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On the Robustness of Right Truncated Esscher Transformed Laplace Distribution

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Abstract: Truncation arises in many practical situations such as Epidemiology, Material science, Psychology, Social Sciences and Statistics where one wants to study about data which lie above or below a certain threshold or within a specified range. Right truncation often happens when an event/source is detected if its measurement is less than a truncation variable. The present study aims to introduce a right truncated version of an asymmetric and heavy tailed distribution namely, Esscher transformed Laplace distribution beyond the interval $(-\infty, b)$. Various distributional and reliability properties of the proposed distribution are investigated. The performance of η, the parameter, is estimated using nlm method and the robustness of the RTETL(η; b) distribution with respect to b, where $b(>0)$, the truncation point, is illustrated using simulation study. A real data analysis of breaking stress of carbon fiber is also carried out.

Keywords: Esscher transformed Laplace distribution, estimation, right truncation, real data analysis, robustness, simulation.

1 Introduction

Truncated distributions are inevitable parts of practical statistics where the distributional values are limited to lie above or below a given threshold or within a specified range. Usually truncation happens because there is no interest beyond the truncation point. Also, if the distribution is not valid beyond the truncation point, truncation may occur. There exists many applied fields where the concept of truncation arises especially in medicine, industry, reliability etc. This attracted the attention of many researchers and a lot of truncated distributions were emerged in statistical literature. In industry the concept of truncation will improve the quality of the produced items through effective process optimization. The basic notions of left and right truncation were introduced by Galton [1] and later by Pearson and Lee [2]. Based on their studies, several truncated distributions were derived viz. truncated Gamma distribution, truncated exponential distribution, truncated t and F distributions, truncated Cauchy distribution, truncated Pareto distribution, truncated Weibull distribution, left truncated beta distribution, right and left truncated generalized Gaussian distribution, truncated Frechet-Weibull and Frechet distributions, Weibull truncated exponential distribution, right truncated mixture Topp-Leone with exonential distribution, [0,1] truncated Gompertz exponential distribution, simplex truncated multivariate normal distribution, truncated Cauchy power Weibull-G class of distributions, exponentiated left truncated power distribution, truncated bivariate Kumaraswamy exponential distribution, Weibull generalized truncated Poisson distribution and truncated exponentiated exponential distribution. For details see, Chapman [3], Cosentino et al. [4], Kotz and Nadarajah [5], Nadarajah and Kotz [6], Masoom Ali and Nadarajah [7], Zhang and Xie [8], Zaninetti [9], Anithakumari et al. [10], Anithakumari et al. [11], Abid and Abdulrazak [12], Gul et al. [13], Hashim and Al-kadim Al-Khafaji [14], Hussein and Ahmed [15], Adams [16], Alotaibi et al. [17], Arshad et al. [18], El-Damrawy et.al [19], Orabi-et al. [20] and Ribeiro-Reis [21].

The manuscript is arranged in the following way. In section 2, a review of Esscher transformed Laplace distribution is done. Right truncated Esscher transformed Laplace distribution is developed and studied in section 3. A simulation study for illustrating the performance of the

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parameter as well as the robustness of the distribution with respect to the truncation point are done in section 4. In section 5, the credibility of the proposed model is illustrated by analyzing a real data. Considering all this, the article ended by section 6.

2 Esscher Transformed Laplace (ETL) Distribution

Esscher transformed Laplace (ETL) distribution introduced by George and George [22] is a new asymmetric distribution having tail heaviness, which is developed through Esscher transformation, a concept introduced by Esscher [23]. This novel distribution is a member of exponential family(one parameter) and is obtained from classical Laplace distribution through exponential tilting.
The probability density function and cumulative distribution function of Esscher transformed Laplace $(ETL(\eta))$ distribution are

$$f(x,\eta)=\begin{cases}\frac{1-\eta^2}{2}e^{x(1+\eta)}; & x<0\\ \frac{1-\eta^2}{2}e^{-x(1-\eta)}; & x\geq 0,\quad |\eta|<1\end{cases} \tag{1}$$

and

$$F(x)=\begin{cases}\frac{1-\eta}{2}e^{x(1+\eta)]}; & x<0\\ \frac{1-\eta}{2}+\frac{1+\eta}{2}[1-e^{-x(1-\eta)]}]; & x\geq 0\end{cases}. \tag{2}$$

This type of distributions are asymmetric and having tails heavier than normal one. They are useful in modeling financial and flood level data which are having heavy-tailed nature and high skewness. For details (see, Dais George and Sebastian George [24], Dais et al.[25] and Rimsha, H. and Dais George [26]).

3 Right Truncated Esscher transformed Laplace Distribution

In this section, we consider the right truncation of (1) beyond the interval $(-\infty,b)$. Let f(x) and F(x) denote the probability density function and distribution function of a continuous random variable X. If we remove the values of X above a specified point b, the distribution of the resulting values can be written as

$$f(x,b)=\frac{f(x)}{F(b)}; \quad -\infty<x<b \tag{3}$$

and the distribution is said to be right truncated at b. The probability density function of right truncated Esscher transformed Laplace $(RTETL(\eta,b))$ distribution is obtained as

$$g(x,\eta,b)=\begin{cases}\frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1+\eta)e^{-b(1-\eta)}}; & -\infty<x<b;\ b\geq 0\\ \frac{(1+\eta)e^{x(1+\eta)}}{e^{b(1+\eta)}}; & -\infty<x<b;\ b<0\end{cases} \tag{4}$$

Graphs of the pdf of $RTETL(\eta,b)$ for different values of η and b=10 are shown in Figure 1(a) and 1(b).

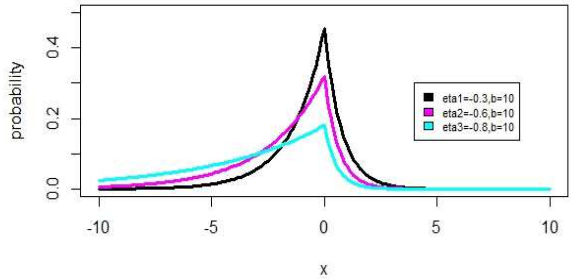

Fig.1(a): Plots of $RTETL(\eta,b)$ distribution for $\eta\in(-1,0)$, b=10

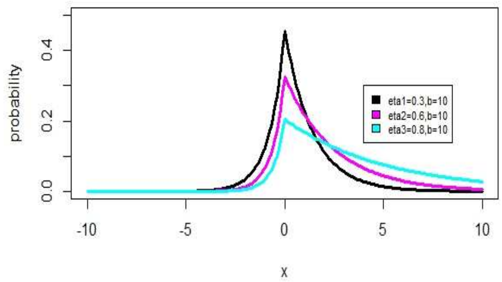

Fig.1(b): Plots of $RTETL(\eta,b)$ distribution for $\eta\in(0,1)$, b=10

The distribution function, characteristic function and moments of (4) are respectively

$$G(x,\eta)=\begin{cases}\frac{2-(1+\eta)e^{-x((1-\eta)}}{2-(1+\eta)e^{-b(1-\eta)}}; & -\infty<x<b;\ b\geq 0\\ \frac{e^{x(1+\eta)}}{e^{b(1+\eta)}}; & -\infty<x<b;\ b<0\end{cases}, \tag{5}$$

$$\phi_X(t) = \begin{cases} \frac{1-\eta^2}{2-(1+\eta)e^{-b(1-\eta)}}[\frac{2}{1-(\eta+it)^2} - \frac{e^{it-(1-\eta)b}}{1-(\eta+it)}]; \quad b \geq 0 \\ \frac{1+\eta}{1+\eta+it}[\frac{e^{b(1+\eta+it)}}{e^{b(1+\eta)}}]; \quad b < 0, \quad t \in R \end{cases} \tag{6}$$

and

$$E(X^r) = \begin{cases} \frac{1-\eta^2}{2-(1+\eta)e^{-b(1-\eta)}}[\frac{1}{(1-\eta)^{r+1}}[\gamma(r+1, b(1-\eta))] + \frac{(-1)^r r!}{(1+\eta)^{r+1}}]; \quad b \geq 0 \\ \frac{1+\eta}{e^{b(1+\eta}}[(\frac{(-1)^{r+1})(\gamma(r+1,-b(1+\eta)))}{(1+\eta)^{r+1}} + \frac{(-1)^r r!}{(1+\eta)^{r+1}}]; \quad b < 0 \end{cases}$$

where $\gamma(a,x)$ is the lower incomplete gamma function. The mean is

$$E(X) = \begin{cases} \frac{1-\eta^2}{2-(1+\eta)e^{-b((1-\eta)}}[\frac{1}{(1-\eta)^2}[\gamma(2, b(1-\eta))] - \frac{1}{(1+\eta)^2}]; \quad b \geq 0 \\ \frac{-1}{1+\eta} + b; \quad b < 0 \end{cases}$$

and

$$E(X^2) = \begin{cases} \frac{1-\eta^2}{2-(1+\eta)e^{-b(1-\eta)}}[\frac{1}{(1-\eta)^3}[\gamma(3, b(1-\eta))] + \frac{2}{(1+\eta)^3}]; b \geq 0 \\ \frac{2}{(1+\eta)^2} + b^2 - \frac{2b}{1+\eta}; \quad b < 0. \end{cases}$$

Using these two moments we can find variance of the distribution.

3.1 Reliability Characteristics

The survival function, failure rate, cumulative hazard rate and second rate of failure of $RTETL(\eta,b)$ distribution are respectively

$$S(x) = \frac{(1+\eta)[e^{-x(1-\eta)} - e^{-b(1-\eta)}]}{2-(1+\eta)e^{-b(1-\eta)}},$$

$$r(x) = \frac{(1-\eta)e^{-x(1-\eta)}}{e^{-x(1-\eta)} - e^{-b(1-\eta)}},$$

$$-logS(x) = -log[\frac{(1+\eta)[e^{-x(1-\eta)} - e^{-b(1-\eta)}]}{2-(1+\eta)e^{-b(1-\eta)}}],$$

and

$$r^*(x) = log[\frac{e^{-x(1-\eta)} - e^{-b(1-\eta)}}{e^{-(x+1)(1-\eta)} - e^{-b(1-\eta)}}].$$

3.2 Order Statistics

Let $X_{(1:n)} \leq X_{(2:n)} \leq X_{(3:n)} \leq ... \leq X_{(n:n)}$ denote the order statistics obtained from a random sample $X_1, X_2, ...X_n$ taken independently from $RTETL(\eta,b)$ distribution. Then the probability mass function of first order statistics is given by

$$f_{X_{(1:n)}}(x) = \begin{cases} n\frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1+\eta)e^{-b((1-\eta)}}[1 - \frac{2-(1+\eta)e^{-x(1-\eta)}}{2-(1+\eta)e^{-b(1-\eta)}}]^{n-1}; b \geq 0 \\ n\frac{(1+\eta)e^{x(1+\eta)}}{e^{b(1+\eta)}}[1 - \frac{e^{x(1+\eta)}}{e^{b(1+\eta)}}]^{n-1}; b < 0. \end{cases}$$

and the probability mass function of the n^{th} order statistics is given by

$$f_{X_{(n:n)}}(x) = \begin{cases} n\frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1+\eta)e^{-b((1-\eta)}}[\frac{2-(1+\eta)e^{-x((1-\eta)}}{2-(1+\eta)e^{-b((1-\eta)}}]^{n-1}; b \geq 0 \\ n\frac{(1+\eta)e^{x(1+\eta)}}{e^{b(1+\eta)}}[\frac{e^{x(1+\eta)}}{e^{b((1+\eta)}}]^{n-1}; b < 0. \end{cases}$$

3.3 Record Value

Records arises in many practical situations such as sports, traffic, industry, medicine etc. Arnold et al.[27] made detailed description on the theory of records. Let X_1, X_2,....X_n be asequence of i.i.d random variables taken from $RTETL(\eta,b)$ distribution. If the value of an observation X_j exceeds that of all previous observations, then X_j is called an upper record value. The pdf of the n^{th} record value say R_n of (4) is

$$f_{R_n}(x) = \begin{cases} \frac{(1-\eta^2)e^{\eta x-|x|}}{2-(1+\eta)e^{-b(1-\eta)}n!} \left[-log\left(1 - \frac{2-(1+\eta)e^{-x(1-\eta)}}{2-(1+\eta)e^{-b(1-\eta)}}\right)\right]^n; \quad b \geq 0 \\ \frac{(1+\eta)e^{x(1+\eta)}}{e^{b(1+\eta)}n!}[-log(1 - \frac{e^{x(1+\eta)}}{e^{b(1+\eta)}})]^n; \quad b < 0. \end{cases}$$

3.4 Entropy

Shannon's entropy is defined by

$$H(X) = E(-logf(x)).$$

For $RTETL(\eta,b)$distribution, Shannon's entropy is obtained as

$$H(X) = \begin{cases} log[\frac{2-(1+\eta)e^{-b((1-\eta)}}{1-\eta^2}] - \frac{(1-\eta^2)be^{-b(1-\eta)}}{2-(1+\eta)e^{-b((1-\eta)}} + 1; \quad b \geq 0 \\ 1 - b(1+\eta) + log[\frac{e^{b(1+\eta)}}{1+\eta}]; \quad b < 0. \end{cases}$$

3.5 Estimation

In this section, we estimate the parameter η of the $RTETL(\eta,b)(b > 0)$ distribution using maximum likelihood estimation method. We take a random sample $X_1, X_2 ... X_n$ of size n from an RTETL distribution. Then the logarithm of likelihood function is

$$logL = nlog(1-\eta^2) + \sum(\eta x - |x|) \\ -nlog[2-(1+\eta)e^{-b((1-\eta)}].$$

$$\frac{\partial logL}{\partial \eta} = \frac{-2\eta}{1-\eta^2} + [b(1+\eta)+1] \\ \frac{e^{-b((1-\eta)}}{2-(1+\eta)e^{-b((1-\eta)}} + \bar{X} = 0.$$

The solution of $\frac{\partial logL}{\partial \eta} = 0$ will provide the MLE of η. But this equation cannot be solved analytically. So we use nlm (Non-Linear Maximization) method for estimating the parameter η.

4 Simulation study

Here, to evaluate the performance of the nlm estimator of the parameter η, we use Monte-Carlo simulation method. We also evaluate the robustness of the $RTETL(\eta, b)$ $(b > 0)$ distribution with respect to b, the truncation point. For that we generate 5000 random samples of sizes n=20, 50, 75 and 100 from the $ETL(\eta)$ distribution for some true values of the parameter η= -0.3, 0.3 and 0.6. We select the right truncated data from each generated sample for different truncation points viz. b=5, 20, 50 and 100 and find out 5000 estimates of η using (4) for each sample sizes. The estimate of the parameter, average bias and mean square error of the estimate (MSE) are computed and it is given in Table 1, Table 2, Table 3 and Table 4.

For different choices of n, the sample size, the average bias and average mean square error(MSE) of the estimates are seemed to be reasonably small. But as the sample size increase there exists a slight decrease in both the measurers. Also we can see that there is no considerable variation in the average bias and average MSE with respect to various choice of the parameters. More over, it is found that there is no significant difference in both the measures for different choice of the truncation points. So $RTETL(\eta, b)$ $(b > 0)$ distribution is robust with respect to different truncation points.

5 Real Data Analysis

Breaking stress study for metals is a very important study in material science since it is needed for safety considerations. Carbon fibers are stiff strong and light. It has automotive applications and used in processes to create excellent building materials. The quality of carbon fibre is ensured by controlling the breaking stress so that its failure will requires expensive repair. Then only the manufactured components meet the required specification. As an application of the proposed model we use the data set given in Nichols and Padgett [28]. The data set consists of 100 observations of breaking stress of carbon fibers in (Gba) and are recorded below.

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57

1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.87, 1.89, 1.92

2.0, 2.03, 2.03, 2.05, 2.12, 2.17,2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50

2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82

2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15

3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60
3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56

Figure 2 depicts histogram of the standardised data .

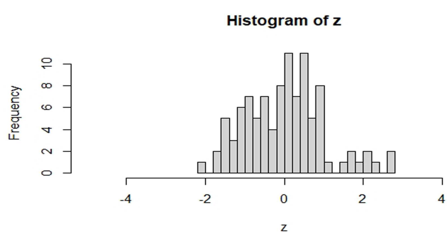

Fig.2: Histogram of the standardised data.

Since it has the same shape as that of $RTETL(\eta, b)$ distribution, we fit the $RTETL(\eta, b)$ distribution using the above data set with truncation point b=1.5.

The embedded pdf is given in Figure 3.

Also, using Kolmogrov-Smirnov (K-S) test,we check the goodness of fit . It is found that the K-S distance for $RTETL(\eta, b)$ is 0.54639 and p-value is 0.9291. In Amal et al. [29], it is shown that the truncated Lomax Frechet(TLF) distribution is a superior model for the breaking stress of carbon fibers data than Frechet,

Table 1: Values of $\hat{\eta}$, average bias and average MSE for b=5 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.2631	0.0369	0.0175	0.2630	-0.0369	0.0176	0.4547	-0.1452	0.0296
50	-0.2696	0.0303	0.0075	0.2702	-0.0298	0.0076	0.4636	-0.1363	0.0219
75	-0.2710	0.0289	0.0053	0.2743	-0.0256	0.0051	0.4656	-0.1343	0.0203
100	-0.2732	0.0267	0.0040	0.2738	-0.0261	0.0041	0.4648	-0.1351	0.0199

Table 2: Values of $\hat{\eta}$, average bias and average MSE for b=20 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.2669	0.0330	0.0173	0.2654	-0.0345	0.0169	0.4558	-0.1441	0.0292
50	-0.2706	0.0293	0.0074	0.2702	-0.0297	0.0076	0.4625	-0.1374	0.0223
75	-0.2732	0.0268	0.0053	0.2722	-0.0277	0.0053	0.4644	-0.1355	0.0207
100	-0.2736	0.0263	0.0040	0.2741	-0.0258	0.0040	0.4646	-0.1353	0.0201

Table 3: Values of $\hat{\eta}$, average Bias and average MSE for b=50 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.2621	0.0378	0.0181	0.2646	-0.0353	0.0178	0.4551	-0.1448	0.0295
50	-0.2691	0.0308	0.0076	0.2720	-0.0280	0.0073	0.4611	-0.1388	0.0227
75	-0.2742	0.0257	0.0051	0.2745	-0.0254	0.0051	0.4649	-0.1351	0.0205
100	-0.2744	0.0255	0.0040	0.2730	-0.0269	0.0040	0.4660	-0.1339	0.0197

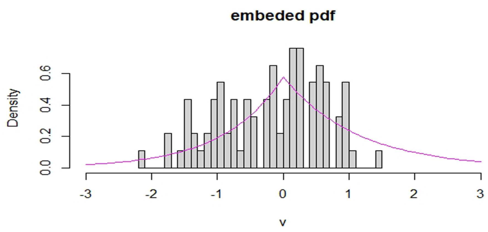

Fig.3: Histogram and embedded pdf of $RTETL(\eta,b)$ distribution

exponentiated Frechet, Marshall-Olkin Frechet, transmuted Frechet, Kumaraswamy Frechet, transmuted Marshall-Olkin Frechet and the Weibull Frechet. So we compare the results of $RTETL(\eta,b)$ distribution with the truncated Lomax Frechet(TLF) distribution and it is seen that for the TLF distribution the K-S distance measure and p-value are 0.6666 and 0.1502 respectively. Again, we calculate the AIC and BIC values corresponding to both the distributions and are tabulated in Table 5.

Table 5 shows that the right truncated Esscher transformed Laplace ($RTETL(\eta,b)$) distribution is a better model when compared to the truncated Lomax Frechet distribution.

6 Conclusion

In this work, we proposed right truncated Esscher transformed Laplace ($RTETL(\eta,b)$) distribution and studied its various distributional and structural properties. The parameter of the distribution η is estimated using nlm method and its accuracy is also illustrated using simulation method. We study the robust nature of the proposed distribution using simulated data with respect to the truncation points and it is obtained that there is no appreciable difference in both average bias and average MSE for different choice of the truncation points. So the $RTETL(\eta,b)$ $(b>0)$ distribution is robust with respect to the different truncation points. The breaking stress of carbon fibre is modelled using the $RTETL(\eta,b)$ distribution and it is compared with truncated Lomax Frechet distribution. It is found that the proposed distribution is a superior model than truncated Lomax Frechet distribution in modeling breaking stress of carbon fibre data.

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Table 4: Values of $\hat{\eta}$, average bias and average MSE for b=100 and different values of n, η

sample size	η=-0.3			η=0.3			η=0.6		
n	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE	$\hat{\eta}$	Bias	MSE
20	-0.2624	0.0375	0.0175	0.2638	-0.0362	0.0177	0.4572	-0.1427	0.0291
50	-0.2688	0.0311	0.0078	0.2711	-0.0288	0.0076	0.4623	-0.1376	0.0224
75	-0.2731	0.0268	0.0051	0.2735	-0.0264	0.0051	0.4629	-0.1370	0.0211
100	-0.2744	0.0255	0.0040	0.2745	-0.0254	0.0040	0.4649	-0.1350	0.0199

Table 5: Values of MLE's, KS, Log likelihood, AIC, and BIC values of breaking stress of carbon fibre data.

Distribution fitted	estimate	KS	LL	AIC	BIC
RTETL	$\hat{\eta} = 0.10956$	0.54639	-113.029	228.058	230.579
TLF	$\hat{\alpha} = 62.70$	0.6666	-143.123	292.247	292.245
	$\hat{\mu} = 146.961$				
	$\hat{\delta} = 0.5180$				

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