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Reflection and Transmission of an Incident Progressive Wave by Obstacles in Homogeneous Shallow Water

K. K. Al Arfaj [1,*], *M. A. Helal* [2] *and M. S. Abou-Dina* [2]

[1]Department of Mathematics, College of Science, King Faisal University, Al-Ahsa, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Cairo University, Cairo, Egypt

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Abstract: The influence of a suspended fixed obstacle on an incident progressive wave inside an ideal homogeneous shallow water is studied in two dimensions. The fluid occupies an infinite channel of a constant depth, and a fixed obstacle of a small horizontal extent is partially submerged without contact with the bottom of the channel. An asymptotic double series expansion for the solution is used. The procedure enables us to calculate analytic expressions for the local perturbations up to the second order. The results of the first-order approximation indicate that no reflections exist. The second-order approximation of the solution is found to be the superposition of a progressive wave and local perturbations. For approximations of order higher than two, a secular term which increases monotonically with time and distance appears in the expressions for the progressive wave. This unacceptable result is due to a certain aspects in the mathematical procedure used. For this reason, the procedure is modified by using a suitable transformation of variables which reduces the determination of the transmitted wave to the solution of the **KdV** equation. As an illustration, the special case of the incident uniform flow is considered and the stream lines of the resulting flow are drawn.

Keywords: Progressive wave, shallow water, reflection, fixed obstacle

1 Introduction

Simulations for the geophysical phenomena of a fluid flow over weirs, under gates and submerged elands were studied in several theoretical and experimental works. These works deal with model problems of free-surface fluid flow over a topography or under floating submerged bodies. the theoretical problem is a nonlinear boundary value problem which may be in certain cases, constrained by initial conditions (see [3,13]).

The two-dimensional fluid flow over an obstacle or under a floating body, within the frame of the linearized theory of motion, has been investigated by several authors, for instance [4,9,12,14]. The mathematical theory used in these investigations is inadequate to describe the important nonlinear aspects of the phenomenon. Using a certain procedure the solution for the velocity potential of the nonlinear problem is expressed as a power series in a certain small parameter [13]. The above-mentioned linearized theory assumes the first term of such a series as a first approximation of the solution. the radius of convergence of this series is shown by Gouyon [6] to be of the same order as that of the ratio of the free surface amplitude to the wave length. Hence, this theory is inadequate to deal with the propagation of long waves.

Different numerical techniques were developed to solve the nonlinear system of equations to which the original problem is reduced. Yeung [15] present an exhaustive review of the numerical techniques which are widely applied to this system.

Analytical techniques, within the frame of the shallow-water theory, were used by several authors to investigate free-surface flows over certain non-horizontal bottoms, see [1,2,7].

Guli [7] and Abou-dina and Helal [1] studied the problem of the reflection and transmission of an incident progressive wave over a topography in shallow water using both of the Lagrangean an Eulerian description of the problem, respectively.

In the present work, we investigate the effect of a fixed vertical submerged barrier on the propagation of an incident wave inside a homogeneous fluid. Euler's description is used and the problem is studied within the frame of the two-

* Corresponding author e-mail: kkarfaj@kfu.edu.sa

dimensional shallow water theory. The fluid is supposed to occupy an infinite channel of constant depth and the horizontal extent of the submerged barrier is assumed to be small, see fig. (1).

The analysis enables to separate progressive waves from local perturbations and shows the absence of reflected waves in the first order of approximation. These results are similar to those obtained for the case of nonhorizontal topograpgy by Ogilvie [9], Guli [7] and Abou-Dina and Helal [1]. The second order approximation of the solution is found to be the superposition of progressive wave and local perturbations. Analytical expressions are calculated for the local perturbations of the second order. For approximations of order higher than two, the expressions for the progressive waves contain a secular term which increases monotonically with time and distance. This unacceptable result is due to certain aspects of the mathematical used procedure. For this reason, the procedure is modified by utilizing a suitable transformation of variables. The modification reduces the determination of the transmitted wave to be the solution of the equation of Korteweg and de Vries (**KdV**).

As an illustration, the special case of the incident uniform flow is considered and the stream lines of the resulting flow are drawn.

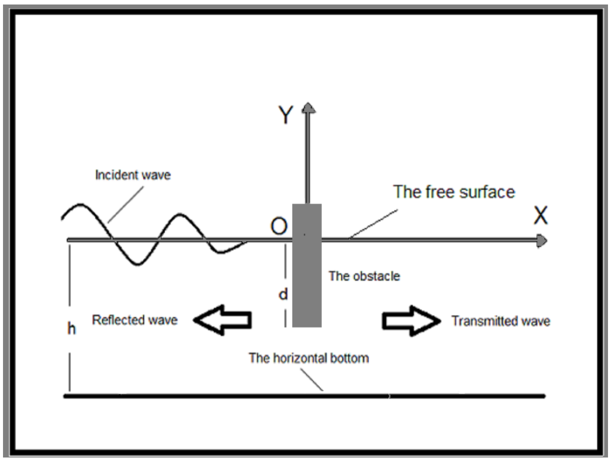

Fig. 1: Explanatory diagram of an upstream wave inside a fluid with a fixed immersed obstacle penetrating the free surface

The origin of the Cartesian system of coordinates is fixed in the submerged obstacle. The axis **Ox** points along the direction of the incident-wave velocity, the axis **Oy** is vertical upwards and the plane **Oxz** coincides with the free surface at rest. The bottom of the channel is impermeable and horizontal.

2 Main problem

Consider an incident upstream wave inside a fluid layer with free surface and finite depth in an infinitely long channel. The bottom of the channel is horizontal and a fixed submerged obstacle penetrating the free surface without contact with the bottom of the channel is present fig (1). It is required to determine the reflected and transmitted parts of the incident wave.

To simplify the mathematics, the problem is assumed two-dimensional, and the fluid is taken to be ideal and homogeneous. Also, the motion is assumed irrotational and sufficiently slow. Hence, the free surface remains always in the neighbourhood of its position at rest. The horizontal extent of the submerged obstacle is taken to be small compared with the dimensions of the channel.

The motion will be referred to a fixed orthogonal Cartesian system of coordinates $O(x,y)$ fig (1).

For the rest of this work, we consider the following notations:

- d Depth of the submerged part of the obstacle (fig (1)),
- ρ The constant density of the fluid,
- g The acceleration of gravity,
- c_0 The critical velocity $(= \sqrt{gh}\,)$,
- ε A small parameter,

■t The parameter of time,
■(x,y) The rectangular coordinates of a point,
■$x = k^{\pm}(y) = \varepsilon k_0^{\pm}(y)$ The equation of the obstacle's boundary,
■$y = \eta(x,t)$ The equation of the free surface,
■$P(x,y,t)$ The pressure applied to the fluid particle occupying the position (x,y) at the instant of time t,
■$\mathbf{W}(x,y,t)$ The velocity of the particle, which occupies the position (x,y) at the instant of time t,
■$\Phi(x,y,t)$ The velocity potential function $[\mathbf{W}(x,y,t) = \nabla\Phi(x,y,t)]$,
■$\Psi(x,y,t)$ The stream function.
■$+,-$ The regions on the right and on the left of the obstacle, respectively.
■$\prime, \prime\prime$ First and second derivatives with respect to the argument of the superscripted function,
■i,t,r Incident, transmitted and reflected waves, respectively,
■$\frac{\partial}{\partial n}$ The derivative directed along the outward normal to the considered surface.

3 The equations of motion

The irrotationality condition reduces the problem to the search for a velocity potential function $\Phi(x,y,t)$ with finite and continuous gradient $\mathbf{W}(x,y,t)$ in addition to the free surface elevation function $\eta(x,t)$, as described in [13]. Note that the physical and simplifying conditions, the equations satisfied by $\Phi(x,y,t)$ are found to be presented as follows
In the fluid mass with constant density : The continuity equation reads

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \tag{1}$$

and Bernoulli's equation for the unsteady flow is

$$P(x,y,t) = -\rho\left\{\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi'}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right] + gy\right\}.$$

On the free surface $(y = \eta(x,t))$: The impermeability of this boundary is expressed as

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial t} \quad \text{at} \quad y = \eta(x,t),$$

witch leads to

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right] + gy = 0 \quad \text{at} \quad y = \eta(x,t).$$

On the horizontal bottom: The impermeability requires that

$$\frac{\partial\Phi}{\partial y} = 0 \quad \text{at} \quad y = -h.$$

On the impermeable submerged obstacle: The boundary conditions, expressing the impermeability of the submerged obstacle take the following forms

$$\frac{\partial\Phi^{\pm}}{\partial x} - \frac{d}{dy}k^{\pm}(y)\frac{\partial\Phi^{\pm}}{\partial y\,|} = 0$$

$$\text{at} \quad x = k^{\pm}(y), -d \leqslant y \leqslant \eta^{\pm}(x,t).$$

The velocity $\mathbf{W}(x,y,t)$ in the fluid mass under the submerged obstacle is continuous.
The radiation condition: The radiation condition (see Stoker (1956) and Wehasen & Laitone (1966)) states that
(a) At the upstream extremity of the channel only the reflected wave and the incident wave are found.
(b) At the downstream extremity of the channel only the transmitted wave propagates.

The Cauchy-Riemann conditions: These conditions combine the velocity potential Φ and the stream function Ψ by the following relations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \tag{2}$$

$$\frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}. \tag{3}$$

4 The shallow water theory

Following Germain [5], we introduce the set of distorted variables

$$\tilde{x} = \varepsilon x, \quad \tilde{y} = y \quad \text{and} \quad \tilde{t} = \varepsilon t, \tag{4}$$

where ε is the small parameter defined above. The system of equations of the problem (1)-(3) is written in terms of $\tilde{x}, \tilde{y}$ and $\tilde{t}$ as

$$\varepsilon^2 \frac{\partial^2 \tilde{\Phi}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\Phi}}{\partial \tilde{y}^2} = 0, \tag{5}$$

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{y}} = \varepsilon^2 \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} \frac{\partial \tilde{\eta}}{\partial \tilde{x}} + \varepsilon \frac{\partial \tilde{\eta}}{\partial \tilde{t}} \quad \text{for} \quad \tilde{y} = \tilde{\eta}(\tilde{x}, \tilde{t}), \tag{6}$$

$$\varepsilon \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} + \frac{1}{2}\left[\varepsilon^2 \left(\frac{\partial \tilde{\Phi}}{\partial \tilde{x}}\right)^2 + \left(\frac{\partial \tilde{\Phi}}{\partial \tilde{y}}\right)^2\right] + g\tilde{y} = 0 \quad \text{for} \quad \tilde{y} = \tilde{\eta}(\tilde{x}, \tilde{t}), \tag{7}$$

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{y}} = 0 \quad \text{at} \quad \tilde{y} = -h, \tag{8}$$

$$\varepsilon \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} \frac{d}{d\tilde{y}} k^{\pm}(\tilde{y}) = 0 \quad \text{at} \quad \tilde{x} = \varepsilon k^{\pm}(\tilde{y}), \tag{9}$$

$$\varepsilon \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} = \frac{\partial \tilde{\Psi}}{\partial \tilde{y}}, \tag{10}$$

and

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{y}} = -\varepsilon \frac{\partial \tilde{\Psi}}{\partial \tilde{x}}, \tag{11}$$

where $\tilde{\Phi}(\tilde{x}, \tilde{y}, \tilde{t}), \tilde{\Psi}(\tilde{x}, \tilde{y}, \tilde{t})$ and $\tilde{\eta}(\tilde{x}, \tilde{t})$ denote the functions $\Phi(\tilde{x}/\varepsilon, \tilde{y}, \tilde{t}/\varepsilon)$ $\Psi(\tilde{x}/\varepsilon, \tilde{y}, \tilde{t}/\varepsilon)$ and $\eta(\tilde{x}/\varepsilon, \tilde{t}/\varepsilon)$ respectively. The functions $\tilde{\Phi}, \tilde{\Psi}$ and $\tilde{\eta}$ are expressed in the frame of the shallow water theory in powers of the small parameter ε as follows

$$\tilde{\Phi}(\tilde{x}, \tilde{y}, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \exp\left(\frac{-m\lambda|\tilde{x}|}{\varepsilon}\right) \tilde{\Phi}_{n,m}(\tilde{x}, \tilde{y}, \tilde{t}), \tag{12}$$

$$\tilde{\Psi}(\tilde{x}, \tilde{y}, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \exp\left(\frac{-m\lambda|\tilde{x}|}{\varepsilon}\right) \tilde{\Psi}_{n,m}(\tilde{x}, \tilde{y}, \tilde{t}), \tag{13}$$

and

$$\tilde{\eta}(\tilde{x}, \tilde{t}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \exp\left(\frac{-m\lambda|\tilde{x}|}{\varepsilon}\right) \tilde{\eta}_{n,m}(\tilde{x}, \tilde{t}), \tag{14}$$

where λ is a real positive constant and $\tilde{\Phi}_{n,m}, \tilde{\Psi}_{n,m}$ and $\tilde{\eta}_{n,m}$ are unknown functions to be determined. The above system of equations must be verified at each order (n,m). For simplification, the hats $(\sim)$ over the symbols will be omitted in the sequel.

5 Solution of the system of equations

In this section, we shall limit ourselves to the verification of the homogeneous equations in the fluid mass, on the free surface, on the horizontal bottom and at each order (n,m).

The order of approximation $(1,m)$**:** Substituting for the functions Φ and η given by formulas (12)- (14) in expressions (5)-(8), we get the following system of equations satisfied at the order of approximation $(1,0)$

$$\frac{\partial^2 \Phi_{1,m}}{\partial y^2} + m^2\lambda^2\Phi_{1,m} = 0,$$

$$\frac{\partial \Phi_{1,m}}{\partial y} = 0, \quad \text{at} \quad y = 0,$$

$$\eta_{1,m} = 0,$$

and

$$\frac{\partial \Phi_{1,m}}{\partial y} = 0, \quad \text{at} \quad y = -h.$$

The solution of the above system is given by

$$\Phi_{1,0}(x,y,t) = \begin{cases} \Phi^0_{1,0}(x,t) & m = 0, \\ 0 & m > 0. \end{cases}$$

and

$$\eta_{1,m}(x,t) = 0 \quad m \geq 0$$

where $\Phi^0_{1,0}(x,t)$ is an arbitrary function.

The order of approximation $(2,m)$**:** The system of equations reduces in the order $(2,m)$ to

$$\frac{\partial^2 \Phi_{2,m}}{\partial y^2} + m^2\lambda^2\Phi_{2,m} = 0,$$

$$\frac{\partial \Phi_{2,m}}{\partial y} = 0, \quad \text{at} \quad y = 0,$$

$$g\eta_{2,m} + \frac{\partial \Phi_{1,m}}{\partial t} = 0 \quad \text{at} \quad y = 0,$$

and

$$\frac{\partial \Phi_{2,m}}{\partial y} = 0 \quad \text{at} \quad y = -h,$$

The solution of the above system is given by

$$\Phi_{2,m}(x,y,t) = \begin{cases} \Phi^0_{2,0}(x,t) & m = 0, \\ A^0_{2,m}(x,t)\cos(m\lambda y) & m > 0, \end{cases}$$

$$\lambda = \frac{\pi}{h},$$

where $\Phi^0_{2,0}(x,t), A^0_{2,m}(x,t), m > 0$ are arbitrary functions.

The order of approximation $(3,m)$**:** The system of equations reduces, in the order $(3,m)$, to

$$\frac{\partial^2 \Phi_{3,m}}{\partial y^2} + m^2\lambda^2\Phi_{3,m} = 2m\lambda\frac{\partial \Phi_{2,m}}{\partial x} - \frac{\partial^2 \Phi_{1,m}}{\partial x^2},$$

$$\frac{\partial \Phi_{3,m}}{\partial y} - \frac{\partial \eta_{2,m}}{\partial t} = 0, \quad \text{at} \quad y = 0,$$

$$\frac{\partial \Phi_{3,m}}{\partial y} = 0, \quad \text{at} \quad y = -h,$$

and

$$g\eta_{3,m} + \frac{\partial \Phi_{2,m}}{\partial t} = 0, \quad \text{at} \quad y = 0.$$

The solution of the above system leads to the following

$$\Phi_{1,0}^{0}(x,t) = \Phi_{1,0}^{t}(x - c_0 t) + \Phi_{1,0}^{r}(x + c_0 t),$$

$$A_{2,m}^{0}(x,t) = A_{2,m}(t), \quad m > 0$$

$$\Phi_{3,m}(x,y,t) = \begin{cases} -\left(\frac{1}{2}y^2 + yh\right)\frac{\partial^2 \Phi_{1,0}^{0}}{\partial x^2} + \Phi_{3,0}^{*}(x,t) & m = 0, \\ A_{3,m}(x,t)\cos(m\lambda y) & m > 0, \end{cases} \tag{15}$$

and

$$\eta_{3,m}(x,t) = \begin{cases} -\frac{1}{g}\frac{\partial}{\partial t}\Phi_{2,0}^{0}(x,t), & m = 0, \\ -\frac{1}{g}\frac{d}{dt}A_{2,m}(t), & m > 0, \end{cases}$$

where $\Phi_{1,0}^{t}, \Phi_{1,0}^{r}, A_{2,m}, \Phi_{3,0}^{*}$ and $A_{3,m}$ are arbitrary functions of their arguments. In the same way, it can be shown that the $(4,0)$ order solution leads to

$$\Phi_{2,0}^{0}(x,t) = \Phi_{2,0}^{t}(x - c_0 t) + \Phi_{2,0}^{r}(x + c_0 t),$$

where $\Phi_{2,0}^{t}$ and $\Phi_{2,0}^{r}$ are arbitrary functions.

6 The second order solution

Enforcing the radiation condition into the results of the present section, the total velocity potential is given (up to the second order of ε) by

$$\begin{aligned} \Phi^{+}(x,y,t) =& \varepsilon\Phi_{1,0}^{t}(x - c_0 t) + \varepsilon^2\Phi_{2,0}^{t}(x - c_0 t) + \\ &+ \varepsilon^2 \sum_{m=1}^{\infty} A_{2,m}^{+}(t)\exp\left(\frac{-m\pi x}{\varepsilon h}\right)\cos\left(\frac{m\pi y}{h}\right) \\ &+ O\left(\varepsilon^3\right), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Phi^{-}(x,y,t) =& \varepsilon\Phi^{i}(x - c_0 t) + \varepsilon\Phi_{1,0}^{r}(x + c_0 t) \\ &+ \varepsilon^2\Phi_{2,0}^{r}(x + c_0 t) \\ &+ \varepsilon^2 \sum_{m=1}^{\infty} A_{2,m}^{-}(t)\exp\left(\frac{m\pi x}{\varepsilon h}\right)\cos\left(\frac{m\pi y}{h}\right) \\ &+ O\left(\varepsilon^3\right). \end{aligned} \tag{17}$$

Here, the velocity potential of the incident wave $\left(\varepsilon\Phi^{i}(x - c_0 t)\right)$, is assumed to be of the first order in ε. The free surface elevation at this order is given by

$$\eta^{\pm}(x,t) = -\frac{\varepsilon}{g}\frac{\partial}{\partial t}\Phi^{\pm}(x,y,t), \quad \text{for} \quad y = 0. \tag{18}$$

The expressions for the stream function of the second order are obtained using the Cauchy-Riemann conditions (10) and (11) together with expressions (16) and (17) in the form

$$\begin{aligned} \Psi^{+}(x,y,t) =& \varepsilon^2 (y + h)\Phi^{i\prime}(x - c_0 t) \\ &- \varepsilon^2 \sum_{m=1}^{\infty} A_{2,m}^{+}(t)\exp\left(\frac{-m\pi x}{\varepsilon h}\right)\sin\left(\frac{m\pi y}{h}\right) \\ &+ O\left(\varepsilon^3\right), \end{aligned} \tag{19}$$

and

$$\begin{aligned}\Psi^{-}(x,y,t) =&\varepsilon^2 (y+h)\left[\Phi^{i}\prime(x-c_0 t)+\Phi^{r'}_{1,0}(x+c_0 t)\right]\\&+\varepsilon^2\sum_{m=1}^{\infty}A^{-}_{2,m}(t)\exp\left(\frac{m\pi x}{\varepsilon h}\right)\sin\left(\frac{m\pi y}{h}\right)\\&+O\left(\varepsilon^3\right).\end{aligned}\tag{20}$$

The pressure is given (up to this order) by

$$P^{\pm}(x,y,t)=-\rho\left[\frac{\partial}{\partial t}\Phi^{\pm}(x,y,t)+gy\right].$$

The continuity of the pressure on the bottom at $x=0$ gives

$$\Phi^{i\prime}(-c_0 t)-\Phi^{r'}_{1,0}(c_0 t)=\Phi^{t'}_{1,0}(-c_0 t).\tag{21}$$

The determination of the unknown functions needs to specify the boundary conditions due to the particular forms of the obstacle.

7 Application of the boundary conditions

The impermeability of the barrier implies

$$\frac{\partial\Phi^{+}}{\partial n}=\frac{\partial\Phi^{-}}{\partial n}=0,\quad \text{at } x=\varepsilon^2 k_0^{\pm}(y), -d\le y\le\eta^{\pm}(x,t).\tag{22}$$

The horizontal extent of the barrier is assumed to be of the first order $\left(k^{\pm}(y)=\varepsilon k_0^{\pm}(y)\right)$. Hence, condition (22) can be developed in the neighbourhood of $x=0$ as

$$\begin{aligned}&\left[\varepsilon\frac{\partial\Phi^{\pm}}{\partial x}-\varepsilon\frac{d}{dy}k_0^{\pm}(y)\frac{\partial\Phi^{\pm}}{\partial y}\right]+\varepsilon^2 k_0^{\pm}(y)\\&\left[\varepsilon\frac{\partial^2\Phi^{\pm}}{\partial x^2}-\varepsilon\frac{d}{dy}k_0^{\pm}(y)\frac{\partial^2\Phi^{\pm}}{\partial x\partial y}\right]+\ldots,\end{aligned}\tag{23}$$

for $x=0$ and $-d\le y\le\eta^{\pm}$.
The continuity of the velocity $\overrightarrow{\mathrm{W}}$ in the homogeneous fluid mass under the barrier at $x=0$ gives

$$\frac{\partial\Phi^{+}}{\partial x}=\frac{\partial\Phi^{-}}{\partial x}\quad\text{at } x=0, -h\le y\le -d,\tag{24}$$

$$\frac{\partial\Phi^{+}}{\partial y}=\frac{\partial\Phi^{-}}{\partial y}\quad\text{at } x=0, -h\le y\le -d,\tag{25}$$

Substituting from (16) and (17) in (24) and making use of (23), one obtains in the first order

$$\begin{aligned}&\Phi^{t\prime}_{1,0}(-c_0 t)-\sum_{m=1}^{\infty}\left(\frac{m\pi}{h}\right)A^{+}_{2,m}(t)\cos\left(\frac{m\pi y}{h}\right)=\Phi^{i\prime}(-c_0 t)\\&+\Phi^{r'}_{1,0}(c_0 t)+\sum_{m=1}^{\infty}\left(\frac{m\pi}{h}\right)A^{-}_{2,m}(t)\cos\left(\frac{m\pi y}{h}\right),-h\le y\le 0.\end{aligned}$$

Then,

$$\Phi^{i\prime}(-c_0 t)+\Phi^{r'}_{1,0}(c_0 t)=\Phi^{t'}_{1,0}(-c_0 t),\tag{26}$$

$$-A^{+}_{2,m}(t)=A^{-}_{2,m}(t)\equiv\left(\frac{h}{\pi}\right)A_m(t),\quad m\geqslant 1,\tag{27}$$

where $A_m(t)$, $m \geqslant 1$, are arbitrary functions to be determined. Equations (21) and (26) leads to the following.

$$\Phi_{1,0}^{t}(-c_0 t) = \Phi(-c_0 t),$$

then,

$$\Phi_{1,0}^{t}(x - c_0 t) = \Phi^{i}(x - c_0 t), \tag{28}$$

$$\Phi_{1,0}^{r}(c_0 t) = 0,$$

which implies that

$$\Phi_{1,0}^{r}(x + c_0 t) = 0. \tag{29}$$

Substituting from (27), (28) and (29) into (16) and (17), the following expressions for the velocity potential in regions to the right and to the left of the submerged obstacle (up to the second order of approximation) are obtained as

$$\begin{aligned} \Phi^{+}(x,y,t) =& \varepsilon \Phi^{i}(x - c_0 t) + \varepsilon^2 \Phi_{2,0}^{t}(x - c_0 t) \\ & - \varepsilon^2 \sum_{m=1}^{\infty} \left(\frac{h}{\pi}\right) A_m(t) \exp\left(\frac{-m\pi x}{\varepsilon h}\right) \cos\left(\frac{m\pi y}{h}\right) \\ & + O\left(\varepsilon^3\right), \end{aligned} \tag{30}$$

and

$$\begin{aligned} \Phi^{-}(x,y,t) =& \varepsilon \Phi^{i}(x - c_0 t) + \varepsilon^2 \Phi_{2,0}^{r}(x + c_0 t) \\ & + \varepsilon^2 \sum_{m=1}^{\infty} \left(\frac{h}{\pi}\right) A_m(t) \exp\left(\frac{m\pi x}{\varepsilon h}\right) \cos\left(\frac{m\pi y}{h}\right) \\ & + O\left(\varepsilon^3\right). \end{aligned} \tag{31}$$

We noted that the considered limitations and geometry do not permit reflection of the first order by the submerged obstacle as seen from the last expression of $\Phi^{-}(x,y,t)$. Further, using equations (23) and (25) together with expressions (30) and (31), the functions $A_m(t)$ are found to satisfy the following dual series equations

$$\sum_{m=1}^{\infty} m A_m(t) \sin\left(\frac{m\pi y}{h}\right) = 0, \quad -h \leqslant y \leqslant -d,$$

and

$$\Phi^{i\prime}(-c_0 t) + \sum_{m=1}^{\infty} m A_m(t) \cos\left(\frac{m\pi y}{h}\right) = 0, \quad -d \leqslant y \leqslant 0.$$

The linearity and uniformity of the above dual series equations impose.

$$A_m(t) = a_m \Phi^{i\prime}(-c_0 t), \quad m \geqslant 1, \tag{32}$$

where $a_m > 1$ are constant coefficients satisfying the dual series equations

$$\Phi_{1,0}^{t}(x - c_0 t) = \Phi^{i}(x - c_0 t), \tag{33}$$

and

$$1 + \sum_{m=1}^{\infty} m a_m \cos\left(\frac{m\pi y}{h}\right) = 0, \quad -d \leqslant y \leqslant 0. \tag{34}$$

The solution of the dual series equations (33) and (34)) is given by Noble and Whiteman (1970) in the following form.

$$a_m = \frac{P_m(\gamma) - P_{m-1}(\gamma)}{m}, \quad m \geqslant 1, \tag{35}$$

where $P_m(\gamma)$ denotes the Legendre polynomial of degree m and argument γ with

$$\gamma = \cos\left(\frac{\pi d}{h}\right). \tag{36}$$

Substituting (35) and (36) into (32), we obtain

$$A_m(t) = \Phi^{i\prime}(-c_0 t)\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right], \quad m \geqslant 1. \tag{37}$$

The total velocity potentials are obtained from (30), (31) and (37) as

$$\begin{aligned}\Phi^{\pm}(x,y,t) = \varepsilon\Phi^{i}(x-c_0 t) \mp \varepsilon^2\left(\frac{h}{\pi}\right)\Phi^{i\prime}(-c_0 t)\times \\ \sum_{m=1}^{\infty}\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right]\exp\left(\frac{\mp m\pi x}{\varepsilon h}\right)\cos\left(\frac{m\pi y}{h}\right)+\ldots\end{aligned} \tag{38}$$

Expressions (38) show that the effect of such a barrier on the incident progressive wave, at the first order, is to produce local perturbations vanishing far from the barrier and that at this level of approximation, no reflections exist. The free surface elevation is found from (18) ,(38) to be

$$\begin{aligned}\eta^{\pm}(x,t) =& \frac{c_0\varepsilon^2}{g}\Phi^{i\prime}(x-c_0 t) \\ &\mp \frac{c_0\varepsilon^3}{g}\left(\frac{h}{\pi}\right)\Phi^{i\prime\prime}(-c_0 t)\sum_{m=1}^{\infty}\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right] \\ &\exp\left(\frac{\mp m\pi x}{\varepsilon h}\right)+\ldots\end{aligned}$$

Finally, substituting equations (27), (36) and (37) into (19) and (20)), we obtain the stream function $\Psi(x,y,t)$ in the following form

$$\begin{aligned}\Psi^{\pm}(x,y,t) = \varepsilon^2(y+h)\Phi^{i\prime}(x-c_0 t)+\varepsilon^2\left(\frac{h}{\pi}\right)\Phi^{i\prime}(-c_0 t)\times \\ \sum_{m=1}^{\infty}\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right]\exp\left(\frac{\mp m\pi x}{\varepsilon h}\right)\sin\left(\frac{m\pi y}{h}\right)+\ldots\end{aligned} \tag{39}$$

8 Numerical application

To illustrate the theoretical results obtained above, we consider in the remaining part of this section, the particular case of an incident uniform stream with velocity

$$W = e^2 W_0,$$

We consider,

$$\Phi^{i\prime}(x-c_0 t) = W_0. \tag{40}$$

Substituting (40) into (39), we get the stream function for this particular case as

$$\begin{aligned}\Psi^{\pm}(x,y,t) = \varepsilon^2 W_0 h\Bigg\{1+\frac{y}{h}+\frac{1}{\pi}\sum_{m=1}^{\infty}\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right] \\ \exp\left(\frac{\mp m\pi x}{\varepsilon h}\right)\sin\left(\frac{m\pi y}{h}\right)\Bigg\}+\ldots\end{aligned}$$

The corresponding system of stream-lines is obtained as the solution of the transcendental equation

$$\Psi^{\pm}(x,y,t) = \varepsilon^2 W_0 h C,$$

where c is a real constant with values between 0 and 1. Hence, the equation of the system of stream lines is

$$\begin{aligned}\Bigg\{1+\frac{y}{h}+\frac{1}{\pi}\sum_{m=1}^{\infty}\left[\frac{P_m(\gamma)-P_{m-1}(\gamma)}{m}\right] \\ \exp\left(\frac{\mp m\pi x}{\varepsilon h}\right)\sin\left(\frac{m\pi y}{h}\right)\Bigg\} = C, \quad 0 \leq C \leq 1.\end{aligned}$$

The following figure exhibits the system of stream-lines calculated for the particular value of $d/h = 0.5$.

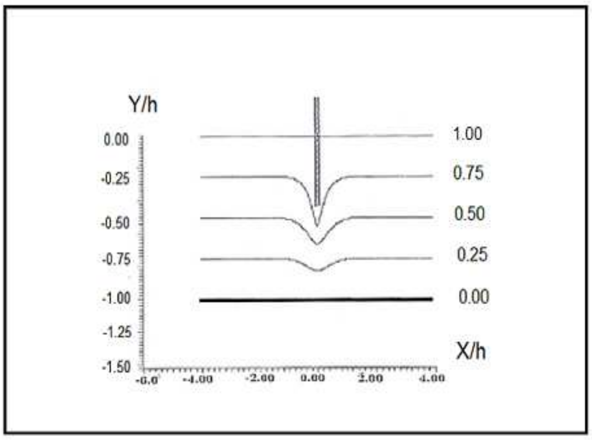

Fig. 2: The system of stream-lines corresponding to $d/h = 0.5$

The system of stream-lines (drawn in the non-distorted space) corresponding to an incident uniform flow with velocity $W = \varepsilon^2 W_0$. The number assigned to each line refers to the corresponding value of $\Psi^{\pm}(x,y,t)/(\varepsilon^2 W_0 h)$. The local effect of the submerged obstacle on the incident wave is illustrated: the fluid particles avoid penetrating the obstacle and accelerate as they approach the obstacle and then they decelerate to their original velocity.

9 The secular term

Applying the procedure used in sect 5 to the region on the right of the barrier for the order $(5,0)$ we can see, using (38), that the function $\Phi^{*}_{3,0}(x,t)$, which appears in (15), satisfies

$$\frac{\partial^2}{\partial x^2}\Phi^{*+}_{3,0} - \left(1/c_0^2\right)\frac{\partial^2}{\partial t^2}\Phi^{*+}_{3,0} = -\left(h^2/3\right)\frac{\partial^4}{\partial x^4}\Phi^i\left(x-c_0t\right)$$
$$+\left(1/c_0^2\right)\frac{\partial}{\partial t}\Phi^i\left(x-c_0t\right)\frac{\partial^2}{\partial x^2}\Phi^i\left(x-c_0t\right)+$$
$$+\left(2/c_0^2\right)\frac{\partial}{\partial x}\Phi^i\left(x-c_0t\right)\frac{\partial^2}{\partial x\partial t}\Phi^i\left(x-c_0t\right),$$

which has a solution given by

$$\Phi^{*+}_{3,0}(x,t) = R_3^+\left(x-c_0t\right) - t\left\{\left(\frac{3}{2}\right)\left[\Phi^{i\prime}\left(x-c_0t\right)\right]^2 + \left(\frac{h^2c_0}{3}\right)\Phi^{i\prime\prime\prime}\left(x-c_0t\right)\right\}, \tag{41}$$

where $R_3^+\left(x-c_0t\right)$ is an arbitrary function of $(x-c_0t)$. The function $\Phi^{*+}_{3,0}(x,t)$ represents a progressive wave traveling downstream. The second term on the right hand side of (41) is a secular term increasing monotonically with time. This secular term is not accepted physically. The expression (41) can be presented in the form

$$\Phi^{*+}_{3,0}(x,t) = S_3^+\left(x-c_0t\right) - x\left\{\left(\frac{3}{2c_0}\right)\left[\Phi^{i\prime}\left(x-c_0t\right)\right]^2 + \left(\frac{h^2}{3}\right)\Phi^{i\prime\prime\prime}\left(x-c_0t\right)\right\}, \tag{42}$$

where $S_3^+(x-c_0t)$ is an arbitrary function of $(x-c_0t)$ with

$$S_3^+(x-c_0t) = (x-c_0t)\left\{\left(\frac{3}{2c_0}\right)\left[\Phi^{i\prime}(x-c_0t)\right]^2 + \left(\frac{h^2}{3}\right)\Phi^{i\prime\prime\prime}(x-c_0t)\right\} + R_3^+(x-c_0t),$$

relation (42) shows that the secular term vanishes at $x=0$.

9.1 The origin of the secular terms

The appearance of a secular term in the progressive wave is a purely mathematical artifact, it is due to the choice of the distortion process given by (4). This distortion gives rise to waves propagating with critical velocity. However, since the early observations of Russel (1845), it is known that the velocity of propagation of long waves exceeds the critical velocity by a term of the same order as that of the relative free surface amplitude. This gives the physical justification of the appearance of secular terms in the above calculations.

From a mathematical point of view, the distortion formula (4) and expressions (12), (13) and (14) are nothing more than a sort of infinite series expansions of the functions Φ, Ψ and η representing the exact solution of the distorted system of equations (5)-(11). This type of representation always has a certain domain of validity. If we consider, for example, Maclaurin's expansion of the function $\sin(\varepsilon x) := \varepsilon x + O\left(\varepsilon^3\right)$ we note that it contains a secular term at the first order. However, this representation gives the exact value of $\sin(\varepsilon x)$ at the point $x=0$, and provides a reasonable approximation, for the function, in the narrow neighborhood enclosing the point. This secular term may disappear if we use another representation for the same function as, for example, the Fourier series representation. The procedure based on (4) and (12)-(14), should be situated within this context.

The above mathematical procedure provides independent expressions for local perturbations and progressive waves. The expression of progressive waves is valid only at $x=0$. Hence this procedure needs to be modified in order to deal with progressive waves, taking into account its behavior at $x=0$.

10 Modification of the mathematical procedure

Laboratory observations [10] show that long waves are slowly modulated, and that their celerity slightly exceeds the critical velocity. To take these properties into account, we follow Temperville [11] and use the set of variables u, v and y defined by means of the following relations

$$u = (x-c_0t), \quad v = \varepsilon^2 x, \quad y = y,$$

where u is the fast variable used for describing the basic wave, and v is the slow variable used for describing the modulation.

The transformed version of the equations in the fluid mass is consequently written as

$$\varepsilon^6\frac{\partial^2\Phi^+}{\partial v^2} + 2\varepsilon^4\frac{\partial^2\Phi^+}{\partial u\partial v} + \varepsilon^2\frac{\partial^2\Phi^+}{\partial u^2} + \frac{\partial^2\Phi^+}{\partial y^2} = 0,$$

and the conditions on the free surface are transformed into the forms

$$\frac{\partial\Phi^+}{\partial y} = \varepsilon^6\frac{\partial\Phi^+}{\partial v}\frac{\partial\eta^+}{\partial v} + \varepsilon^4\left[\frac{\partial\Phi^+}{\partial v}\frac{\partial\eta^+}{\partial u} + \frac{\partial\Phi^+}{\partial u}\frac{\partial\eta^+}{\partial v}\right] + \varepsilon^2\frac{\partial\Phi^+}{\partial u}\frac{\partial\eta^+}{\partial u} - \varepsilon c_0\frac{\partial\eta^+}{\partial u}, \quad \text{at} \quad y = \eta^+(u,v),$$

and

$$g\eta^+ - \varepsilon c_0\frac{\partial\Phi^+}{\partial u} + \frac{1}{2}\left\{\varepsilon^2\left[\frac{\partial\Phi^+}{\partial u} + \varepsilon^2\frac{\partial\Phi^+}{\partial v}\right]^2 + \left[\frac{\partial\Phi^+}{\partial y}\right]^2\right\} = 0 \quad \text{at} \quad y = \eta^+(u,v).$$

The condition on the impermeable horizontal bottom remains the same

$$\frac{\partial \Phi^+}{\partial y} = 0 \quad \text{at} \quad y = -h.$$

In the absence of local perturbations, far from the obstacle, the following representation for the functions Φ^+ and η^+ is used

$$\Phi^+(u,v,y) = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \Phi^+_{2n-1}(u,v,y)$$

and

$$\eta^+(u,v) = \sum_{n=1}^{\infty} e^{2n} \eta^+_{2n}(u,v).$$

The above system is assumed to be satisfied at each order of the small parameter ε.

We can see, after considerable calculations, that the results of the first five order approximations contain no secular term, and that the function η_2^+ satisfies the Korteweg and de Vries equation equation (KdV) of the form

$$\frac{h^2}{6}\frac{\partial^3 \eta_2^+}{\partial u^3} + \frac{3}{2h}\eta_2^+ \frac{\partial \eta_2^+}{\partial u} + \frac{\partial \eta_2^+}{\partial v} = 0. \tag{43}$$

In order to solve the KDV equation (43) one has to specify an initial condition at $v = 0$ (i.e. at $x = 0$). An accurate value for the unknown function η_2^+ can be obtained at $x = 0$, using the procedure of the double series expansion used earlier, since the secular terms contaminating the obtained expressions disappear at the origin $x = 0$ for all instants of time. This initial condition is given by (18) in the form

$$\eta_2^+(u,0) = -\frac{c_0}{g}\Phi^i/(u),$$

There are several methods devoted for the analytical solution of the KdV equation. Among others, we cite the method of Bargmann, the perturbation method and the inverse scattering method. Each one of these methods provides a solution suitable for certain particular form of the specified initial condition. We are not aware of any other method which can give analytical solution for a general form of the initial condition. In such case, numerical methods are recommended for obtaining the features of the resulting flow. We propose to follow the scheme of Zabusky and Kruscal (1965) [16] for the numerical study of the solution of the KdV equation.

Work is in progress to extend the above results to cover the case of stratified fluids. We expect, in this case, to obtain reflected waves by the obstacle at the first order of approximation.

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Conflicts of Interest

There are no conflicts of interest declared by the authors for the publication of this paper.

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