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The Fuzzy Conformable Integro-Differential Equations

Atimad Harir, Said Melliani and Lalla Saadia Chadli*

Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University P.O. Box 523, Beni Mellal, 23000, Morocco

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Abstract: The fuzzy generalized conformable fractional derivative is a novel fuzzy fractional derivative based on the basic limit definition of the derivative in [1]. We introduce the convolution product of fuzzy mapping and a crisp function. The conformable Laplace convolution formula is proved under the generalized conformable fractional derivatives concept and used to solve fuzzy integro-differential equations with a kernel of convolution type. The method is demonstrated by solving two examples, and the related theorems and properties are proved in detail.

Keywords: Fuzzy conformable Laplace transform, generalized conformable derivatives, fuzzy fractional differential, fuzzy valued function.

1 Introduction

Various researchers have proposed many definitions of fuzzy fractional derivatives over the years. The Riemann-Liouville fuzzy fractional derivative is one of them, and the so-called fuzzy Caputo derivative is the other. However, they aren't the only definitions available. A new fractional derivative has recently been discovered. established in [2,3,4,5,6,7,8,9], and it can be seen that the new derivative proposed in this study meets all of the requirements. Instead of the normal one fuzzy Conformable fractional derivatives are a new type of defined fuzzy fractional derivative that can be used in a variety of ways. In papers [10,11,12,13,14,15,16,17,18], fuzzy derivatives are investigated. On the other hand, the literature's definitions are solely for the real world.

In [1] used the conformable derivative to create the concept of the fuzzy conformable derivative. This was the starting point for the study of set differential equations and fuzzy conformable differential equations later on. Bede and Gal presented the weakly generalized differential of a fuzzy-valued function to address some of the drawbacks of this technique. In addition, Harir and colleagues [19] developed The lateral Hukuhara derivatives are used to describe generalized conformable differentiability.

A fuzzy conformable differential equation has no unique solution, which is an advantage of generalized conformable differentiability over the Hukuhara differentiability of a function. Stefanini and Bede generalized the Hukuhara difference [11,12] and the derivative for interval-valued functions. They demonstrated that conformable differentiability has connections to weakly generalized conformable differentiability see [20,21,22].

This research used the conformable Laplace transform method to solve FCDEs with a conformable convolution-type kernel under generalized conformable differentiability. Clearly, the proposed formula allows us to use the Laplace method to solve harder FCDEs. And we dealt with a variety of scenarios with this kernel.

In the first and second examples, g(t) was constant (positive or negative) and non-constant functions of t, respectively

* Corresponding author e-mail: atimad.harir@gmail.com

2 Preliminaries

Let us denote by $\mathbb{R}_{\mathscr{F}} = \{\mu : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties:

(i) μ is normal i.e, there exists an $\xi_0 \in \mathbb{R}$ such that $\mu(\xi_0) = 1$,
(ii) μ is fuzzy convex i.e for $\xi, \eta \in \mathbb{R}$ and $0 < \lambda \leq 1$,

$$\mu(\lambda\xi + (1-\lambda)\eta) \geq \min[\mu(\xi), \mu(\eta)]$$

(iii) μ is upper semi-continuous,
(iv) $[\mu]^0 = cl\{\xi \in \mathbb{R} | \mu(\xi) > 0\}$ is compact.

Then $\mathbb{R}_{\mathscr{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathscr{F}}$. For $0 < \alpha \leq 1$ denote $[\mu]^\alpha = \{\xi \in \mathbb{R} | \mu(\xi) \geq \alpha\}$, then from (i) to (iv) it follows that the α-level sets $[\mu]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which is denoted by $[\mu]^\alpha = [\mu_1^\alpha, \mu_2^\alpha]$. By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$, and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual.

Theorem 1.*[23] If $\mu \in \mathbb{R}_{\mathscr{F}}$, then*

(i) $[\mu]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$
(ii) $[\mu]^{\alpha_2} \subset [\mu]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
(iii) $\{\alpha_k\} \subset [0,1]$ is a non-decreasing sequence which converges to α then

$$[\mu]^\alpha = \bigcap_{k \geq 1} [\mu]^{\alpha_k}$$

Conversely, if $A_\alpha = \{[\mu_1^\alpha, \mu_2^\alpha]; \alpha \in (0,1]\}$ is a family of closed real intervals verifying (i) and (ii), then $\{A_\alpha\}$ defined a fuzzy number $\mu \in \mathbb{R}_{\mathscr{F}}$ such that
$[\mu]^\alpha = A_\alpha$ for $\quad 0 < \alpha \leq 1$ and $[\mu]^0 = \overline{\bigcup_{0<\alpha\leq 1} A_\alpha} \subset A_0$.

Lemma 1.*[24] Let $\mu, \nu : \mathbb{R} \to [0,1]$ be the fuzzy sets.*
Then $\mu = \nu$ if and only if $[\mu]^\alpha = [\nu]^\alpha$ for all $\alpha \in [0,1]$.

Definition 1.*[25] A fuzzy number μ in parametric form is a pair $(\mu_1^\alpha, \mu_2^\alpha)$ of functions $\mu_1^\alpha, \mu_2^\alpha$, $\alpha \in [0,1]$, which satisfy the following requirements:*

1. μ_1^α is a bounded increasing left continuous function in $(0,1]$, and right continuous at 0,
2. μ_2^α is a bounded decreasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\mu_1^\alpha \leq \mu_2^\alpha$, $0 \leq \alpha \leq 1$.

A crisp number k is simply represented by $\mu_1^\alpha = \mu_2^\alpha = k$.
For arbitrary $\mu = (\mu_1^\alpha, \mu_2^\alpha)$, $\nu = (\nu_1^\alpha, \nu_2^\alpha)$ and $\lambda > 0$ we define addition and scalar multiplication by λ see [11,24]:

$$[\mu + \nu]^\alpha = [\mu_1^\alpha + \nu_1^\alpha, \mu_2^\alpha + \nu_2^\alpha]$$

$$[\lambda\mu]^\alpha = \lambda[\mu]^\alpha = \begin{cases} [\lambda\mu_1^\alpha, \lambda\mu_2^\alpha] \; if \; \lambda \geq 0 \\ [\lambda\mu_2^\alpha, \lambda\mu_1^\alpha] \; if \; \lambda < 0, \end{cases}$$

Definition 2. *Let $\mu, \nu \in \mathbb{R}_{\mathscr{F}}$. If there exists $w \in \mathbb{R}_{\mathscr{F}}$ such as $\mu = \nu + w$ then w is called the H-difference of μ, ν and it is denoted $u \ominus \nu$.*

Define $d : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_+ \cup \{0\}$ by the equation

$$d(\mu, \nu) = \sup_{\alpha \in [0,1]} d_H([\mu]^\alpha, [\nu]^\alpha), \quad for\ all\ \mu, \nu \in \mathbb{R}_{\mathscr{F}}$$

where d_H is the Hausdorff metric .

$$d_H([\mu]^\alpha, [\nu]^\alpha) = \max\{|\mu_1^\alpha - \nu_1^\alpha|, |\mu_2^\alpha - \nu_2^\alpha|\}$$

where $\mu = (\mu_1^\alpha, \mu_2^\alpha)$, $\nu = (\nu_1^\alpha, \nu_2^\alpha) \subset \mathbb{R}$ is utilized in Bede and Gal [11]. Then, it is easy to see that d is a metric in $\mathbb{R}_{\mathscr{F}}$ and has the following properties [25]

(i) $d(\mu+w,\nu+w)=d(\mu,\nu), \quad \forall \mu,\nu,w\in\mathbb{R}_{\mathscr{F}}$,
(ii) $d(k\mu,k\nu)=|k|d(\mu,\nu)$, $\forall k\in\mathbb{R}$, $\mu,\nu\in\mathbb{R}_{\mathscr{F}}$,
(iii) $d(\mu+\nu,w+e)\leq d(\mu,w)+d(\nu,e)$, $\forall \mu,\nu,w,e\in\mathbb{R}_{\mathscr{F}}$
$(i\nu)$ $(d,\mathbb{R}_{\mathscr{F}})$ is a complete metric space.

Definition 3.[26] *Let $f:\mathbb{R}\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy-valued function. If for arbitrary fixed $\xi_0\in\mathbb{R}$ and $\varepsilon>0$ a $\delta>0$ such that*

$$|\xi-\xi_0|<\delta \Longrightarrow d(f(\xi),f(\xi_0))<\varepsilon$$

f is said to be continuous.

3 The fuzzy conformable fractional differentiability

Definition 4.[1] *Let $f:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be a fuzzy function. q^{th} order " fuzzy conformable fractional derivative " of F is defined by*

$$T_q(f)(\xi)=\lim_{\varepsilon\to0^+}\frac{f(\xi+\varepsilon\xi^{1-q})\ominus f(\xi)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{f(\xi)\ominus f(\xi-\varepsilon\xi^{1-q})}{\varepsilon}.$$

for all $\xi>0$, $q\in(0,1)$. Let $f^{(q)}(\xi)$ stands for $T_q(f)(\xi)$. Hence

$$f^{(q)}(\xi)=\lim_{\varepsilon\to0^+}\frac{f(\xi+\varepsilon\xi^{1-q})\ominus f(\xi)}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{f(\xi)\ominus f(\xi-\varepsilon\xi^{1-q})}{\varepsilon}.$$

If f is q-differentiable in some $(0,a)$, and $\lim_{\xi\to0^+}f^{(q)}(\xi)$ exists, then

$$f^{(q)}(0)=\lim_{\xi\to0^+}f^{(q)}(\xi)$$

and the limits (in the metric d)

Remark. From the definition, it directly follows that if f is q-differentiable, then the multi-valued mapping f_α is q-differentiable for all $\alpha\in[0,1]$ and

$$T_qf_\alpha=[f^{(q)}(\xi)]^\alpha \tag{1}$$

Here T_qf_α is denoted the conformable fractional derivative of f_α of order q.

Theorem 2.[1] *Let $f:(0,a)\to\mathbb{R}_{\mathscr{F}}$ be fuzzy function, where $f_\alpha(\xi)=\left[f_1^\alpha(\xi),f_2^\alpha(\xi)\right]$, $\alpha\in[0,1]$.*
(i)If f is $q_{(1)}$-differentiable, then $f_1^\alpha(\xi)$ and $f_2^\alpha(\xi)$ are q-differentiable and

$$\left[f^{(q_{(1)})}(\xi)\right]^\alpha=\left[(f_1^\alpha)^{(q)}(\xi),(f_2^\alpha)^{(q)}(\xi)\right].$$

(ii)If f is $q_{(2)}$-differentiable, then $f_1^\alpha(\xi)$ and $f_2^\alpha(\xi)$ are q-differentiable and

$$\left[f^{(q_{(2)})}(\xi)\right]^\alpha=\left[(f_2^\alpha)^{(q)}(\xi),(f_1^\alpha)^{(q)}(\xi)\right].$$

Theorem 3.[1] *Let $q\in(0,1]$*
(i)If f is (1)-differentiable and f is $q_{(1)}$-differentiable then

$$T_{q_{(1)}}f(\xi)=\xi^{1-q}D_1^1f(\xi)$$

(ii)If f is (2)-differentiable and f is $q_{(2)}$-differentiable then

$$T_{q_{(2)}}f(\xi)=\xi^{1-q}D_2^1f(\xi)$$

Note that the definition of (n)-differentiable or $\left(D_n^1\right)$ for $n\in1,2$ see [11,27,28,5,29].

4 Fuzzy fractional integral

Let $q \in (0,1]$ and $f:(0,a) \to \mathbb{R}_{\mathscr{F}}$ be such that $[f(\xi)]^{\alpha} = [f_1^{\alpha}(\xi), f_2^{\alpha}(\xi)]$ for all $\xi \in (0,a)$ and $\alpha \in [0,1]$. Suppose that $f_1^{\alpha}, f_2^{\alpha} \in C((0,a),\mathbb{R}) \cap L^1((0,a),\mathbb{R})$ for all $\alpha \in [0,1]$ and let

$$A_{\alpha} =: \left[\int_0^{\xi} \frac{f_1^{\alpha}(\tau)}{\tau^{1-q}} d\tau, \int_0^{\xi} \frac{f_2^{\alpha}(\tau)}{\tau^{1-q}} d\tau\right], \quad \xi \in (0,a). \tag{2}$$

Lemma 2.[30] *The family $\{A_{\alpha}; \alpha \in [0,1]\}$, given by Eq(2), defined a fuzzy number $f \in \mathbb{R}_{\mathscr{F}}$ such that $[f]^{\alpha} = A_{\alpha}$*

Definition 5. *Let $f \in C((0,a),\mathbb{R}_{\mathscr{F}}) \cap L^1((0,a),\mathbb{R}_{\mathscr{F}})$, define the fuzzy fractional integral for $q \in (0,1]$.*

$$I_q(f)(\tau) = I(\xi^{q-1} f)(\tau) = \int_0^{\tau} \frac{f(\xi)}{\xi^{1-q}} d\xi,$$

by

$$\begin{aligned}[I_q(f)(\tau)]^{\alpha} &= \left[I(\xi^{q-1} f)(\tau)\right]^{\alpha} = \left[\int_0^{\tau} \frac{f(\xi)}{\xi^{1-q}} d\xi\right]^{\alpha} \\ &= \left[\int_0^{\tau} \frac{f_1^{\alpha}(\xi)}{\xi^{1-q}} d\xi, \int_0^{\tau} \frac{f_2^{\alpha}(\xi)}{\xi^{1-q}} d\xi\right].\end{aligned}$$

where the integral $\int_0^{\tau} \frac{f_i^{\alpha}}{\xi^{1-q}}(\xi) d\xi$, for $i = 1,2$ is the usual Riemann improper integral.

For $q = 1$, we obtain $If(\tau) = \int_0^{\tau} f(\xi) d\xi$, that is the integral operator. Also, the following properties are obvious.

$(i) I_q cf(\xi) = cI_q f(\xi)$ for each $c \in \mathbb{R}_+$
$(ii) I_q(f+G)(\xi) = I_q f(\xi) + I_q G(\xi)$.

Theorem 4.[30] *$T_q I_q(f)(\xi) = f(\xi)$, for $\xi \geq 0$, where f is any continuous fuzzy-value function in the domain of I_q.*

5 Fuzzy conformable Laplace transform

Definition 6.[3] *The conformable fractional exponential function is defined for every $\xi \geq 0$ by:*

$$E_q(p,\xi) = e^{p\frac{\xi^q}{q}}, \tag{3}$$

where $p \in \mathbb{R}$ and $0 < q \leq 1$.

Definition 7. *Let $0 < q \leq 1$ and $f(\xi)$ be continuous fuzzy-value function. Suppose that $E_q(-p,\xi) f(\xi)$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^{\infty} E_q(-p,\xi) f(\xi) d_q\xi$ is called fractional fuzzy conformable Laplace transform of order q starting from zero of f and is defined as:*

$$\mathbf{L}_q[f(\xi)] = \int_0^{\infty} E_q(-p,\xi) f(\xi) d_q\xi, \quad p > 0 \text{ and integer}. \tag{4}$$

$$= \int_0^{\infty} E_q(-p,\xi) f(\xi) \xi^{q-1} d\xi. \tag{5}$$

Denote by $\mathscr{L}_q[g(\xi)]$ the classical fractional Laplace transform of order q starting from zero of crisp function $g(\xi)$. Since from proposition 2.1 see [26], we have

$$\int_0^{\infty} E_q(-p,\xi) f(\xi) d_q\xi = \left(\int_0^{\infty} E_q(-p,\xi) f_1^{\alpha}(\xi) d_q\xi, \int_0^{\infty} E_q(-p,\xi) f_2^{\alpha}(\xi) d_q\xi\right),$$

then, we follow:

$$\mathbf{L}_q[f(\xi)] = (\mathscr{L}_q[f_1^{\alpha}(\xi)], \mathscr{L}_q[f_2^{\alpha}(\xi)]).$$

where $q \in (0,1]$ and

$$\mathscr{L}_q[f_1^{\alpha}(\xi)] = \int_0^{\infty} E_q(-p,\xi) f_1^{\alpha}(\xi) d_q\xi \quad \text{and } \mathscr{L}_q[f_2^{\alpha}(\xi)] = \int_0^{\infty} E_q(-p,\xi) f_2^{\alpha}(\xi) d_q\xi$$

Theorem 5.[30] *Let $0 < q \leq 1$ and $f^{(q)}(\xi)$ be a conformable fractional integral fuzzy-value function, and $f(\xi)$ is the primitive of $f^{(q)}(\xi)$ on $[0,\infty)$. Then*

(i)if f is $q_{(1)}$-differentiable:

$$\mathbf{L}_q\left[f^{(q)}(\xi)\right] = p\mathbf{L}_q\left[f(\xi)\right] \ominus f(0) \tag{6}$$

(ii)if f is $q_{(2)}$-differentiable:

$$\mathbf{L}_q\left[f^{(q)}(\xi)\right] = (-f(0)) \ominus ((-p)\mathbf{L}_q\left[f(\xi)\right]) \tag{7}$$

Theorem 6.[30] *Let $f(\xi),\ g(\xi)$ be continuous fuzzy-valued functions, $q \in (0,1]$ and c_1, c_2 two real constants, then*

$$\mathbf{L}_q\left[c_1 f(\xi) + c_2 g(\xi)\right] = c_1\mathbf{L}_q\left[f(\xi)\right] + c_2\mathbf{L}_q\left[g(\xi)\right]. \tag{8}$$

Lemma 3.[30] *Let $q \in (0,1]$ and $f(\xi)$ be continuous fuzzy-value function on $[0,\infty)$, suppose that $\lambda \geq 0$, then*

$$\mathbf{L}_q\left[\lambda f(\xi)\right] = \lambda\mathbf{L}_q\left[f(\xi)\right]$$

Remark. Let $f(\xi)$ be continuous fuzzy-value function and $g(\xi) \geq 0$. Suppose that $(f(\xi)g(\xi))E_q(-p,\xi)$ is improper fuzzy Rimann-integrale on $[0,\infty)$, then

$$\int_0^\infty (f(\xi)g(\xi))E_q(-p,\xi)d_q\xi = \left(\int_0^\infty (f_1^\alpha(\xi)g(\xi))E_q(-p,\xi)d_q\xi, \int_0^\infty (f_2^\alpha(\xi)g(\xi))E_q(-p,\xi)d_q\xi\right).$$

Theorem 7.[30] *Let $0 < q \leq 1$ and $f(\xi)$ is continuous fuzzy-value function and $\mathbf{L}_q\left[f(\xi)\right] = F(p)$, then*

$$\mathbf{L}_q\left[E_q(a,\xi)f(\xi)\right] = F(p-a)$$

where $E_q(a,\xi)$ is real value function and $p-a > 0$.

The relation between the fuzzy Laplace transform and the fractional fuzzy conformable Laplace transforms is given below.

Theorem 8.[30] *Let $0 < q \leq 1$ and $f(\xi)$ be continuous fuzzy-value function such that $\mathbf{L}_q\left[f(\xi)\right] = F_q(p)$ exist. Then*

$$F_q(p) = \mathbf{L}\left[f\left((q\xi)^{\frac{1}{q}}\right)\right] \tag{9}$$

where $\mathbf{L}\left[g(\xi)\right] = \int_0^\infty e^{-p\xi}g(\xi)d\xi$

Remark. We calculate the fractional Laplace for certain functions see [3,31,32,33,34]

–$\mathbf{L}_q\left[1\right] = \frac{1}{p},\ p > 0$
–$\mathbf{L}_q\left[\xi\right] = \mathbf{L}\left[(q\xi)^{1/q}\right] = q^{\frac{1}{q}}\frac{\Gamma(1+\frac{1}{q})}{p^{1+\frac{1}{q}}},\ p > 0.$
–$\mathbf{L}_q\left[e^{\frac{\xi^q}{q}}\right] = \frac{1}{p-1},\ \ p > 1.$
–$\mathbf{L}_q\left[e^{-k\frac{\xi^q}{q}}f(\xi)\right] = \mathbf{L}\left[e^{-k\xi}f\left((q\xi)^{\frac{1}{q}}\right)\right].$

For example $\mathbf{L}_q\left[e^{\lambda\frac{\xi^q}{q}}\right] = \mathbf{L}\left[e^{\lambda\xi}\right] = \frac{1}{p-\lambda}.$

6 Fuzzy conformable Laplace convolution

Definition 8. *Let $g:[0,\infty[\to\mathbb{R}$ be a crisp continuous function and $f:[0,\infty[\to\mathbb{R}_{\mathscr{F}}$ a fuzzy-valued continuous mapping. We define the convolution product of g and f on $[0,\infty[$ as follows:*

$$\begin{aligned}(g*f)(\xi) &= \int_0^{\xi} g(\xi^q - s^q) f(s) d_q s, \quad \xi \geq 0 \\ &= \int_0^{\xi} g(\xi^q - s^q) f(s)\, s^{q-1} ds, \quad \xi \geq 0\end{aligned} \tag{10}$$

*Remark.*Suppose that $E_q(-p,\xi)f(\xi)$ and $E_q(-p,\xi)g(\xi)$ are integrable on $[0,\infty[$. We examine the two following alternatives:

(a)If the function g is nonnegative on $[0,\infty[$, then

$$(g*f)(\xi) = \left(\int_0^{\xi} g(\xi^q - s^q) f_1^{\alpha}(s) d_q s, \int_0^{\xi} g(\xi^q - s^q) f_2^{\alpha}(s) d_q s\right) \tag{11}$$

Therefore,

$$(g*f)(\xi) = ((g*f_1^{\alpha})(\xi), (g*f_2^{\alpha})(\xi)). \tag{12}$$

f_1 and f_2 are two crisp functions defined from $[0,\infty]$ into $\mathbb{R}$, then, we recall the well-known classical convolution Laplace formula:

$$\mathscr{L}_q[(f_1*f_2)(\xi)] = \mathscr{L}_q[f_1(\xi^q)]\,\mathscr{L}_q[f_2(\xi)] \tag{13}$$

Then using (12)-(13) and the fact that $\mathscr{L}_q[g(\xi^q)] \geq 0$, we get

$$\begin{aligned}\mathbf{L}_q[\left(g*f\right)(\xi)] &= \left(\mathbf{L}_q\left[\left(g*f_1^{\alpha}\right)(\xi)\right], \mathbf{L}_q\left[\left(g*f_2^{\alpha}\right)(\xi)\right]\right) \\ &= \left(\mathscr{L}_q[g(\xi^q)]\mathscr{L}_q[f_1^{\alpha}(\xi)], \mathscr{L}_q[g(\xi^q)]\mathscr{L}_q[f_2^{\alpha}(\xi)]\right) \\ &= \mathscr{L}_q[g(\xi^q)]\, \left(\mathscr{L}_q[f_1^{\alpha}(\xi)], \mathscr{L}_q[f_2^{\alpha}(\xi)]\right) \\ &= \mathscr{L}_q[g(\xi^q)]\, \mathbf{L}_q[f(\xi)]\end{aligned} \tag{14}$$

(b)If the function g is non-positive on $[0,\infty[$, then

$$(g*f)(\xi) = \left(\int_0^{\xi} g(\xi^q - s^q) f_2^{\alpha}(s) d_q s, \int_0^{\xi} g(\xi^q - s^q) f_1^{\alpha}(s) d_q\right) \tag{15}$$

Therefore,

$$(g*f)(\xi) = ((g*f_2^{\alpha})(\xi), (g*f_1^{\alpha})(\xi)). \tag{16}$$

Then from (13)-(16) and since $\mathscr{L}_q[g(\xi^q)] \leq 0$, we deduce

$$\begin{aligned}\mathbf{L}_q[\left(g*f\right)(\xi)] &= \left(\mathbf{L}_q\left[\left(g*f_2^{\alpha}\right)(\xi)\right], \mathbf{L}_q\left[\left(g*f_1^{\alpha}\right)(\xi)\right]\right) \\ &= \left(\mathscr{L}_q[g(\xi^q)]\mathscr{L}_q[f_2^{\alpha}(\xi)], \mathscr{L}_q[g(\xi^q)]\mathscr{L}_q[f_1^{\alpha}(\xi)]\right) \\ &= \mathscr{L}_q[g(\xi^q)]\, \left(\mathscr{L}_q[f_1^{\alpha}(\xi)], \mathscr{L}_q[f_2^{\alpha}(\xi)]\right) \\ &= \mathscr{L}_q[g(\xi^q)]\, \mathbf{L}_q[f(\xi)]\end{aligned} \tag{17}$$

Theorem 9. *Let $f:[0,\infty[\longrightarrow\mathbb{R}_{\mathscr{F}}$ be a fuzzy valued continuous mapping and let $g:[0,\infty[\longrightarrow\mathbf{R}$ be crisp continuous function, such that g is the function of ξ^q for $0<q\leq 1$. Assume that the mapping $E_q(-p,\xi)f(\xi)$, $E_q(-p,\xi)g(\xi)$ and $E_q(-p,\xi)(g*f)(\xi)$ are integrable over $[0,\infty[$ for all $p>0$; then*

$$\mathbf{L}_q\left[(g*f)(\xi)\right] = \mathscr{L}_q[g(\xi^q)]\,\mathbf{L}_q[f(\xi)]. \tag{18}$$

*Proof.*Let $q \in]0,1]$, $\xi \geq 0$ and $p > 0$. It is obvious that

$$[(g*f)(\xi)]^{\alpha} = \left[\left(g*f_1^{\alpha}\right), \left(g*f_2^{\alpha}\right)\right]$$

For $\mathbf{L}_q\left(\left(g*f_1^{\alpha}\right), \left(g*f_2^{\alpha}\right)\right)$ α-cut see demonstration of Theorem 25 in [9] is similar.
Now we show that

$$\mathbf{L}_q\left[(g*f)(\xi)\right] = \mathscr{L}_q[g(\xi)]\ \mathbf{L}_q[f(\xi)]$$

We apply the conformable Laplace transform to Eq (10)

$$\begin{aligned}\mathbf{L}_q\left[(g*f)(\xi)\right]^{\alpha} &= \mathbf{L}_q\left[\left(g*f_1^{\alpha}\right), \left(g*f_2^{\alpha}\right)\right] \\ &= \left[\mathbf{L}_q\left(g*f_1^{\alpha}\right), \mathbf{L}_q\left(g*f_2^{\alpha}\right)\right]\end{aligned}$$

$$= \left[\int_0^{\infty} E(-p,\xi)\left(\int_0^{\xi} g(\xi^q - s^q) f_1^{\alpha}(s) d_q s\right) d_q\xi, \int_0^{\infty} E(-p,\xi)\left(\int_0^{\xi} g(\xi^q - s^q) f_2^{\alpha}(s) d_q s\right) d_q\xi\right]$$
$$= \left[\int_0^{\infty} E(-p,\xi)\left(\int_0^{\xi} g(\xi^q - s^q) f_1^{\alpha}(s) s^{q-1} ds\right) \xi^{q-1} d\xi, \int_0^{\infty} E(-p,\xi)\left(\int_0^{\xi} g(\xi^q - s^q) f_2^{\alpha}(s) s^{q-1} ds\right) \xi^{q-1} d\xi\right]$$

By changing the order of integration we get

$$= \left[\int_0^{\infty}\int_s^{\infty} e^{-p\frac{\xi^q}{q}} g(\xi^q - s^q) f_1^{\alpha}(s) \xi^{q-1} s^{q-1} d\xi ds, \int_0^{\infty}\int_s^{\infty} e^{-p\frac{\xi^q}{q}} g(\xi^q - s^q) f_2^{\alpha}(s) \xi^{q-1} s^{q-1} d\xi ds\right]$$

Then we substitute $\tau^q = \xi^q - s^q$ into the above integral and obtain

$$= \left[\int_0^{\infty}\int_0^{\infty} e^{-p\frac{\tau^q+s^q}{q}} g(\tau^q) f_1^{\alpha}(s) \tau^{q-1} d\tau s^{q-1} ds, \int_0^{\infty}\int_0^{\infty} e^{-p\frac{\tau^q+s^q}{q}} g(\tau^q) f_2^{\alpha}(s) \tau^{q-1} d\tau s^{q-1} ds\right]$$
$$= \left[\int_0^{\infty} e^{-p\frac{\tau^q}{q}} g(\tau^q)\tau^{q-1} d\tau \int_0^{\infty} e^{-p\frac{s^q}{q}} f_1^{\alpha}(s) s^{q-1} ds, \int_0^{\infty} e^{-p\frac{\tau^q}{q}} g(\tau^q)\tau^{q-1} d\tau \int_0^{\infty} e^{-p\frac{s^q}{q}} f_2^{\alpha}(s) s^{q-1} ds\right]$$
$$= \left[\int_0^{\infty} e^{-p\frac{\tau^q}{q}} g(\tau^q)\tau^{q-1} d\tau \int_0^{\infty} e^{-p\frac{s^q}{q}} f_1^{\alpha}(s) s^{q-1} ds, \int_0^{\infty} e^{-p\frac{\tau^q}{q}} g(\tau^q)\tau^{q-1} d\tau \int_0^{\infty} e^{-p\frac{s^q}{q}} f_2^{\alpha}(s) s^{q-1} ds\right]$$
$$= \int_0^{\infty} E(-p,\tau) g(\tau^q)\tau^{q-1} d\tau \left(\left[\int_0^{\infty} E(-p,s) f_1^{\alpha}(s) s^{q-1} ds, \int_0^{\infty} E(-p,s) f_2^{\alpha}(s) s^{q-1} ds\right]\right)$$
$$= \mathscr{L}_q[g(\xi^q)]\left[\mathscr{L}_q[f_1^{\alpha}(\xi)], \mathscr{L}_q[f_2^{\alpha}(\xi)]\right]$$
$$= \mathscr{L}_q g(\xi^q)\ \mathbf{L}_q f(\xi)$$

Our current goal is to use the fuzzy conformable Laplace transform method to solve the following fuzzy integro-differential equation under generalized conformable differentiability:

$$\begin{aligned} y^{(q)}(\tau) &= f(\tau) + \int_0^{\tau} g\left(\frac{\tau^q}{q} - \frac{\xi^q}{q}\right) y(\xi) d_q\xi \\ y(0) &= y_0 = (y_{01}, y_{02}) \in \mathbb{R}_{\mathscr{F}} \end{aligned}$$

where the unknown function $y(\xi) = (y_1^{\alpha}(\xi), y_2^{\alpha}(\xi))$ is a fuzzy function of $\xi \geq 0$, provided that $\bar{f} : [0,\infty[\to \mathbb{R}_{\mathscr{F}}$ is a continuous fuzzy-valued function and $g : [0,\infty[\to \mathbb{R}$ is a crisp continuous function.
Assume in a first time that $\mathscr{L}_q[g(\xi)] \geq 0$. By using the fuzzy conformable Laplace transform and Theorem (9) , we have

$$\mathbf{L}_q\left[y^{(q)}(\xi)\right] = \mathbf{L}_q[f(\xi)] + \mathscr{L}_q[g(\xi)] \cdot \mathbf{L}_q[y(\xi)]. \tag{19}$$

Then, we have the following alternatives for solving (19).
Case 1. If y is $q_{(1)}$-differentiable, then

$$\begin{aligned} y^{(q)}(\xi) &= \left((y_1^{\alpha})^{(q)}(\xi), (y_2^{\alpha})^{(q)}(\xi)\right), \\ \mathbf{L}_q\left[y^{(q)}(\xi)\right] &= p\mathbf{L}_q[y(\xi)] \ominus y(0) \end{aligned}$$

Then from (19), it follows that

$$p\mathbf{L}_q[y(\xi)] = y(0) + \mathbf{L}_q[f(\xi)] + \mathscr{L}_q[g(\xi)] \cdot \mathbf{L}_q[y(\xi)].$$

Using $\mathscr{L}_q[g(\xi)] \geq 0$, we deduce

$$\begin{aligned} p\mathscr{L}_q[y_1^\alpha(\xi)] &= y_{01}^\alpha(\alpha) + \mathscr{L}_q[f_1^\alpha(\xi)] + \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)], \\ p\mathscr{L}_q[y_2^\alpha(\xi)] &= y_{02}^\alpha(\alpha) + \mathscr{L}_q[f_2^\alpha(\xi)] + \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)]. \end{aligned} \tag{20}$$

Therefore,

$$\mathscr{L}_q[y_1^\alpha(\xi)] = \frac{y_{01}^\alpha + \mathscr{L}_q[f_1^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}$$

$$\mathscr{L}_q[y_2^\alpha(\xi)] = \frac{y_{02}^\alpha + \mathscr{L}_q[f_2^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}.$$

By using the inverse conformable Laplace transform, we get

$$y_1^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{y_{01}^\alpha + \mathscr{L}_q[f_1^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}\right],$$

$$y_2^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{y_{02}^\alpha(\alpha) + \mathscr{L}_q[f_2^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}\right].$$

Case 2. If y is $q_{(2)}$-differentiable, then

$$y^{(q)}(\xi) = \left((y_2^\alpha)^{(q)}(\xi), (y_1^\alpha(\xi))^{(q)}(\xi)\right),$$

$$\mathbf{L}_q\left[y^{(q)}(\xi)\right] = -y(0) \ominus (-p\mathbf{L}_q[y(\xi)]).$$

Then from (19), it follows that

$$-y(0) \ominus (-p\mathbf{L}_q[y(\xi)]) = \mathbf{L}_q[f(\xi)] + \mathscr{L}_q[g(\xi)]\mathbf{L}_q[y(\xi)] \tag{21}$$

Using $\mathscr{L}_q[g(\xi)] \geq 0$, we deduce

$$-y_{02}^\alpha + p\mathscr{L}_q[y_2^\alpha(\xi)] = \mathscr{L}_q[f_1^\alpha(\xi)] + \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] \tag{22}$$

$$-y_{01}^\alpha + p\mathscr{L}_q[y_1^\alpha(\xi)] = \mathscr{L}[f_2^\alpha(\xi)] + \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] \tag{23}$$

That is,

$$\begin{aligned} \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)] &= -y_{02}^\alpha - \mathscr{L}_q[f_1^\alpha(\xi)] \\ \mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)] &= -y_{01}^\alpha - \mathscr{L}_q[f_2^\alpha(\xi)] \end{aligned} \tag{24}$$

Then by solving the linear system (24), we have

$$\mathscr{L}_q[y_1^\alpha(\xi)] = \frac{\mathscr{L}[g(\xi)]\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]\right) + p\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]\right)}{(\mathscr{L}_q[g(\xi)])^2 - p^2}$$

$$\mathscr{L}_q[y_2^\alpha(\xi)] = \frac{\mathscr{L}_q[g(\xi)]\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]\right) + p\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]\right)}{(\mathscr{L}_q[g(\xi)])^2 - p^2}$$

By using the inverse conformable Laplace transform, we get

$$y_1^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{\mathscr{L}[g(\xi)]\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]\right) + p\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]\right)}{(\mathscr{L}_q[g(\xi)])^2 - p^2}\right]$$

$$y_2^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{\mathscr{L}_q[g(\xi)]\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]\right) + p\left(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]\right)}{(\mathscr{L}_q[g(\xi)])^2 - p^2}\right]$$

Similarly, if we assume that $\mathscr{L}[g(\xi)] < 0$, we obtain the following results.

1.If y is $q_{(1)}$-differentiable, then

$$\mathscr{L}_q[y_1^\alpha(\xi)] = \frac{\mathscr{L}_q[g(\xi)]\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]) + p\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)])}{p^2 - (\mathscr{L}[g(\xi)])^2}$$
$$\mathscr{L}_q[y_2^\alpha(\xi)] = \frac{\mathscr{L}_q[g(\xi)]\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]) + p\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)])}{p^2 - (\mathscr{L}[g(\xi)])^2}$$

By using the inverse conformable Laplace transform, we get

$$y_1^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{\mathscr{L}_q[g(\xi)]\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)]) + p\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)])}{p^2 - (\mathscr{L}[g(\xi)])^2}\right]$$
$$y_2^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{\mathscr{L}_q[g(\xi)]\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_2^\alpha(\xi)] - p\mathscr{L}_q[y_2^\alpha(\xi)]) + p\,(\mathscr{L}_q[g(\xi)]\mathscr{L}_q[y_1^\alpha(\xi)] - p\mathscr{L}_q[y_1^\alpha(\xi)])}{p^2 - (\mathscr{L}[g(\xi)])^2}\right]$$

2.If y is $q_{(2)}$-differentiable, then

$$\mathscr{L}_q[y_1^\alpha(\xi)] = \frac{y_{01}^\alpha + \mathscr{L}_q[f_2^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}$$
$$\mathscr{L}_q[y_2^\alpha(\xi)] = \frac{y_{02}^\alpha + \mathscr{L}_q[f_1^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}$$

By using the inverse conformable Laplace transform, we obtain

$$y_1^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{y_{01}^\alpha + \mathscr{L}_q[f_2^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}\right]$$
$$y_2^\alpha(\xi) = \mathscr{L}_q^{-1}\left[\frac{y_{02}^\alpha + \mathscr{L}_q[f_1^\alpha(\xi)]}{p - \mathscr{L}_q[g(\xi)]}\right]$$

*Example 1.*The fuzzy Volterra integro-differential equation as follows:

$$y^{(q)}(\xi) = \left(1 + \frac{\xi^q}{q}\right)\sigma + \int_0^t y(\xi)d_q\xi \tag{25}$$
$$y^\alpha(0) = (0,0)$$

where $q \in (0.1]$, $f(\xi) = (1 + \frac{\xi^q}{q})\,\sigma$, $\quad \sigma = [\alpha - 1, 1 - \alpha]$ $\alpha \in [0,1]$ and $g(\xi) = 1$ is non-negative.
Case 1: If $y(\xi)$ is $(q_{(1)})$-differentiable, then from (17) we have

$$\mathscr{L}_q[y_1^\alpha(\xi)] = \frac{\alpha-1}{p(p-1)}$$
$$\mathscr{L}_q[y_2^\alpha(\xi)] = \frac{1-\alpha}{p(p-1)}$$

By the inverse Laplace transform, we get the lower and upper functions of solution of (25) for $\xi \geq 0$

$$y_1^\alpha(\xi) = (\alpha - 1)\left(\exp(\tfrac{\xi^q}{q}) - 1\right)$$
$$y_2^\alpha(\xi) = (1 - \alpha)\left(\exp(\tfrac{\xi^q}{q}) - 1\right)$$

In this case, since $y(\xi)$ is $(q_{(1)})$-differentiable, the solution is valid.

Case 2: If $y(\xi)$ is $(q_{(2)})$-differentiable, then from (17) we obtain

$$\mathscr{L}_q[y_1^\alpha(\xi)] = (1 - \alpha)\frac{p+1}{p(p^2+1)}$$
$$\mathscr{L}_q[y_2^\alpha(\xi)] = (\alpha - 1)\frac{p+1}{p(p^2+1)}$$

Then by the inverse Laplace transform the lower and upper functions of solution of (25) are given for $\xi \in [3\pi/2, 2\pi]$ as follows:

$$y_1^\alpha(\xi) = (\alpha - 1)(\cos(\tfrac{\xi^q}{q}) - \sin(\tfrac{\xi^q}{q}) - 1)$$
$$y_2^\alpha(\xi) = (1 - \alpha)(\cos(\tfrac{\xi^q}{q}) - \sin(\tfrac{\xi^q}{q}) - 1)$$

In this case, $y(\xi)$ is $(q_{(2)})$-differentiable only for $\xi \in [7\pi/4, 2\pi]$ and the solution is acceptable only over this interval.

*Example 2.*We consider the following fuzzy integro-differential equation:

$$y^{(q)}(\xi)+y(\xi)=\int_0^{\xi}\sin(\xi^q-s^q)y(s)ds, \quad (26)$$
$$y(0,\alpha)=[\alpha-1,1-\alpha].$$

Case 1: If $y(\xi)$ is $(q_{(1)})$-differentiable, then from Theorems (5) and (6) we have

$$\mathscr{L}_q[y_1^{\alpha}(\xi)]=(\alpha-1)\frac{(p^2+1)}{p^3+p^2+p}$$
$$\mathscr{L}_q[y_2^{\alpha}(\xi)]=(1-\alpha)\frac{(p^2+1)}{p^3+p^2+p}$$

By the inverse Laplace transform we get the lower and upper functions of solution of (26) for $\xi\geq 0$

$$y_1^{\alpha}(\xi)=(\alpha-1)\left[1-\frac{2\sqrt{3}}{3}\exp\left(-\frac{\xi^q}{2q}\right)\sin\left(\frac{\sqrt{3}\xi^q}{2q}\right)\right], \quad (27)$$

$$y_2^{\alpha}(\xi)=(1-\alpha)\left[1-\frac{2\sqrt{3}}{3}\exp\left(-\frac{\xi^q}{2q}\right)\sin\left(\frac{\sqrt{3}\xi^q}{2q}\right)\right]. \quad (28)$$

In this case, the solution is invalid over $[0,\infty[$, since $y(\xi)$ is not $(q_{(1)})$-differentiable.

Case 2: If $y(\xi)$ is $(q_{(2)})$-differentiable, then Theorems (5) and (7) yield

$$p\mathscr{L}_q[y_1^{\alpha}(\xi)]+\left(p^2+1\right)p\mathscr{L}_q[y_2^{\alpha}(\xi)]=(1-\alpha)\frac{p^2+1}{p} \quad (29)$$

$$\left(p^2+1\right)\mathscr{L}_q[y_1^{\alpha}(\xi)]+p\mathscr{L}_q[y_2^{\alpha}(\xi)]=(\alpha-1)\frac{p^2+1}{p} \quad (30)$$

By solving the linear system (26) and using the inverse Laplace transform, we get

$$y_1^{\alpha}(\xi)=(\alpha-1)\left[1+\frac{2\sqrt{3}}{3}\exp\left(\frac{\xi^q}{2q}\right)\sin\left(\frac{\sqrt{3}\xi^q}{2q}\right)\right]$$
$$y_2^{\alpha}(\xi)=(1-\alpha)\left[1+\frac{2\sqrt{3}}{3}\exp\left(\frac{\xi^q}{2q}\right)\sin\left(\frac{\sqrt{3}\xi^q}{2q}\right)\right].$$

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