\int_k^h \lambda(\delta)\Delta^\alpha\delta.$
(iv) $\int_h^k \lambda(\delta)\Delta^\alpha\delta = \int_h^c \lambda(\delta)\Delta^\alpha\delta + \int_c^k \lambda(\delta)\Delta^\alpha\delta.$
(v) $\int_h^h \lambda(\delta)\Delta^\alpha\delta = 0.$

Lemma 4.*Let $h,\ k \in \mathbb{T}$ with $h < k$ and $\lambda,\ \mu$ are α-fractional differentiable, then the integration by parts form is presented by*

$$\int_h^k \lambda(\delta)T_\alpha^\Delta \mu(\delta)\Delta^\alpha\delta = \lambda(\delta)\mu(\delta)|_h^k - \int_h^k \mu^\sigma(\delta)T_\alpha^\Delta\lambda(\delta)\Delta^\alpha\delta. \quad (13)$$

Lemma 5.*Let $h,\ k \in \mathbb{T}$, $\alpha \in (0,1]$ and $\lambda,\ \mu : \mathbb{T} \to \mathbb{R}$ are rd-continuous, then inequality of Hölder is presented by*

$$\int_h^k |\lambda(\eta)\mu(\eta)|\Delta^\alpha\eta \leq \left[\int_h^k |\lambda(\eta)|^\beta\Delta^\alpha\eta\right]^{\frac{1}{\beta}} \left[\int_h^k |\mu(\eta)|^\gamma\Delta^\alpha\eta\right]^{\frac{1}{\gamma}}, \quad (14)$$

where $\beta > 1$ and $1/\beta + 1/\gamma = 1$. This inequality is reversed if $0 < \beta < 1$ and if $\beta < 0$ or $\gamma < 0$.

3 Generalization of Hardy's type inequalities on fractional time scales

Theorem 3. *Assume $\xi \geq 0$, non-decreasing and $\Omega(\eta) = \int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s$. Let $\phi \geq 0$, non-decreasing and $0 < \zeta \leq \infty$. Then*

$$\int_0^\zeta \phi\left(\frac{\Omega(\eta)}{\eta}\right)\Delta^\alpha\eta \leq \int_0^\zeta \phi(\xi(\eta))\Delta^\alpha\eta.$$

When $\phi(\eta) = \eta^\beta$, $\beta \geq 1$, we have

$$\int_0^\zeta \left(\frac{\Omega(\eta)}{\eta}\right)^\beta \Delta^\alpha\eta \leq \int_0^\zeta \xi^\beta(\eta)\Delta^\alpha\eta.$$

*Proof.*Since

$$\begin{aligned}\int_0^\zeta \phi\left(\frac{\Omega(\eta)}{\eta}\right)\Delta^\alpha\eta &= \int_0^\zeta \phi(\eta^{-1}\Omega(\eta))\Delta^\alpha\eta \\ &= \int_0^\zeta \phi\left(\eta^{-1}\int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s\right)\Delta^\alpha\eta,\end{aligned}$$

since ξ is non-decreasing, then

$$\begin{aligned}\int_0^\zeta \phi\left(\frac{\Omega(\eta)}{\eta}\right)\Delta^\alpha\eta &\leq \int_0^\zeta \phi\left(\eta^{-1}\xi(\eta)\int_0^\eta s^{1-\alpha}\Delta^\alpha s\right)\Delta^\alpha\eta \\ &= \int_0^\zeta \phi\left(\eta^{-1}\xi(\eta)\eta\right)\Delta^\alpha\eta \\ &= \int_0^\zeta \phi\left(\xi(\eta)\right)\Delta^\alpha\eta.\end{aligned}$$

The next therory be a generalization to inequality of Hardy.

Theorem 4. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu > 0$, $\frac{\eta}{\mu(\eta)}$ is non-increasing and*

$$\Omega(\eta) = \int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s$$

where $0 < \alpha \leq 1$, $0 < \gamma < 1$ and $\beta > \frac{\gamma}{1+\gamma-\alpha}$. Then

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta \leq \frac{1}{(\alpha-\gamma)^{\beta-1}((\alpha-\gamma)(1-\beta)+\beta-\alpha)} \times \int_0^\infty \left(\frac{\eta\xi(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta. \quad (15)$$

*Proof.*Since

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta = \int_0^\infty \left(\frac{\left(\int_0^\eta s^{1-\alpha}s^{\gamma\left(\frac{\beta-1}{\beta}\right)}s^{-\gamma\left(\frac{\beta-1}{\beta}\right)}\xi(s)\Delta^\alpha s\right)}{(\mu(\eta))}\right)^\beta \Delta^\alpha\eta.$$

Applying inequality of Hölder with indices β and $\beta/(\beta-1)$, we get

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta \le$$

$$\int_0^\infty \mu^{-\beta}(\eta)\left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^\beta$$

$$\left(\left(\int_0^\eta s^{-\gamma}\Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^\beta \Delta^\alpha\eta$$

$$= \int_0^\infty \mu^{-\beta}(\eta)\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s\right)$$

$$\left(\frac{\eta^{\alpha-\gamma}}{\alpha-\gamma}\right)^{\beta-1}\Delta^\alpha\eta$$

$$= \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s$$

$$\int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)}\mu^{-\beta}(\eta)\Delta^\alpha\eta,$$

since $\left(\frac{\eta}{\mu(\eta)}\right)^\beta$ is non- increasing, we have

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta$$

$$\le \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\left(\frac{s}{\mu(s)}\right)^\beta$$

$$\int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)-\beta}\Delta^\alpha\eta\Delta^\alpha s$$

$$= \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\left(\frac{s}{\mu(s)}\right)^\beta$$

$$\frac{-s^{(\alpha-\gamma)(\beta-1)-\beta+\alpha}}{(\alpha-\gamma)(\beta-1)-\beta+\alpha}\Delta^\alpha s$$

$$= \frac{1}{(\alpha-\gamma)^{\beta-1}((\alpha-\gamma)(1-\beta)+\beta-\alpha)}$$

$$\times\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)+(\alpha-\gamma)(\beta-1)-\beta+\alpha}\left(\frac{s\xi(s)}{\mu(s)}\right)^\beta\Delta^\alpha s,$$

then

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta \le \frac{1}{(\alpha-\gamma)^{\beta-1}((\alpha-\gamma)(1-\beta)+\beta-\alpha)} \times\int_0^\infty \left(\frac{\eta\xi(\eta)}{\mu(\eta)}\right)^\beta\Delta^\alpha\eta.$$

Corollary 1. *If $\alpha=1$ in theorem 4. We obtain inequality (5) which is inequality (6) of theorem (2.2) in [4].*

Corollary 2. *If $\alpha=1, \gamma=\frac{1}{\beta}$ and $\mu(\eta)=\eta$ in theorem 4. We get classical inequality of Hardy*

$$\int_0^\infty \left(\frac{\int_0^\eta \xi(s)ds}{\eta}\right)^\beta d\eta \le \left(\frac{\beta}{\beta-1}\right)^\beta\int_0^\infty \xi^\beta(\eta)d\eta.$$

Theorem 5. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu>0$, $\frac{\eta}{\mu(\eta)}$ be non-decreasing and*

$$\Omega(\eta) = \int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s$$

where $0<\alpha\le 1, \gamma>0$ and $0<\beta<\frac{\gamma}{\gamma+\alpha-1}$. Then

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta \ge \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+\beta-\alpha)} \times\int_0^\infty \left(\frac{\eta\xi(\eta)}{\mu(\eta)}\right)^\beta\Delta^\alpha\eta. \quad (16)$$

Proof. Since

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta =$$

$$\int_0^\infty \left(\frac{\left(\int_0^\eta s^{1-\alpha}s^{\gamma\left(\frac{\beta-1}{\beta}\right)}s^{-\gamma\left(\frac{\beta-1}{\beta}\right)}\xi(s)\Delta^\alpha s\right)}{\mu(\eta)}\right)^\beta\Delta^\alpha\eta.$$

Applying reversed inequality of Hölder with indices β and $\beta/(\beta-1)$, we get

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^\beta \Delta^\alpha\eta$$

$$\ge \int_0^\infty \mu^{-\beta}(\eta)\left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^\beta$$

$$\left(\left(\int_0^\eta s^{\gamma}\Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^\beta \Delta^\alpha\eta$$

$$= \int_0^\infty \mu^{-\beta}(\eta)\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)$$

$$\left(\frac{\eta^{\alpha+\gamma}}{\alpha+\gamma}\right)^{\beta-1}\Delta^\alpha\eta$$

$$= \frac{1}{(\alpha+\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s$$

$$\int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)}\mu^{-\beta}(\eta)\Delta^\alpha\eta,$$

since $\left(\frac{\eta}{\mu(\eta)}\right)^{\beta}$ is non-decreasing, we have

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^{\beta} \Delta^\alpha \eta \geq \frac{1}{(\alpha-\gamma)^{\beta-1}} \int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)} \xi^{\beta}(s) \left(\frac{s}{\mu(s)}\right)^{\beta} \int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)-\beta} \Delta^\alpha \eta \Delta^\alpha s$$
$$= \frac{1}{(\alpha-\gamma)^{\beta-1}} \int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)} \xi^{\beta}(s) \left(\frac{s}{\mu(s)}\right)^{\beta} \frac{-s^{(\alpha+\gamma)(\beta-1)+\alpha-\beta}}{(\alpha+\gamma)(\beta-1)-\beta+\alpha} \Delta^\alpha s$$
$$= \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+\beta-\alpha)} \times \int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)+(\alpha+\gamma)(\beta-1)+\alpha-\beta} \left(\frac{s\xi(s)}{\mu(s)}\right)^{\beta} \Delta^\alpha s,$$

then

$$\int_0^\infty \left(\frac{\Omega(\eta)}{\mu(\eta)}\right)^{\beta} \Delta^\alpha \eta \geq \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+\beta-\alpha)} \times \int_0^\infty \left(\frac{\eta\xi(\eta)}{\mu(\eta)}\right)^{\beta} \Delta^\alpha \eta.$$

Corollary 3. *If $\alpha = 1$ in theorem 5. We obtain inequality (6) which is inequality (7) of theorem (2.3) in [4].*

Theorem 6. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu > 0$, $\frac{\eta}{\mu(\eta)}$ be non-increasing and*

$$\Omega(\eta) = \int_0^\eta s^{1-\alpha} \xi(s) \Delta^\alpha s$$

where $0 < \alpha \leq 1$, $0 < \gamma < 1$, $q > \alpha\beta - \gamma(\beta-1)$ and $\beta > 1$. Then

$$\int_0^\infty \frac{\Omega^{\beta}(\eta)}{\mu^{q}(\eta)} \Delta^\alpha \eta \leq \frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+q-\alpha)} \int_0^\infty \frac{(\eta\xi(\eta))^{\beta}}{\mu^{q}(\eta)} \Delta^\alpha \eta. \qquad (17)$$

Proof. Since

$$\int_0^\infty \frac{\Omega^{\beta}(\eta)}{\mu^{q}(\eta)} \Delta^\alpha \eta = \int_0^\infty \frac{\left(\int_0^\eta s^{1-\alpha} s^{\gamma\left(\frac{\beta-1}{\beta}\right)} s^{-\gamma\left(\frac{\beta-1}{\beta}\right)} \xi(s) \Delta^\alpha s\right)^{\beta}}{\mu^{q}(\eta)} \Delta^\alpha \eta.$$

Applying inequality of Hölder with indices β and $\beta/(\beta-1)$, we get

$$\int_0^\infty \frac{\Omega^{\beta}(\eta)}{\mu^{q}(\eta)} \Delta^\alpha \eta \leq \int_0^\infty \mu^{-q}(\eta) \left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)} \xi^{\beta}(s) \Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^{\beta} \left(\left(\int_0^\eta s^{-\gamma} \Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^{\beta} \Delta^\alpha \eta$$
$$= \int_0^\infty \mu^{-q}(\eta) \left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)} \xi^{\beta}(s) \Delta^\alpha s\right) \left(\frac{\eta^{\alpha-\gamma}}{\alpha-\gamma}\right)^{\beta-1} \Delta^\alpha \eta$$
$$= \frac{1}{(\alpha-\gamma)^{\beta-1}} \int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)} \xi^{\beta}(s) \Delta^\alpha s \int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)} \mu^{-q}(\eta) \Delta^\alpha \eta,$$

since $\left(\frac{\eta}{\mu(\eta)}\right)^{q}$ is non- increasing, we have

$$\int_0^\infty \frac{\Omega^{\beta}(\eta)}{\mu^{q}(\eta)} \Delta^\alpha \eta \leq \frac{1}{(\alpha-\gamma)^{\beta-1}} \int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)} \xi^{\beta}(s) \left(\frac{s}{\mu(s)}\right)^{q} \int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)-q} \Delta^\alpha \eta \Delta^\alpha s$$
$$= \frac{1}{(\alpha-\gamma)^{\beta-1}} \int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)} \xi^{\beta}(s) \left(\frac{s}{\mu(s)}\right)^{q} \frac{-s^{(\alpha-\gamma)(\beta-1)-q+\alpha}}{(\alpha-\gamma)(\beta-1)-q+\alpha} \Delta^\alpha s$$
$$= \frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+q-\alpha)} \times \int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)+(\alpha-\gamma)(\beta-1)-\beta+\alpha} \frac{(s\xi(s))^{\beta}}{\mu^{q}(s)} \Delta^\alpha s,$$

then

$$\int_0^\infty \frac{\Omega^{\beta}(\eta)}{\mu^{q}(\eta)} \Delta^\alpha \eta \leq \frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+q-\alpha)} \times \int_0^\infty \frac{(\eta\xi(\eta))^{\beta}}{\mu^{q}(\eta)} \Delta^\alpha \eta.$$

Corollary 4. *If $\alpha = 1$ in theorem 6. We obtain inequality (7) which is inequality (3) of theorem (2.1) in [5].*

Corollary 5. *If $\alpha = 1,\ \gamma = \frac{1}{\beta},\ \beta = q$ and $\mu(\eta) = \eta$ in theorem 6. We obtain classical inequality of Hardy*

$$\int_0^\infty \left(\frac{\int_0^\eta \xi(s)ds}{\eta}\right)^\beta d\eta \le \left(\frac{\beta}{\beta-1}\right)^\beta \int_0^\infty \xi^\beta(\eta)d\eta.$$

Theorem 7. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu > 0$, $\frac{\eta}{\mu(\eta)}$ be non-decreasing and*

$$\Omega(\eta) = \int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s$$

where $0 < \alpha \le 1$, $\gamma > 0$, $q > \gamma(\beta-1) + \alpha\beta$ and $0 < \beta < 1$. Then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\mu^q(\eta)}\Delta^\alpha\eta \ge \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+q-\alpha)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\mu^q(\eta)}\Delta^\alpha\eta. \quad (18)$$

Proof. Since

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\mu^q(\eta)}\Delta^\alpha\eta = \int_0^\infty \mu^{-q}(\eta)\left(\int_0^\eta s^{1-\alpha}s^{\gamma\left(\frac{\beta-1}{\beta}\right)}s^{-\gamma\left(\frac{\beta-1}{\beta}\right)}\xi(s)\Delta^\alpha s\right)^\beta \Delta^\alpha\eta.$$

Applying reversed inequality of Hölder with indices β and $\beta/(\beta-1)$, we get

$$\begin{aligned}\int_0^\infty \frac{\Omega^\beta(\eta)}{\mu^q(\eta)}\Delta^\alpha\eta \ge{}& \int_0^\infty \mu^{-q}(\eta)\left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^\beta \left(\left(\int_0^\eta s^\gamma\Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^\beta \Delta^\alpha\eta\\ ={}& \int_0^\infty \mu^{-q}(\eta)\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)\left(\frac{\eta^{\alpha+\gamma}}{\alpha+\gamma}\right)^{\beta-1}\Delta^\alpha\eta\\ ={}& \frac{1}{(\alpha+\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s \int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)}\mu^{-q}(\eta)\Delta^\alpha\eta,\end{aligned}$$

since $\left(\frac{\eta}{\mu(\eta)}\right)^q$ is non-decreasing, we have

$$\begin{aligned}\int_0^\infty \frac{\Omega^\beta(\eta)}{\mu^q(\eta)}\Delta^\alpha\eta \ge{}& \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s) \left(\frac{s}{\mu(s)}\right)^q \int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)-q}\Delta^\alpha\eta\Delta^\alpha s\\ ={}& \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s) \left(\frac{s}{\mu(s)}\right)^q \frac{-s^{(\alpha+\gamma)(\beta-1)+\alpha-q}}{(\alpha+\gamma)(\beta-1)-q+\alpha}\Delta^\alpha s\\ ={}& \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+2q-\alpha)}\\ &\times\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)+(\alpha+\gamma)(\beta-1)+\alpha-\beta}\frac{(s\xi(s))^\beta}{\mu^q(s)}\Delta^\alpha s,\end{aligned}$$

then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\mu^q(\eta)}\Delta^\alpha\eta \ge \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+q-\alpha)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\mu^q(\eta)}\Delta^\alpha\eta.$$

Corollary 6. *If $\alpha = 1$ in theorem 7. We obtain inequality (8) which is inequality (4) of theorem (2.2) in [5].*

Theorem 8. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu > 0$, $\frac{\eta}{\mu(\eta)}$ be a non-increasing,*

$$\Lambda(\eta) = \int_0^\eta s^{1-\alpha}\mu(s)\Delta^\alpha s \text{ and } \Omega(\eta) = \int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s,$$

where $0 < \alpha \le 1$, $0 < \gamma < 1$, $q > \frac{\alpha\beta-\gamma(\beta-1)}{2}$ and $\beta > 1$. Then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \le \frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+2q-\alpha)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}\Delta^\alpha\eta. \quad (19)$$

Proof. Since

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta = \int_0^\infty \Lambda^{-q}(\eta)\left(\int_0^\eta s^{1-\alpha}s^{\gamma\left(\frac{\beta-1}{\beta}\right)}s^{-\gamma\left(\frac{\beta-1}{\beta}\right)}\xi(s)\Delta^\alpha s\right)^\beta \Delta^\alpha\eta.$$

Applying inequality of Hölder with indices β and $\beta/(\beta-1)$, we get

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \leq$$
$$\int_0^\infty \Lambda^{-q}(\eta)\left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^\beta$$
$$\left(\left(\int_0^\eta s^{-\gamma}\Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^\beta \Delta^\alpha\eta$$
$$=\int_0^\infty \Lambda^{-q}(\eta)\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s\right)$$
$$\left(\frac{\eta^{\alpha-\gamma}}{\alpha-\gamma}\right)^{\beta-1}\Delta^\alpha\eta$$
$$=\frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)\Delta^\alpha s$$
$$\int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)}\Lambda^{-q}(\eta)\Delta^\alpha\eta,$$

since $\left(\frac{\eta^2}{\Lambda(\eta)}\right)^q$ is non- increasing, we have

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \leq$$
$$\frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)$$
$$\left(\frac{s^2}{\Lambda(s)}\right)^q\int_s^\infty \eta^{(\alpha-\gamma)(\beta-1)-2q}\Delta^\alpha\eta\Delta^\alpha s$$
$$=\frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)}\xi^\beta(s)$$
$$\left(\frac{s^2}{\Lambda(s)}\right)^q\frac{-s^{(\alpha-\gamma)(\beta-1)-2q+\alpha}}{(\alpha-\gamma)(\beta-1)-2q+\alpha}\Delta^\alpha s$$
$$=\frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+2q-\alpha)}$$
$$\times\int_0^\infty s^{\beta(1-\alpha)+\gamma(\beta-1)+(\alpha-\gamma)(\beta-1)-\beta+\alpha}\frac{(s\xi(s))^\beta}{\Lambda^q(s)}\Delta^\alpha s,$$

then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \leq$$
$$\frac{1}{(\alpha-\gamma)^{\beta-1}((\gamma-\alpha)(\beta-1)+2q-\alpha)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}\Delta^\alpha\eta.$$

Corollary 7. *If $\alpha=1$ in theorem 8. We obtain inequality*

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}d\eta \leq$$
$$\frac{1}{(1-\gamma)^{\beta-1}((\gamma-1)(\beta-1)+2q-1)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}d\eta,$$

which is inequality (6) of theorem (1) in [6].

Corollary 8. *If $\alpha=1,\ \gamma=\frac{1}{\beta},\ q=\frac{\beta}{2}$ and $\Lambda(\eta)=\eta^2$ in theorem 8. We get classical inequality of Hardy*

$$\int_0^\infty\left(\frac{\int_0^\eta \xi(s)ds}{\eta}\right)^\beta d\eta \leq \left(\frac{\beta}{\beta-1}\right)^\beta\int_0^\infty \xi^\beta(\eta)d\eta.$$

Theorem 9. *Assume ξ be a non-negative α-integrable over $(0,\infty)$, $\mu>0$, $\frac{\eta}{\mu(\eta)}$ be a non-decreasing,*

$$\Lambda(\eta)=\int_0^\eta s^{1-\alpha}\mu(s)\Delta^\alpha s \text{ and } \Omega(\eta)=\int_0^\eta s^{1-\alpha}\xi(s)\Delta^\alpha s,$$

where $0<\alpha\leq 1,\ \gamma>0,\ q>\frac{\alpha\beta+\gamma(\beta-1)}{2}$ and $0<\beta<1$. Then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \geq \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+2q-\alpha)} \times\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}\Delta^\alpha\eta. \tag{20}$$

Proof. Since

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta =$$
$$\int_0^\infty \frac{\left(\int_0^\eta s^{1-\alpha}s^{\gamma\left(\frac{\beta-1}{\beta}\right)}s^{-\gamma\left(\frac{\beta-1}{\beta}\right)}\xi(s)\Delta^\alpha s\right)^\beta}{\Lambda^q(\eta)}\Delta^\alpha\eta.$$

Applying reversed inequalityof Hölder with indices β and $\beta/(\beta-1)$, we get

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \geq$$
$$\int_0^\infty \Lambda^{-q}(\eta)\left(\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)^{\frac{1}{\beta}}\right)^\beta$$
$$\left(\left(\int_0^\eta s^{\gamma}\Delta^\alpha s\right)^{\frac{\beta-1}{\beta}}\right)^\beta \Delta^\alpha\eta$$
$$=\int_0^\infty \Lambda^{-q}(\eta)\left(\int_0^\eta s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s\right)$$
$$\left(\frac{\eta^{\alpha+\gamma}}{\alpha+\gamma}\right)^{\beta-1}\Delta^\alpha\eta$$
$$=\frac{1}{(\alpha+\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s)\Delta^\alpha s$$
$$\int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)}\Lambda^{-q}(\eta)\Delta^\alpha\eta,$$

since $\left(\frac{\eta^2}{\Lambda(\eta)}\right)^q$ is non-decreasing, we have

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \geq \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s) \left(\frac{s^2}{\Lambda(s)}\right)^q \int_s^\infty \eta^{(\alpha+\gamma)(\beta-1)-2q}\Delta^\alpha\eta\Delta^\alpha s,$$
$$= \frac{1}{(\alpha-\gamma)^{\beta-1}}\int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)}\xi^\beta(s) \left(\frac{s^2}{\Lambda(s)}\right)^q \frac{-s^{(\alpha+\gamma)(\beta-1)+\alpha-2q}}{(\alpha+\gamma)(\beta-1)-2q+\alpha}\Delta^\alpha s,$$
$$= \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+2q-\alpha)} \times \int_0^\infty s^{\beta(1-\alpha)+\gamma(1-\beta)+(\alpha+\gamma)(\beta-1)+\alpha-\beta}\frac{(s\xi(s))^\beta}{\Lambda^q(s)}\Delta^\alpha s,$$

then

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}\Delta^\alpha\eta \geq \frac{1}{(\alpha+\gamma)^{\beta-1}((\alpha+\gamma)(1-\beta)+2q-\alpha)} \times \int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}\Delta^\alpha\eta.$$

Corollary 9.*If $\alpha = 1$ in theorem 9. We get inequality*

$$\int_0^\infty \frac{\Omega^\beta(\eta)}{\Lambda^q(\eta)}d\eta \geq \frac{1}{(1+\gamma)^{\beta-1}((1+\gamma)(1-\beta)+2q-1)}\int_0^\infty \frac{(\eta\xi(\eta))^\beta}{\Lambda^q(\eta)}d\eta,$$

which is inequality (8) of theorem (2) in [6].

4 Conclusion

First, we introduced to calculus of time scales and calculus of conformable fractional via time scales. Second, we extended some inequalities of Hardy and reversed Hardy over fractional time scales versions. The strategy depend on new tools over fractional calculus on time scales. The obtained inequalities through this paper will lead to the classical inequalities which are established earlier in [4], [5]and [6]. Some future works remain open like:

• Developing results on various forms of fractional integral operators.

• Creating new outcomes to inequalities like Opial, Copson, Hilber inequalities.

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