Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170218

Survey on Linear Integral Equations and the Laguerre Polynomials

Muhammad Hakim Ali Qasmi [1], *Ali Hasan*[2], *Ali Akgül* [3,4,⋆], *Muhammad Farman* [5] *Salem M. Aljazzar* [6] *and Faisal Al-Showaikh* [7]

[1]Department of Mathematics, Lahore College of Arts,Science and Technoloy, Pakistan
[2]Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
[3]Siirt University, Art and Science Faculty, Department of Mathematics, 56100 Siirt, Turkey
[4]Department of Electronics and Communication Engineering, Saveetha School of Engineering, SIMATS, Chennai, Tamilnadu, India
[5]Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
[6]Jeddah College of Engineering, University of Business and Technology, Jeddah 21432, Saudi Arabia
[7]College of Arts & Science, Ahlia University, Kingdom of Bahrain

Received: 18 Aug. 2022, Revised: 15 Jan. 2023, Accepted: 21 Jan. 2023
Published online: 1 Mar. 2023

Abstract: In this paper, we dealt with Integral Equations (IEs). Many real-world problems are modeled in the form of IEs. Nowadays, IEs are used frequently as a replacement for Differential Equations. There are various methods to deal with IEs such as ADM, VIM, and HPM, etc. Here in this work, we used Galerkin Method. In this method, we considered Laguerre Polynomials (some other polynomials may also be used). In the method, the Galerkin equation is solved by using Maple Code for getting unknown constants, solutions, and Graphs.

Keywords: Integral Equations, Galerkin Method, Laguerre Polynomials, Maple Code.

1 Introduction

Integral equations are used in the modeling of various problems and coming from the different fields of engineering and sciences. Some integral equations transform from the differential equations by suitable conversions, but some integral equations come without any transformation, directly from the problems, for instance Abel's integral equation [1,2]. In present days Integral Equations have got importance among the engineers and scientists. Now, many researchers used integral equations in place of differential equations. The reasons for using the integral equation is to avoid the initial and boundary value conditions, as the given problems are condensed and reduced in the dimension when transform in integral equations, and the problems come in the function of an unknown variable, which can easily handle. In the comparison of numerical integral and numerical differentiation, numerical integral gives relatively better approximation and least error than the numerical differentiation [3,4].

The Solution of Integral equation in closed form plays a vital role for verifying consistency of different methods. The above mentioned problems, which are presented in the mathematical forms or equations, so called integral equations are solved by using different and new methods such as successive approximation and successive substitution method, direct computational method, method of Decomposition (Adomian and Modified), Laplace transform, Elzaki transform, Simpson's quadrature, VIM, HPM, Galerkin's, B-spline wavelet, Rationalized Haar functions method, Taylor series expansion, undetermined coefficient approximation and the method of regularization [5,6,7,9]. For finding the approximate solutions some polynomials, Taylor polynomials, Bernstein polynomials, Laguerre polynomials, Hermite polynomials, Legendre polynomials etc. are used in some of these above methods [10,11].

Mathematicians are providing new and useful techniques in engineering and scientific fields to find out the solutions of numerical problems in easy and faster

* Corresponding author e-mail: aliakgul@siirt.edu.tr

way. Mathematicians keep trying to facilitate the world by their efforts and hardworking, and whose are exploring the more convenient and appropriate tools and methods from the previous existed tools and methods. There are many examples of such extraordinary works; one of them is dealing with the integral equations. Many daily life problems (physical, biological and others) are represented in the shape of integral equations. Solutions of these problems are important, and for these solutions (which may be exact or approximate) researchers are continuously working on it.

An Italian mathematician N.H. Abel introduced the first integral equation in 1825 and the D. Bois-Reymond used term integral equation in 1888 [1,2]. For closed form (or exact) and numerical (or approximate) solutions, the researchers are applying various techniques on integral equations such as Salih Yalcinbas used Taylor's polynomials for calculating solution of nonlinear Volterra-Fredholm IEs [10], Dehghan and Shakeri solved the Integro-differential equation which comes from oscillating magnetic field by using HPM (homotopy perturbation method) [11], Amjad and shahrokh used Gaussian radial basis function for solving numerically Fredholm integral equation [12], Maleknejad, Rostami and Khodabin determined numerically the solution of Volterra IEs with using Stochastic Operational Matrix(SOM) which based on the Block Pulse Function(BPF) [13], Hashmi et.al solved numerically those Singular Volterra integral equation in which weak singular kernels are involved by using Optimal Homotopy Asymptotic method (OHAM) [14], Mohamed and Behiry solved the nonlinear High Order Volterra-Fredholm Integro-Differential Equations(HOVFIDE) with method of differential transform (DTM) [15], Shali, Darania and Ali Asgar used the Collocation method to calculate solution of nonlinear Volterra-Fredholm IEs [16], Jafarian and Measoomy got the solution of system of FIE(Fredholm IEs) with using the feed-back neural network approach [17], Nemati, Ordokhani and Lima got the numerical solution of VIEs of nonlinear type with the Legendre polynomials [18], Ghasemi et.al calculate the VFIEs which are mixed with the Analytical method [19], Mirzaee and Hoseini used Taylor series, Hybrid of BPF for getting solution of VFIEs [20], Li-Hong et.al applied the method of Modified Reproducing Kernel for getting solutions of system of Linear VIEs [21], Calio and Marchetti approximated the solution of Linear VFIEs of second kind using method of Numerical Collocation [22], Yousef and Lin solved the FVIEs using Scaling Function Interpolation Method(SFIM) [23], Haifa and Fawzi used the technique of modified Adomian for obtaining solution of VIEs of nonlinear Kind [24], Almasieh and Nazari used multiquadratic radial basis functions with solution of non-linear VFIEs [25], Behzadi et.al using by generalization of Euler-Maclaurin summation formula solved numerically weak singular Fredholm integral equation [26], Huaiqing, Chen and Nie solved those integral equations which are linear using the method of Radial basis function [27], Ghomanjani et.al used the Bezier curves for getting solution of second kind Fredholm integral equation [28], Heydari et.al used a Computational method which is based on generalize basis function for attaining the solution of VIEs [29], Dardery and Allan using Chebyshev polynomial solved the Singular integral equations [30], Kumar et.al solved analytically Abel integral equation of astrophysics problem by using Laplace transform [31], for getting Numerical solution of second kind nonlinear Volterra IEs Duan and Chen used Picard iteration method [32], Liu et.al for attaining solution of Volterra IEs with cardinal spline introduced the latest numerical process [33], Samad, Zarei and Hasan applied homotopy analysis transform process for getting the solution of first kind Abel integral equations [34], Mamadu and Njoseh numerically solved Volterra IEs with orthogonal polynomials in Galerkin Method [35], Sara and Rashidinia applied Boubaker polynomial Collocation Approach for getting solution of system of nonlinear VFIEs [36], Irin et.al used Simpson's Quadrature rule with the solution of second kind VIEs [37], Zarnan solved Urysohn integral equation using Hermite polynomials method (a new approach for finding integral equations solution) [38], Mariam and Almuhalbedi solved the Volterra Population Model (VPM) with restarted ADM [39], Naila, Zulfiqar, Aslam and Zahida used Laplace transform to find solution of ODEs and Second Kind VIEs with bulge and logarithmic functions [40], Ali et.al solved the Volterra integral equations using the ADM [41], and etc.

2 Methodology

The whole methodology is explained below step by step.

2.1 Laguerre Polynomials

A French mathematician E.N. Laguerre developed the Laguerre polynomials. We got these polynomials by solving following Laguerre Equation

$$xz'' + (1-x)z' + nz = 0 \tag{1}$$

where $n \in z$ and $n \geq 0$
For calculating Laguerre polynomials following relation is utilized.

$$L_n(x) = \sum_{m=0}^{n} \frac{(-1)^m (n!)}{\{(m!)^2\}\{(n-m)!\}} x^m \tag{2}$$

Some Laguerre polynomials are listed below:

$$L_o(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = \frac{-4x + x^2 + 2}{2}$$

$$L_3(x) = \frac{-x^3 + 9x^2 - 18x + 6}{6}$$

2.2 Methodology

We use the Laguerre polynomials in the algorithm for getting solution of integral equation, and proposed algorithm may apply for finding the solution of IEs. In the Algorithm higher degree polynomials are utilized for better accuracy of required solution. First of all compute the Laguerre polynomials till required degree.
Assuming general integral equation i.e.

$$g(x)w(x) = h(x) + \beta \int_{q(x)}^{p(x)} k(x,t)w(t)dt \tag{3}$$

Where $h(x)$,$g(x)$ and $k(x,t)$ are known functions,β is a constant parameter while $w(x)$ is an unknown function.
Now, consider the solution of equation (3)

$$w(x) = \sum_{i=0}^{n} a_i L_i(x) \tag{4}$$

Where a_i, $i = 0,1,2,\cdots,n$ are unknown constants and $L_i(x)$, $i = 0,1,2,\cdots,n$ are Laguerre polynomials of degree n. Using equation (4) in equation (3)

$$g(x)\sum_{i=0}^{n} a_i L_i(x) = h(x) + \beta \int_{q(x)}^{p(x)} k(x,t) \sum_{i=0}^{n} a_i L_i(t)dt$$

$$g(x)\sum_{i=0}^{n} a_i L_i(x) - \beta \int_{q(x)}^{p(x)} k(x,t) \sum_{i=0}^{n} a_i L_i(t)dt = h(x) \tag{5}$$

Multiply both sides of equation (5) by weight function $L_j(x)$ also integrating w.r.t. x from a to b, and getting the following Galerkin equation, which is a system of equations.

$$\sum_{i=0}^{n} a_i \int_a^b [g(x)L_i(x) - \beta \int_{q(x)}^{p(x)} k(x,t) \sum_{i=0}^{n} a_i L_i(t)dt] L_j(x)dx$$
$$= \int_a^b h(x)L_j(x)dx$$

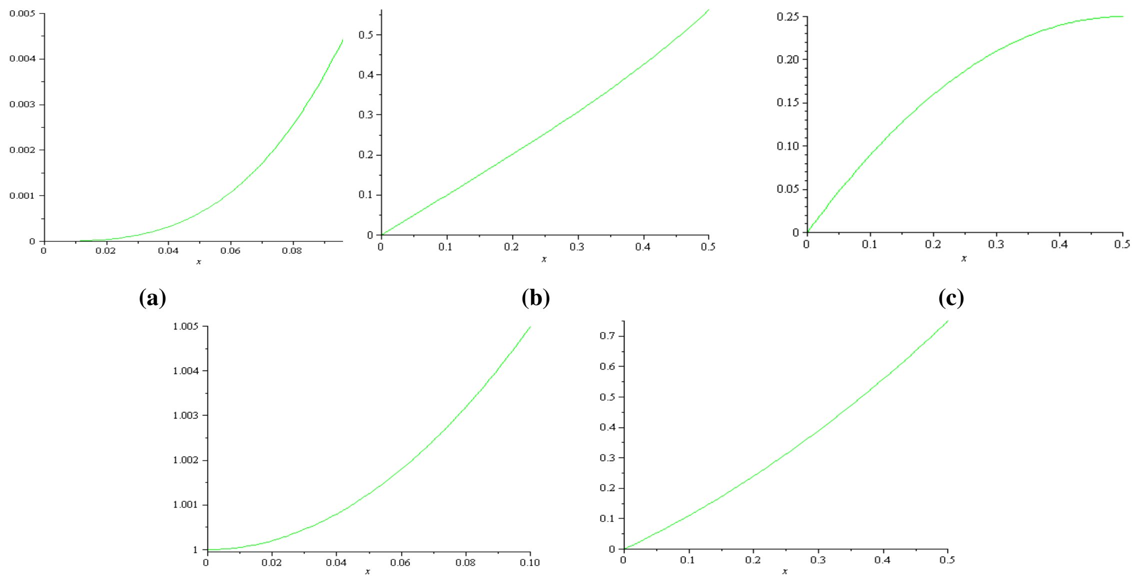
$\forall i,j \in \{0,1,2,\cdots,n\}$
After solving above system for unknown constants and putting the values of these constants in equation (4), we obtain the solution $w(x)$ (which may be approximate or exact).

Table 1: Volterra Integral Equation (IE), Constant and Coefficients (CC) and Exact Solution (ES)

Sr No.	IE	CC	ES
1	$w(x) = 5x^3 - x^5 + \int_0^x tw(t)dt$	$a_o = 30, a_1 = -90, a_2 = 90, a_3 = -30$ $a_4 = 0$	$w(x) = 5x^3$
2	$w(x) = x + x^4 + \frac{x^2}{2} + \frac{x^5}{5} - \int_0^x w(t)dt$	$a_o = 25, a_1 = -97, a_2 = 144, a_3 = -96$ $a_4 = 24, a_5 = 0$	$w(x) = x + x^4$
3	$w(x) = x - \frac{2x^3}{3} - 2\int_0^x w(t)dt$	$a_o = -1, a_1 = 3, a_2 = -2$ $a_3 = 0$	$w(x) = x - x^2$
4	$w(x) = 1 + \int_0^x Sin(x-t)w(t)dt$	$a_o = 2, a_1 = -2, a_2 = 1$ $a_3 = 0$	$w(x) = 1 + \frac{x^2}{2}$
5	$w(x) = -x^4 - 2x^3 + x^2 + x + 12\int_0^x (x-t)w(t)dt$	$a_o = 3, a_1 = -5, a_2 = 2$ $a_3 = 0$	$w(x) = x + x^2$

Table 2: Singular Integral Equation (IE), Constant and Coefficients (CC) and Exact Solution (ES)

Sr No.	IE	CC	ES
1	$w(x) = \frac{2x^3}{3} + \int_0^x \frac{1}{(x^4-t^4)^{\frac{1}{4}}} w(t)dt$	$a_o = 6, a_1 = -18, a_2 = 18, a_3 = -6$ $a_4 = 0$	$w(x) = x^3$
2	$w(x) = x^5 - \frac{16x^{\frac{21}{4}}}{63} + \int_0^x \frac{1}{(x^3-t^3)^{\frac{1}{4}}} w(t)dt$	$a_o = 120, a_1 = -600, a_2 = 1200, a_3 = -1200$ $a_4 = 600, a_5 = -120, a_6 = 0$	$w(x) = x^5$
3	$w(x) = 1 + x^2 - 2\sqrt{x} - \frac{16x^{\frac{5}{2}}}{15} + \int_0^x \frac{1}{\sqrt{x-t}} w(t)dt$	$a_o = 3, a_1 = -4, a_2 = 2$ $a_3 = 0$	$w(x) = 1 + x^2$
4	$w(x) = x^2 - \frac{16x^{\frac{5}{2}}}{15} + \int_0^x \frac{1}{\sqrt{x-t}} w(t)dt$	$a_o = 2, a_1 = -4, a_2 = 2$ $a_3 = 0$	$w(x) = x^2$
5	$w(x) = x(x+1) - \frac{4x^{\frac{3}{2}}}{3}(1 + \frac{4x}{5}) + \int_0^x \frac{1}{\sqrt{x-t}} w(t)dt$	$a_o = 3, a_1 = -5, a_2 = 2$ $a_3 = 0$	$w(x) = x + x^2$



(a) (b) (c)

(d) (e)

Fig. 1: Simulations with Volterra Integral Equation.

2.3 Maple Code

The maple code is used for finding the unknown constants or coefficients of linear or nonlinear terms, manually evaluating the constants and coefficients is very difficult to handle and time-consuming work.

3 Integral Equation

We may define integral equation as "An equation in which a function $w(x)$ which is to be evaluated appear inside integral operator", called an integral equation. Generally integral equation expressed as following

$$g(x)w(x) = h(x) + \beta \int_{q(x)}^{p(x)} k(x,t)w(t)dt \tag{6}$$

In above equation (4), $w(x)$ is unknown while $g(x)$,$h(x)$ and $k(x,t)$ are known functions, β is constant parameter and $p(x)$ and $q(x)$ both may be constant, variable or mixed.
Now, we give some solution of distinct types of IEs.

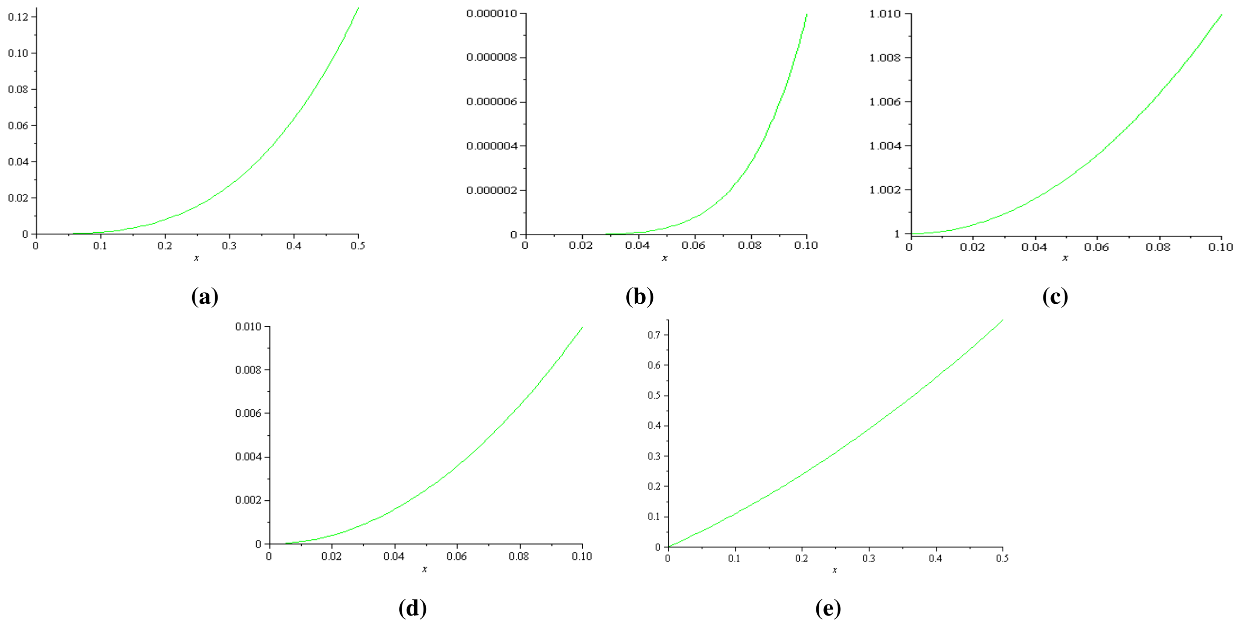


Fig. 2: Simulations with Singular Integral Equation.

3.1 Volterra Integral Equation

An IE having one limit of integration must be variable or both are variables categorized as Volterra integral equation. Generally expressed as follow

$$g(x)w(x) = h(x) + \beta \int_{q(x)}^{p(x)} k(x,t)w(t)dt \tag{7}$$

In above equation (7) both $p(x)$ and $q(x)$ may be variable, otherwise one of them is necessarily variable.

3.2 Singular Integral Equation

An integral equation having both or one of integration limits is infinity, or if the kernel becomes singular for one or more points with in integration limits. Generally expressed as follows:

$$h(x) = \int_0^x \frac{1}{(x-a)^\alpha} w(t)dt \qquad 0 < \alpha < 1 \tag{8}$$

$$w(x) = h(x) \int_0^x \frac{1}{(x-a)^\alpha} w(t)dt \qquad 0 < \alpha < 1 \tag{9}$$

4 Conclusion

The purpose of this work is to solve the Integral Equations with the Laguerre polynomials in Galerkin method which is suitable method to deal IEs, for getting unknown constants and solution Galerkin equation is solve with the help of maple Code because manually finding the constants is not an easy task.
Nowadays, many works are completed using computer applications for reducing manual difficulties and saving the time. The idea of this work is based on reduction of manual calculations and for quick results, which is successfully achieved.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] M. Rahman, Integral Equations and their Applications. WIT press, 2007.
[2] R. Kress, Linear Integral Equations. Edition 3rd ,Vol. 82. Berlin: Springer, 1989.
[3] E. C. Titchmarsh, Introduction to the theory of Fourier integrals (Clarendon Press, Oxford, 1937), pp. 334–339.
[4] F. Erdogan, G.D. Gupta, T. S. Cook, Numerical solution of singular integral equations. In: Sih, G.C. (eds) Methods of analysis and solutions of crack problems. Mechanics of fracture, vol 1. Springer, Dordrecht, 1973.
[5] M. E. Cardinali, C. Giomini, G. Marrosu, An always-converging, successive-substitution, fast algorithm to perform chemical equilibrium calculations, International Journal of Mathematical Education in Science and Technology 1997, 28, 381-391.
[6] M. Y. Ongun, The Laplace Adomian decomposition method for solving a model for HIV infection of $CD4^{+}T$ cells, Mathematical and Computer Modelling, 53, 597, 2011.
[7] M. Suleman, D. Lu, J. H. Heet al., Elzaki projected differential transform method for fractional order system of linear and nonlinear fractional partial differential equation, Fractals, 26, 1850041, 2018.
[8] A.-H. Abdel-Aty, M. M. Khater, D. Baleanu, S. M. Abo-Dahab, J. Bouslimi, M. Omri, Oblique explicit wave solutions of the fractional biological population (BP) and equal width (EW) models, Adv. Differ. Equ., 2020, 552, 2020.
[9] M. M. A. Khater, R. A. M. Attia, A.-H. Abdel-Aty, Computational analysis of a nonlinear fractional emerging telecommunication model with higher–order dispersive cubic–quintic, Information Sciences Letters, 9, 83, 2020.
[10] S. Yalcinba, Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations, Applied Mathematics and Computation, 127, 195, 2002.
[11] M. Dehghan, and F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method, Progress in Electromagnetics Research, 78, 361, 2008.
[12] A. Alipanah, and S. Esmaeili, Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function, Journal of Computational and Applied Mathematics, 235, 5342, 2008.
[13] K. Maleknejad, M. Khodabin, and M. Rostami, Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions, Mathematical and Computer Modelling, 55, 791, 2011.
[14] M. S. Hashmi, N. Khan, and S. Iqbal, Numerical solutions of weakly singular Volterra integral equations using the optimal homotopy asymptotic method, Computers and Mathematics with Applications, 64, 1567, 2011.
[15] S. H. Behiry, and S. I. Mohamed, Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method, Natural Science, 4, 581-587,2012.
[16] J.A. Shali, P. Darania, and A. A. J. Akbarfam, Collocation method for nonlinear Volterra-Fredholm integral equations, Open Journal of Applied Sciences, 2, 115, 2012.
[17] A. Jafarian, and S. M. Nia, Utilizing feed-back neural network approach for solving linear Fredholm integral equations system, Applied Mathematical Modelling, 37, 5027, 2012.
[18] S. Nemati, P. M. Lima, and Y. Ordokhani, Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials, Journal of Computational and Applied Mathematics, 242, 53, 2012.
[19] M. Ghasemi, M. Fardi, and R. K. Ghaziani, Solution of system of the mixed Volterra-Fredholm integral equations by an analytical method, Mathematical and Computer Modelling, 58, 1522-1530, 2013.
[20] F. Mirzaee, and A. A. Hoseini, Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series, Alexandria Engineering Journal, 52, 551, 2013.
[21] L. H. Yang, H. Y. Li, and J. R. Wang, Solving a system of linear Volterra integral equations using the modified reproducing kernel method, Abstract and Applied Analysis, 2013, 1, 2013.
[22] F. Calio, and E. Marchetti, Cubic spline approximation for weakly singular integral models." Applied Mathematics, 4, 1563, 2013.
[23] Y. Al-Jarrah, and En-Bing Lin, Numerical Solution of Freholm-Volterra Integral Equations by Using Scaling Function Interpolation Method, Applied Mathematics, 4, 204-209, 2013.
[24] H. H. Ali, and F. Abdelwahid, Modified Adomian Techniques Applied to Non-Linear Volterra Integral Equations, Open Journal of Applied Sciences, 3, 202, 2013.
[25] H. Almasieh, and J. N. Meleh, Numerical solution of a class of mixed two-dimensional nonlinear Volterra-Fredholm integral equations using multiquadric radial basis functions, Journal of Computational and Applied Mathematics, 260, 173, 2013.
[26] R. Behzadi, E. Tohidi, and F. Toutounian, Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler-Maclaurin summation formula, Journal of Taibah University for Science, 8, 200, 2013.
[27] H. Zhang, Y. Chen, and X. Nie, Solving the linear integral equations based on radial basis function interpolation, Journal of Applied Mathematics, 2014, 1, 2014.
[28] F. Ghomanjani, M. H. Farahi, and A. Klçman, Bezier curves for solving Fredholm integral equations of the second kind, Mathematical Problems in Engineering, 2014, 1, 2014.
[29] M. H. Heydari, et al., A computational method for solving stochastic Ito-Volterra integral equations based on stochastic operational matrix for generalized hat basis functions, Journal of Computational Physics, 270, 402, 2014.
[30] S. M. Dardery, and M. M. Allan, Chebyshev polynomials for solving a class of singular integral equations, Applied Mathematics, 5, 753, 2013.
[31] S. Kumar, et al., Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, Journal of the Egyptian Mathematical Society, 23, 102, 2014.
[32] L. Chen, and J. Duan, Multistage numerical picard iteration methods for nonlinear Volterra integral equations of the second kind, Advances in Pure Mathematics, 5, 672, 2015.
[33] L. Xiaoyan, Z. Liu, and J. Xie, Solving Systems of Volterra Integral Equations with Cardinal Splines, Journal of Applied Mathematics and Physics, 3, 1422, 2015.
[34] S. Noeiaghdam, E. Zarei, and H. B. Kelishami, Homotopy analysis transform method for solving Abel's integral equations of the first kind, Ain Shams Engineering Journal, 7, 483, 2015.

[35] J. E. Mamadu, and I. N. Njoseh, Numerical solutions of Volterra equations using Galerkin method with certain orthogonal polynomials, Journal of Applied Mathematics and Physics, 4, 376, 2016.
[36] S. Davaeifar, and J. Rashidinia, Boubaker polynomials collocation approach for solving systems of nonlinear Volterra-Fredholm integral equations, Journal of Taibah University for Science, 11, 1182, 2017.
[37] I. Rahman, M. M. Parvez, and S. Ghosh, A New Technique for Numerical Solution of System of Volterra Integral Equations of the Second Kind by Simpson's Quadrature Rule, Journal of Computer and Mathematical Sciences, 8, 332, 2017.
[38] J. A. Zarnan, A Novel Approach For The Solution Of Urysohn Integral Equations Using Hermite Polynomials, International journal of applied Engineering Research, 12, 14391, 2017.
[39] M. Al-Mazmumy, and S. O. Almuhalbedi, Restarted Adomian Decomposition Method for Solving Volterra's Population Model, American Journal of Computational Mathematics, 7, 175, 2017.
[40] N. Bilqees, et al, Solution of ordinary differential equations and Volterra integral equation of first and second kind with bulge and logarithmic functions using Laplace transform." International Journal of Advanced and Applied Sciences, 5, 82, 2018.
[41] A. E. Abaoub, A. S. Shkheam, and S. M. Zali, The Adomian Decomposition Method of Volterra Integral Equation of Second Kind, American Journal of Applied Mathematics, 6, 141, 2018.