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Low Order Nonconforming Expanded Characteristic-Mixed Finite Element Method for the Convection-Diffusion Problem

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Abstract: A low order nonconforming finite element method is proposed for the convection-diffusion equations with the expanded characteristic-mixed finite element scheme. The method is a combination of characteristic approximation to handle the convection part in time and a expanded nonconforming mixed finite element spatial approximation to deal with the diffusion part. In the process, the interpolation operator is employed instead of the so-called elliptic projection which is an indispensable tool used for the convergence analysis in the previous literature. When the exact solutions belong to $H^2(\Omega)$ instead of $H^3(\Omega)$, the corresponding optimal order error estimates in L^2-norm are obtained by use of some distinct properties of the nonconforming finite elements.

Keywords: Convection diffusion problems, expanded characteristic mixed finite element method, nonconforming finite elements

1 Introduction

We consider the convection-diffusion equations

$$\begin{cases} (a) c_t + u(X,t)\cdot\nabla c \\ -\nabla\cdot(a(X,t)\nabla c) = f(X,t),\ in\ \Omega\times(0,T), \\ (b)\quad c(X,0) = c_0(X),\ \ in\ \ \Omega, \\ (c)\quad c(X,t) = 0,\ \ on\ \ \partial\Omega\times(0,T). \end{cases} \tag{1}$$

where $\Omega \subset R^2$ denotes an open bounded domain with boundary Γ, $(0,T]$ is the time interval, $X = (x,y)$ and the parameters appearing in (1) satisfy the following assumptions [1].

1) $c(X,t)$ denotes, for example, the concentration of a possible substance;

2) $u(X,t)$ represents the velocity of the flow satisfying

$$|u(X,t)| + |\nabla\cdot u(X,t)| \le C_1, \quad \forall X \in \Omega, \tag{2}$$

here $C_1 > 0$ is a constant;

3) $a(X,t)$ is sufficiently smooth and there exist constants a_1 and a_2, such that

$$0 < a_1 \le a(X,t) \le a_2 < +\infty, \quad \forall X \in \Omega; \tag{3}$$

4) f denotes a source term;

5) ∇ and $\nabla\cdot$ denote the gradient and the divergence operators, respectively.

In many diffusion processes arising in physical problems, convection essentially dominates diffusion, it is natural to seek numerical methods for such problems in order to reflect their almost hyperbolic nature [2]. Many such schemes have been developed, such as the streamline diffusion method [3], the least-squares mixed finite element method [4], the modified method of characteristic-Galerkin finite element procedure [2,5–7], the characteristic finite element methods [8] and the characteristic finite volume element methods [9].

The modified characteristic finite element method was first formulated for scalar parabolic equation by J. Douglas and T. F. Russell in [2]. The method is a combination of characteristic approximation to handle the convection part in time and a finite element spatial approximation to deal with the diffusion part. For convection-dominated problems, the modified characteristic finite element schemes have much smaller time-truncation errors than those of standard methods such as finite difference or Galerkin discretizations in the

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space variables combined with Crank-Nicolson or backward difference scheme in the time variable [10]. Because the solution changes more slowly in the characteristic τ direction than in the t direction. Then the scheme will permit the use of larger time steps. [1] described an expanded characteristic-mixed finite element method which is a combination of characteristic approximation to handle the convection part in time and an expanded mixed finite element spatial approximation to deal with diffusion part. This formulation expands the standard mixed formulation in the sense that three variables are explicitly treated. However, it is only for conforming finite elements and the higher regularity of the exact solution $c \in H^3$ is required.

In this paper, a low order nonconforming expanded characteristic mixed element scheme is proposed for equation(1). In problems with significant convection, nonconforming finite elements with the degrees of freedom defined on the element or the element's edges are appropriate [11]. In our process, the interpolation operator is used instead of the elliptic projection as in the previous literature [1,2] which has considerable practical difficulties in solving simultaneous equations, and the exact solution c is only required to belong to $H^2(\Omega)$ instead of $H^3(\Omega)$ in the convergence analysis thus the results of [1] are improved. At last, by using the special property of the element considered (see Lemmas 1-2 below), the same error estimate orders of the scalar unknown, its gradient and its flux in space are obtained as in conforming finite element case of [1].

2 The Expanded Characteristic-Mixed Finite Element Method

We denote by $W^{k,p}(S)$ the standard Sobolev space of k-differential functions in $L^p(S)$. Let $\| \cdot \|_{k,p,S}$ be its norm and $\| \cdot \|_{k,S}$ be the norm of $H^k(S)$. When $k=0$, we let $L^2(\Omega)$ denote the corresponding space defined on Ω with norm $\| \cdot \|$.

Let $[a,b] \subset [0,T]$, Y be a Sobolev space, and $f(X,t)$ be smooth function on $\Omega \times [a,b]$, also we define $L^p(a,b;Y)$ and $\| f \|_{L^p(a,b;Y)}$ as follows

$$L^p(a,b;Y) = \{f : \int_b^a \| f(\cdot,t) \|_Y^p \, dt < \infty\},$$

$$\| f \|_{L^p(a,b;Y)} = (\int_b^a \| f(\cdot,t) \|_Y^p \, dt)^{\frac{1}{p}},$$

where if $p=\infty$, the integral is replaced by the essential supremum.

Under the above assumptions, we begin to discretize the problem (1). Let

$$\psi(X,t) = (1+|u|^2)^{\frac{1}{2}} \tag{4}$$

and the characteristic direction associated with the operator $c_t + u \cdot \nabla c$ be denoted by $\tau = \tau(X)$, where

$$\frac{\partial}{\partial \tau} = \frac{1}{\partial \psi(X,t)} \frac{\partial}{\partial t} + \frac{u}{\psi(X,t)} \cdot \nabla. \tag{5}$$

Then the equation (1a) can be put in the form:

$$\psi(X,t)\frac{\partial c}{\partial \tau} - \nabla \cdot (a(X,t)\nabla c) = f(X,t), (X,t) \in \Omega \times (0,T). \tag{6}$$

Let $\lambda = -\nabla c$, $\sigma = -a(X,t)\nabla c = a(X,t)\lambda$, then (6) can be rewritten as

$$\begin{cases} (a) \quad \psi(X,t)\dfrac{\partial c}{\partial \tau} + div\sigma = f, \\ (b) \quad \lambda + \nabla c = 0, \\ (c) \quad \sigma - a(X,t)\lambda = 0. \end{cases} \tag{7}$$

Define the following Sobolev spaces:

$$V = H(div,\Omega) = \{v \in (L^2(\Omega))^2 : \nabla \cdot v \in L^2(\Omega)\},$$

$$\Lambda = (L^2(\Omega))^2, W = L^2(\Omega).$$

Then the expanded characteristic-mixed variational problem corresponding to (1) is to find $(\sigma,\lambda,c) : [0,T] \to V \times \Lambda \times W$, such that

$$\begin{cases} (a) \quad (\psi\dfrac{\partial c}{\partial \tau}, w) + (div\sigma, w) = (f,w), & \forall w \in W, \\ (b) \quad (\lambda, v) - (c, divv) = 0, & \forall v \in V, \\ (c) \quad (a(X,t)\lambda,\mu) - (\sigma,\mu) = 0, & \forall \mu \in \Lambda, \\ (d) \quad c(X,0) = c_0(X), & \forall X \in \Omega. \end{cases} \tag{8}$$

This form will be discretized in details below.

From now on, let $\Omega \subset R^2$ be a polygon with boundaries parallel to the axes, T^h be an axis parallel rectangular meshes of Ω satisfying the regularity assumption [12]. For $K \in T^h$, let $h_K = diam\{K\}$ and $h = \max\limits_{K \in T^h}\{h_K\}$.

For $v \in H^1(K)$, the shape function spaces P_K^j and the interpolators I_K^j on K are defined as follows:

$$P_K^1 = span\{1,x,y,y^2\}, P_K^2 = span\{1,x,y,x^2\},$$

$$\frac{1}{|l_k|}\int_{l_k}(v - I_K^j v)ds = 0 \; (j=1,2; k=1,2,3,4),$$

where l_1,l_2,l_3,l_4 are four edges of ∂K.

The associated finite element spaces are defined as

$$V_h = \{\mathbf{v} = (v_1,v_2), v_j|_K \in P_K^j, \forall K \in T^h, \int_F [v_j]ds = 0,$$
$$\forall F \subset \partial K, F \not\subseteq \partial\Omega, j=1,2\},$$

$$\Lambda_h = V_h, \quad W_h = \{w \in L^2(\Omega), w|_K \in Q_{0,0}(K), \forall K \in T^h\},$$

where $[v_j]$ stands for the jump of v_j across the edge F if F is an internal edge, and it is equal to v_j itself if F belongs to

$\partial\Omega$, $Q_{0,0}(K)$ is a space of polynomials with zero degrees for x and y respectively.

For $\mathbf{v}=(v_1,v_2)$ and $w\in L^2(K)$, let $\Pi^1: V\to V_h$, $\Pi^1: \Lambda\to\Lambda_h$ and $\Pi^2: W\to W_h$ satisfy

$$\Pi^1\mathbf{v}=(I^1v_1,I^2v_2), I^j|_K=I^j_K (j=1,2),$$

$$\Pi^2|_K=\Pi^2_K, \Pi^2_K w=\frac{1}{|K|}\int_K wdxdy.$$

Let $\varepsilon_h=\lambda_h-\Pi^1\lambda$, $\zeta_h=\sigma_h-\Pi^1\sigma, e_h=c_h-\Pi^2c$, $\rho=\Pi^2c-c$.

In the procedure to be used, we consider a time step $\triangle t>0$ to approximate the solution at times $t^n=n\triangle t$, and the characteristic derivative will be approximated basically in the following manner.

Setting

$$\overline{X}=X-u(X,t^n)\triangle t,$$

then we have the following approximation [2]

$$\psi(X,t^n)\frac{\partial c}{\partial\tau}|_{t^n}\approx\psi(X,t^n)\frac{c(X,t^n)-c(\overline{X},t^{n-1})}{\sqrt{(X-\overline{X})^2+(\triangle t)^2}}$$
$$=\frac{c(X,t^n)-c(\overline{X},t^{n-1})}{\triangle t}.$$

Our expanded characteristics-mixed finite element method is the determination of $(\sigma_h,\lambda_h,c_h):\{t^0,t^1,...,t^N\}\to V_h\times\Lambda_h\times W_h$, satisfying the relations

$$\begin{cases}(a)\ (\frac{c_h^n-\overline{c}_h^{n-1}}{\triangle t},w_h)+(div\sigma_h^n,w_h)_h=(f^n,w_h),\ \forall w_h\in W_h,\\(b)\ (\lambda_h^n,v_h)-(c_h^n,divv_h)_h=0,\ \forall v_h\in V_h,\\(c)\ (a(X,t^n)\lambda_h^n,\mu_h)-(\sigma_h^n,\mu_h)=0,\ \forall\mu_h\in\Lambda_h,\\(d)\ c_h^0=\Pi^2c_0,\ \forall X\in\Omega,\end{cases}\tag{9}$$

where

$$c_h^n=c_h(t^n),\ \overline{X}=X-u_h^{n-1}(X,\ t^{n-1})\triangle t,$$

$$\overline{c}_h^{n-1}=c_h^{n-1}(\overline{X},\ t^{n-1})=c_h^{n-1}(X-u_h^{n-1}(X,t^{n-1})\triangle t,$$

$$(u,v)_h=\sum_K\int_K uvdxdy,\ f^n=f(X,t^n).$$

3 Existence and Uniqueness of the Solution of Discrete Problem

Theorem 3.1 Under the assumption of (3), there exists a unique solution $(\sigma_h,\lambda_h,c_h)\in V_h\times\Lambda_h\times W_h$ to the expanded characteristic-mixed finite element scheme (9).

Proof. The linear system generated by (9) is square, so the existence of the solution is implied by its uniqueness. Let c_h^{n-1} and f be zero, thus $\overline{c}_h^{n-1}$ is zero too, then we have

$$\begin{cases}(a)\quad (\frac{c_h^n}{\triangle t},w_h)+(div\sigma_h^n,w_h)_h=0,\quad\forall w_h\in W_h,\\(b)\quad (\lambda_h^n,v_h)-(c_h^n,divv_h)_h=0,\quad\forall v_h\in V_h,\\(c)\quad (a(X,t^n)\lambda_h^n,\mu_h)-(\sigma_h^n,\mu_h)=0,\quad\forall\mu_h\in\Lambda_h.\end{cases}\tag{10}$$

Choosing $w_h=c_h^n$ in $(10a)$, $v_h=\sigma_h^n$ in $(10b)$, $\mu_h=\lambda_h^n$ in $(10c)$, and summing them together gives

$$\frac{1}{\triangle t}\parallel c_h^n\parallel^2+(a(X,t^n)\lambda_h^n,\lambda_h^n)=0.$$

According to (3), we get $c_h^n=\lambda_h^n=0$, then with (10c) we have $(\sigma_h^n,\mu_h)=0$, on the other hand, choosing $\mu_h=\sigma_h^n$ gives $\sigma_h^n=0$. The proof is completed.

To get error estimates, we state the following important lemmas.

Lemma 3.1 For $c\in L^2(\Omega),\sigma\in V$, we have

$$(c-\Pi^2c,divv_h)_h=0,\quad\forall v_h\in V_h,$$

$$(div(\sigma-\Pi^1\sigma),w_h)_h=0,\ \forall w_h\in W_h.$$

Proof. By the definition of Π^2 and noting that $divv_h|_K$ is a constant, we get

$$(c-\Pi^2c,divv_h)_h=\sum_{K\in T^h}\int_K(c-\Pi^2c)divv_hdxdy=0.$$

Similarly, since $w_h|_K$ is a constant, we have

$$(div(\sigma-\Pi^1\sigma),w_h)_h=\sum_{K\in T^h}\int_K div(\sigma-\Pi^1\sigma)w_hdxdy$$
$$=\sum_{K\in T^h}w_h|_K\int_K div(\sigma-\Pi^1\sigma)dxdy$$
$$=\sum_{K\in T^h}w_h|_K\int_{\partial K}(\sigma-\Pi^1\sigma)\cdot nds=0.$$

Here and later, $n=(n_1,n_2)$ denotes the unit outward norm on ∂K. The proof is completed.

Lemma 3.2 [15,16] For $c\in H^2(\Omega)\cap H_0^1(\Omega)$, we have

$$\sum_{K\in T^h}\int_{\partial K}c\,n\cdot v_hds\le Ch|c|_2\parallel v_h\parallel,\quad\forall v_h\in V_h.$$

Here and later, the positive constant C is independent of h.

Lemma 3.3 [1] For a given function $\tau\in L^2(\Omega)$, there exists a $q_\tau\in(H^1(\Omega))^2$ such that

$$\begin{cases}(a)\ divq_\tau=\tau,\\(b)\parallel q_\tau\parallel_1\le C\parallel\tau\parallel,\\(c)\parallel q_\tau\parallel\le C\parallel\tau\parallel_{-1}.\end{cases}$$

Lemma 3.4 Let $\tau\in L^2(\Omega),c\in H^2(\Omega)\cap H_0^1(\Omega)$ and $\lambda\in H^1(\Omega)^2$, if $(\xi,\varepsilon_h)\in W_h\times\Lambda_h$ satisfies

$$(\varepsilon_h,q_h)-(\xi,divq_h)_h-\sum_{K\in T^h}\int_{\partial K}c\,n\cdot q_hds$$

$$+(\Pi^1\lambda-\lambda,q_h)=0,\ \forall\, q_h\in V_h,$$

then we have

$$|(\tau,\xi)|\le C(h\parallel\tau\parallel+\parallel\tau\parallel_{-1})\parallel\varepsilon_h\parallel$$

$$+Ch(h\parallel\tau\parallel+\parallel\tau\parallel_{-1})(|\lambda|_1+|c|_2).$$

Proof. Let q_τ be the corresponding function of the given τ. Then by Lemmas 3.1-3.3 and the interpolation theorey

$$\begin{aligned}(\tau,\xi)&=(divq_\tau,\xi)_h=(div(q_\tau-\Pi^1q_\tau),\xi)_h\\&+(div\Pi^1q_\tau,\xi)_h=(div\Pi^1q_\tau,\xi)_h\\&=(\varepsilon_h,\Pi^1q_\tau)-\sum_{K\in T^h}\int_{\partial K}c\,n\cdot\Pi^1q_\tau ds\\&+(\Pi^1\lambda-\lambda,\Pi^1q_\tau)\\&=(\varepsilon_h,\Pi^1q_\tau-q_\tau)+(\varepsilon_h,q_\tau)+(\Pi^1\lambda-\lambda,q_\tau)\\&+(\Pi^1\lambda-\lambda,\Pi^1q_\tau-q_\tau)-\sum_{K\in T^h}\int_{\partial K}c\,n\cdot\Pi^1q_\tau ds\\&\le C(h\parallel\tau\parallel+\parallel\tau\parallel_{-1})\parallel\varepsilon_h\parallel\\&+Ch|c|_2\parallel\Pi^1q_\tau\parallel+Ch^2|\lambda|_1\parallel q_\tau\parallel_1+Ch|\lambda|_1\parallel q_\tau\parallel\\&\le C(h\parallel\tau\parallel+\parallel\tau\parallel_{-1})\parallel\varepsilon_h\parallel\\&+Ch|c|_2\{\parallel\Pi^1q_\tau-q_\tau\parallel+\|q_\tau\|\}\\&+Ch^2|\lambda|_1\parallel q_\tau\parallel_1+Ch|\lambda|_1\parallel q_\tau\parallel\\&=C(h\parallel\tau\parallel+\parallel\tau\parallel_{-1})\parallel\varepsilon_h\parallel\\&+Ch(|c|_2+|\lambda|_1)(\parallel\tau\parallel_{-1}+h\parallel\tau\parallel).\end{aligned}$$

The proof is completed.

Lemma 3.5 Let $\eta\in L^2(\Omega)$, $\bar\eta=\eta(X-u(X)\triangle t)$, $c\in H^2(\Omega)\cap H_0^1(\Omega)$ and $\lambda\in H^1(\Omega)^2$, if $(\xi,\varepsilon_h)\in W_h\times\Lambda_h$ satisfies

$$(\varepsilon_h,q_h)-(\xi,divq_h)_h-\sum_{K\in T^h}\int_{\partial K}c\,n\cdot q_h ds$$

$$+(\Pi^1\lambda-\lambda,q_h)=0,\ \ \forall q_h\in V_h,$$

then there holds

$$|(\eta-\bar\eta,\xi)|\le C(h+\triangle t)\parallel\eta\parallel\parallel\varepsilon_h\parallel$$

$$+Ch(h+\triangle t)\parallel\eta\parallel(|c|_2+|\lambda|_1).$$

Proof. Let $\tau=\eta-\bar\eta\in L^2(\Omega)$. Lemma 3.4 indicates that

$$\begin{aligned}|(\eta-\bar\eta,\xi)|&\le C(h\parallel\eta-\bar\eta\parallel+\parallel\eta-\bar\eta\parallel_{-1})\parallel\varepsilon_h\parallel\\&+Ch(|c|_2+|\lambda|_1)(\parallel\eta-\bar\eta\parallel_{-1}+h\parallel\eta-\bar\eta\parallel)\\&\le C(h+\triangle t)\parallel\eta\parallel\parallel\varepsilon_h\parallel\\&+Ch(h+\triangle t)\parallel\eta\parallel(|c|_2+|\lambda|_1).\end{aligned}$$

By [14], we obtain

$$\parallel\eta-\bar\eta\parallel_{-1}\le C\parallel\eta\parallel\triangle t,\parallel\eta-\bar\eta\parallel\le C\parallel\eta\parallel.$$

Combining the above inequalities yields the desired result.

4 Error Estimates

In this section, we derive the optimal order estimates of (c_h-c), $(\sigma_h-\sigma)$ and $(\lambda_h-\lambda)$ in L^2-norm.

Theorem 4.1. Let (σ_h,λ_h,c_h) and (σ,λ,c) be the solutions of (9) and (7), respectively, $\triangle t=O(h)$. Then for $\triangle t>0$, we have

$$\begin{cases}(a)\max\limits_{0\le n\le N}\parallel(c_h-c)(t^n)\parallel\ \le m_0\triangle t+hm_1,\\(b)\max\limits_{0\le n\le N}\parallel(\lambda_h-\lambda)(t^n)\parallel\ \le m_0\triangle t+hm_1,\\(c)\max\limits_{0\le n\le N}\parallel(\sigma_h-\sigma)(t^n)\parallel\ \le m_0\triangle t+hm_1,\end{cases}\tag{11}$$

where

$$m_0=C\parallel\frac{\partial^2c}{\partial\tau^2}\parallel_{L^2(0,T;L^2)},$$

$$m_1=C(|c_t|_{L^2(0,T;H^2)}+|\lambda_t|_{L^2(0,T;H^1)}+|c_t|_{L^2(0,T;H^1)})$$

$$+C(|\lambda|_{L^\infty(0,T;H^1)}+|c|_{L^\infty(0,T;H^1)}+|\sigma|_{L^\infty(0,T;H^1)}+|c|_{L^\infty(0,T;H^2)}).$$

Proof. For any $v_h\in V_h,\mu_h\in\Lambda_h$ and $w_h\in W_h$, by (8) and (9), we have the following error equations

$$\begin{cases}(a)(\dfrac{e_h^n-\bar e_h^{n-1}}{\triangle t},w_h)+(div\zeta_h^n,w_h)_h\\=(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar c_h^{n-1}}{\triangle t},w_h)-(\frac{\rho^n-\bar\rho^{n-1}}{\triangle t},w_h),\\(b)(\varepsilon_h^n,v_h)-(e_h^n,divv_h)_h-\sum\limits_{K\in T^h}\displaystyle\int_{\partial K}c^nv_h\cdot nds\\+(\Pi^1\lambda^n-\lambda^n,v_h)=0,\\(c)(a(X,t^n)\varepsilon_h^n,\mu_h)-(\zeta_h^n,\mu_h)\\+(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\mu_h)-(\Pi^1\sigma^n-\sigma^n,\mu_h)=0.\end{cases}\tag{12}$$

Then choosing $\mu_h=\zeta_h^n\in V_h=\Lambda_h$ in (12c) yields

$$(a(X,t^n)\varepsilon_h^n,\zeta_h^n)-(\zeta_h^n,\zeta_h^n)+(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\zeta_h^n)$$

$$-(\Pi^1\sigma^n-\sigma^n,\zeta_h^n)=0,$$

which follows

$$\begin{aligned}\parallel\zeta_h^n\parallel&\le C\parallel\varepsilon_h^n\parallel+C\parallel\Pi^1\lambda^n-\lambda^n\parallel+C\parallel\Pi^1\sigma^n-\sigma^n\parallel\\&\le C\parallel\varepsilon_h^n\parallel+Ch|\lambda^n|_1+Ch|\sigma^n|_1.\end{aligned}\tag{13}$$

By (12b), we get

$$\begin{aligned}&(\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t},v_h)-(\frac{e_h^n-e_h^{n-1}}{\triangle t},divv_h)_h\\&-\sum_{K\in T^h}\int_{\partial K}\frac{c^n-c^{n-1}}{\triangle t}v_h\cdot nds\\&+(\frac{(\Pi^1\lambda^n-\lambda^n)-(\Pi^1\lambda^{n-1}-\lambda^{n-1})}{\triangle t},v_h)=0.\end{aligned}\tag{14}$$

Choosing $w_h=\dfrac{e_h^n-e_h^{n-1}}{\triangle t},\mu_h=\dfrac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t},v_h=\zeta_h^n$ in (12a), (12c) and (14), respectively, and summing them to obtain

$$\begin{aligned}&(\frac{e_h^n-\bar{e}_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})+(a(X,t^n)\varepsilon_h^n,\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})\\&-\sum_{K\in T^h}\int_{\partial K}\frac{c^n-c^{n-1}}{\triangle t}\zeta_h^n\cdot nds\\&+(\frac{(\Pi^1\lambda^n-\lambda^n)-(\Pi^1\lambda^{n-1}-\lambda^{n-1})}{\triangle t},\zeta_h^n)\\&+(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})\\&-(\Pi^1\sigma^n-\sigma^n,\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})\\=&(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c^n-\bar{c}^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})\\&-(\frac{\rho^n-\rho^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})\\&-(\frac{\rho^{n-1}-\bar{\rho}^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t}).\end{aligned}\tag{15}$$

Then applying the argument similar to [2], yields

$$\begin{aligned}&|(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c^n-\bar{c}^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})|\\&\le C\parallel\frac{\partial^2c}{\partial\tau^2}\parallel^2_{L^2(t^{n-1},t^n,L^2)}\triangle t+\frac{1}{4}\parallel\frac{e_h^n-e_h^{n-1}}{\triangle t}\parallel^2,\end{aligned}\tag{16}$$

$$\begin{aligned}&|(\frac{\rho^n-\rho^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})|\le\frac{C}{\triangle t}\parallel\rho_t\parallel^2_{L^2(t^{n-1},t^n;L^2)}\\&+\frac{1}{4}\parallel\frac{e_h^n-e_h^{n-1}}{\triangle t}\parallel^2.\end{aligned}\tag{17}$$

On the other hand, the first two terms on the left-hand side of (15) can be estimated as

$$\begin{aligned}&(\frac{e_h^n-\bar{e}_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})+(a(X,t^n)\varepsilon_h^n,\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})\\=&(\frac{e_h^n-e_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})+(\frac{e_h^{n-1}-\bar{e}_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})\\&+(a(X,t^n)\varepsilon_h^n,\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})\\\ge&\parallel\frac{e_h^n-e_h^{n-1}}{\triangle t}\parallel^2+(\frac{e_h^{n-1}-\bar{e}_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})\\&+\frac{1}{2\triangle t}((a(X,t^n)\varepsilon_h^n,\varepsilon_h^n)-(a(X,t^n)\varepsilon_h^{n-1},\varepsilon_h^{n-1})),\end{aligned}\tag{18}$$

$$\begin{aligned}|\sum_{K\in T^h}\int_{\partial K}\frac{c^n-c^{n-1}}{\triangle t}\zeta_h^n\cdot nds|&\le Ch|\frac{c^n-c^{n-1}}{\triangle t}|_2\parallel\zeta_h^n\parallel\\&\le\frac{Ch^2}{\triangle t}\int_{t^{n-1}}^{t^n}|c_t|_2^2ds+C\parallel\zeta_h^n\parallel^2\end{aligned}\tag{19}$$

and

$$\begin{aligned}&|(\frac{(\Pi^1\lambda^n-\lambda^n)-(\Pi^1\lambda^{n-1}-\lambda^{n-1})}{\triangle t},\zeta_h^n)|\\&\le C\parallel\frac{(\Pi^1\lambda^n-\lambda^n)-(\Pi^1\lambda^{n-1}-\lambda^{n-1})}{\triangle t}\parallel\parallel\zeta_h^n\parallel\\&\le\frac{C}{\triangle t}\int_{t^{n-1}}^{t^n}\parallel(\Pi^1\lambda-\lambda)_t\parallel^2ds+C\parallel\zeta_h^n\parallel^2,\end{aligned}\tag{20}$$

respectively.

Thus from (15)-(20), we get

$$\begin{aligned}&\frac{1}{2\triangle t}((a(X,t^n)\varepsilon_h^n,\varepsilon_h^n)-(a(X,t^n)\varepsilon_h^{n-1},\varepsilon_h^{n-1}))+\frac{1}{2}\parallel\frac{e_h^n-e_h^{n-1}}{\triangle t}\parallel^2\\&\le C\parallel\frac{\partial^2c}{\partial\tau^2}\parallel^2_{L^2(t^{n-1},t^n,L^2)}\triangle t+\frac{C}{\triangle t}\parallel\rho_t\parallel^2_{L^2(t^{n-1},t^n,L^2)}\\&-(\frac{\rho^{n-1}-\bar{\rho}^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})-(\frac{e_h^{n-1}-\bar{e}_h^{n-1}}{\triangle t},\frac{e_h^n-e_h^{n-1}}{\triangle t})\\&+\frac{Ch^2}{\triangle t}\int_{t^{n-1}}^{t^n}|c_t|_2^2ds+C\parallel\zeta_h^n\parallel^2+\frac{C}{\triangle t}\int_{t^{n-1}}^{t^n}\parallel(\Pi^1\lambda-\lambda)_t\parallel^2ds\\&-(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t})+(\Pi^1\sigma^n-\sigma^n,\frac{\varepsilon_h^n-\varepsilon_h^{n-1}}{\triangle t}).\end{aligned}\tag{21}$$

To multiply (21) by $2\triangle t$, and sum them in time, we obtain from (15) that

$$\begin{aligned}&a_1\parallel\varepsilon_h^n\parallel^2-a_2\parallel\varepsilon_h^0\parallel^2+\triangle t\sum_{i=1}^n\parallel\frac{e_h^i-e_h^{i-1}}{\triangle t}\parallel^2\\&\le C\parallel\frac{\partial^2c}{\partial\tau^2}\parallel^2_{L^2(0,T;L^2)}\triangle t^2+C\parallel\rho_t\parallel^2_{L^2(0,T;L^2)}+Ch|\sigma|_{L^\infty(0,T,H^1)}\parallel\varepsilon_h^n\parallel\\&-2\sum_{i=1}^n(\frac{\rho^{i-1}-\bar{\rho}^{i-1}}{\triangle t},e_h^i-e_h^{i-1})-2\sum_{i=1}^n(\frac{e_h^{i-1}-\bar{e}_h^{i-1}}{\triangle t},e_h^i-e_h^{i-1})\\&+Ch|\lambda|_{L^\infty(0,T,H^1)}\parallel\varepsilon_h^n\parallel+C\triangle t\sum_{i=1}^n\parallel\varepsilon_h^i\parallel^2+Ch^2|c_t|^2_{L^2(0,T;H^2)}\\&+Ch^3|\lambda|^2_{L^\infty(0,T;H^1)}+Ch^3|\sigma|^2_{L^\infty(0,T;H^1)}+C\parallel(\Pi^1\lambda-\lambda)_t\parallel^2_{L^2(0,T;L^2)}.\end{aligned}\tag{22}$$

By Lemma 3.5 and (12b)

$$\begin{aligned}&\sum_{i=1}^n(\frac{\rho^{i-1}-\bar{\rho}^{i-1}}{\triangle t},e_h^i-e_h^{i-1})\\&=(\frac{\rho^{n-1}-\bar{\rho}^{n-1}}{\triangle t},e_h^n)+\sum_{i=1}^{n-1}(\frac{\rho^{i-1}-\rho^i-(\bar{\rho}^{i-1}-\bar{\rho}^i)}{\triangle t},e_h^i)\\&\le C\frac{h+\triangle t}{\triangle t}\parallel\varepsilon_h^n\parallel\parallel\rho^{n-1}\parallel+Ch\parallel\rho^{n-1}\parallel\frac{h+\triangle t}{\triangle t}(|\lambda^n|_1+|c^n|_2)\\&+C\frac{h+\triangle t}{\triangle t}\sum_{i=1}^{n-1}\parallel\varepsilon_h^i\parallel\parallel\rho^{i-1}-\rho^i\parallel\\&+Ch\sum_{i=1}^{n-1}\parallel\rho^{i-1}-\rho^i\parallel\frac{h+\triangle t}{\triangle t}(|\lambda^i|_1+|c^i|_2)\\&\le\frac{a_1}{8}\parallel\varepsilon_h^n\parallel^2+C\parallel\rho\parallel^2_{L^\infty(0,T;L^2)}+C\parallel\rho_t\parallel^2_{L^\infty(0,T;L^2)}\\&+C\triangle t\sum_{i=1}^n\parallel\varepsilon_h^i\parallel^2+Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)}+|c|^2_{L^\infty(0,T;H^2)}).\end{aligned}\tag{23}$$

Similarly

$$\begin{aligned}
&\sum_{i=1}^{n}(\frac{e_h^{i-1}-\bar{e}_h^{i-1}}{\triangle t},e_h^i-e_h^{i-1})=(\frac{e_h^{n-1}-\bar{e}_h^{n-1}}{\triangle t},e_h^n)\\
&+\sum_{i=1}^{n-1}(\frac{e_h^{i-1}-e_h^i-(\bar{e}_h^{i-1}-\bar{e}_h^i)}{\triangle t},e_h^i)\\
\le\ & C\frac{h+\triangle t}{\triangle t}\parallel \varepsilon_h^n\parallel\parallel e_h^{n-1}\parallel\\
&+Ch\parallel e_h^{n-1}\parallel\frac{h+\triangle t}{\triangle t}(|\lambda^n|_1+|c^n|_2)\\
&+C\frac{h+\triangle t}{\triangle t}\sum_{i=1}^{n-1}\parallel \varepsilon_h^i\parallel\parallel e_h^{i-1}-e_h^i\parallel\\
&+Ch\sum_{i=1}^{n-1}\parallel e_h^{i-1}-e_h^i\parallel\frac{h+\triangle t}{\triangle t}(|\lambda^i|_1+|c^i|_2)\\
\le\ & \frac{a_1}{8}\parallel \varepsilon_h^n\parallel^2+Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)}+|c|^2_{L^\infty(0,T;H^2)})\\
&+C\triangle t\sum_{i=1}^{n}\parallel \varepsilon_h^i\parallel^2+\frac{\triangle t}{3}\sum_{i=1}^{n-1}\parallel\frac{e_h^i-e_h^{i-1}}{\triangle t}\parallel^2+C\parallel e_h^{n-1}\parallel^2.
\end{aligned} \tag{24}$$

Substituting (23) and (24) to (22), we have

$$\begin{aligned}
&a_1\parallel \varepsilon_h^n\parallel^2+\triangle t\sum_{i=1}^{n}\parallel\frac{e_h^i-e_h^{i-1}}{\triangle t}\parallel^2\\
\le\ & C\parallel\frac{\partial^2 c}{\partial\tau^2}\parallel^2_{L^2(0,T;L^2)}\triangle t^2+C\parallel\rho_t\parallel^2_{L^2(0,T;L^2)}+2a_2\parallel\varepsilon_h^0\parallel^2\\
&+Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)}+|c|^2_{L^\infty(0,T;H^2)}+|\sigma|^2_{L^\infty(0,T;H^1)})\\
&+C\parallel\rho\parallel^2_{L^\infty(0,T;L^2)}+C\parallel e_h^{n-1}\parallel^2+C\triangle t\sum_{i=1}^{n}\parallel\varepsilon_h^i\parallel^2\\
&+Ch^2(|\lambda_t|^2_{L^2(0,T;H^1)}+|c_t|^2_{L^2(0,T;H^2)}).
\end{aligned}$$

By (12b)

$$(\varepsilon_h^0,v_h)-\sum_{K\in T^h}\int_{\partial K}c^0 v_h\cdot nds+(\Pi^1\lambda^0-\lambda^0,v_h)=0,\forall v_h\in V_h,$$

choosing $v_h=\varepsilon_h^0\in V_h$, in above equation, we have

$$\parallel\varepsilon_h^0\parallel\le Ch|c^0|_2+Ch|\lambda^0|_1. \tag{25}$$

By Gronwall's lemma, we get

$$\begin{aligned}
\parallel\varepsilon_h^n\parallel\le\ & Ch(|\lambda|_{L^\infty(0,T;H^1)}+|c|_{L^\infty(0,T;H^2)}+|\sigma|_{L^\infty(0,T;H^1)})\\
&+Ch(|\lambda_t|_{L^2(0,T;H^1)}+|c_t|_{L^2(0,T;H^2)})\\
&+C\parallel\rho\parallel_{L^\infty(0,T;L^2)}+C\parallel\rho_t\parallel_{L^2(0,T;L^2)}\\
&+C\triangle t\parallel\frac{\partial^2 c}{\partial\tau^2}\parallel_{L^2(0,T;L^2)}+C\parallel e_n^{n-1}\parallel.
\end{aligned} \tag{26}$$

Taking $w_h=e_h^n,\ v_h=\zeta_h^n,\ \mu_h=\varepsilon_h^n$ in (12), we get

$$\begin{cases}
(a)(\frac{e_h^n-\bar{e}_h^{n-1}}{\triangle t},e_h^n)+(div\zeta_h^n,e_h^n)_h\\
=(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar{c}_h^{n-1}}{\triangle t},e_h^n)-(\frac{\rho^n-\bar{\rho}^{n-1}}{\triangle t},e_h^n),\\
(b)(\varepsilon_h^n,\zeta_h^n)-(e_h^n,div\zeta_h^n)_h-\sum_{K\in T^h}\int_{\partial K}c^n\zeta_h^n\cdot nds\\
+(\Pi^1\lambda^n-\lambda^n,\zeta_h^n)=0,\\
(c)(a(X,t^n)\varepsilon_h^n,\varepsilon_h^n)-(\zeta_h^n,\varepsilon_h^n)+(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\varepsilon_h^n)\\
-(\Pi^1\sigma^n-\sigma^n,\varepsilon_h^n)=0.
\end{cases} \tag{27}$$

Summing the above three equalities yields

$$\begin{aligned}
&(\frac{e_h^n-\bar{e}_h^{n-1}}{\triangle t},e_h^n)+(a(X,t^n)\varepsilon_h^n,\varepsilon_h^n)\\
&+(a(X,t^n)(\Pi^2\lambda^n-\lambda^n),\varepsilon_h^n)+(\Pi^2\lambda^n-\lambda^n,\zeta_h^n)\\
&-\sum_{K\in T^h}\int_{\partial K}c^n\zeta_h^n\cdot nds-(\Pi^1\sigma^n-\sigma^n,\varepsilon_h^n)\\
=\ &(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar{c}_h^{n-1}}{\triangle t},e_h^n)-(\frac{\rho^n-\bar{\rho}^{n-1}}{\triangle t},e_h^n).
\end{aligned} \tag{28}$$

The first term on the right hand of (28) can be estimated in the way analogous to that for (16)

$$\begin{aligned}
&|(\psi^n\frac{\partial c^n}{\partial\tau}-\frac{c_h^n-\bar{c}_h^{n-1}}{\triangle t},e_h^n)|\\
\le\ & C\triangle t\parallel\frac{\partial^2 c}{\partial\tau^2}\parallel^2_{L^2(t^{n-1},t^n;L^2)}+\parallel e_h^n\parallel^2.
\end{aligned} \tag{29}$$

Due to

$$\rho^n-\bar{\rho}^{n-1}=(\rho^n-\rho^{n-1})+(\rho^{n-1}-\bar{\rho}^{n-1})$$

we have

$$|(\frac{\rho^n-\rho^{n-1}}{\triangle t},e_h^n)|\le\frac{C}{\triangle t}\int_{t^{n-1}}^{t^n}\parallel\rho_t\parallel^2 ds+C\parallel e_h^n\parallel^2. \tag{30}$$

By Lemma 3.5 and (12b), the second term on the right hand of (28) can be estimated as

$$\begin{aligned}
&|(\frac{\rho^{n-1}-\bar{\rho}^{n-1}}{\triangle t},e_h^n)|\le C\frac{h+\triangle t}{\triangle t}\parallel\varepsilon_h^n\parallel\parallel\rho^{n-1}\parallel\\
&+Ch\parallel\rho^{n-1}\parallel\frac{h+\triangle t}{\triangle t}(|\lambda^n|_1+|c^n|_2)\\
\le\ &\frac{a_1}{3}\parallel\varepsilon_h^n\parallel^2+C\parallel\rho\parallel^2_{L^\infty(0,T;L^2)}\\
&+Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)}+|c|^2_{L^\infty(0,T;H^2)}).
\end{aligned} \tag{31}$$

Next we estimate the terms on the left hand of (28) one by one.

$$\begin{aligned}
&|(\frac{e_h^n-\bar{e}_h^{n-1}}{\triangle t},e_h^n)+(a(X,t^n)\varepsilon_h^n,\varepsilon_h^n)|\\
\ge\ &\frac{1}{2\triangle t}[(e_h^n,e_h^n)-(\bar{e}_h^{n-1},\bar{e}_h^{n-1})]+a_1\parallel\varepsilon_h^n\parallel^2\\
\ge\ &\frac{1}{2\triangle t}[(e_h^n,e_h^n)-(1+C\triangle t)^2(e_h^{n-1},e_h^{n-1})]+a_1\parallel\varepsilon_h^n\parallel^2,
\end{aligned} \tag{32}$$

where the inequality $\parallel\bar{e}_h^n\parallel\le(1+C\triangle t)\parallel e_h^n\parallel$ $(cf.[1])$ has been used in the last step.

By Lemma 3.2 and the Young inequality, we obtain

$$\begin{aligned}
|(\Pi^1\lambda^n-\lambda^n,\zeta_h^n)|&\le C\parallel\Pi^1\lambda^n-\lambda^n\parallel^2+C\parallel\zeta_h^n\parallel^2\\
&\le Ch^2|\lambda|^2_{L^\infty(0,T;H^1)}+C\parallel\zeta_h^n\parallel^2,
\end{aligned} \tag{33}$$

$$\begin{aligned}
|\sum_{K\in T^h}\int_{\partial K}c^n\zeta_h^n\cdot nds|&\le Ch|c^n|_2\parallel\zeta_h^n\parallel\\
&\le Ch^2|c|^2_{L^\infty(0,T;H^2)}+C\parallel\zeta_h^n\parallel^2,
\end{aligned} \tag{34}$$

$$|(\Pi^1\sigma^n-\sigma^n,\varepsilon_h^n)| \le C \parallel \Pi^1\sigma^n-\sigma^n \parallel^2 + \frac{a_1}{3} \parallel \varepsilon_h^n \parallel^2$$
$$\le Ch^2|\sigma|^2_{L^\infty(0,T;H^1)} + \frac{a_1}{3} \parallel \varepsilon_h^n \parallel^2, \tag{35}$$

$$|(a(X,t^n)(\Pi^1\lambda^n-\lambda^n),\varepsilon_h^n)| \le C \parallel \Pi^1\lambda^n-\lambda^n \parallel^2 + \frac{a_1}{3} \parallel \varepsilon_h^n \parallel^2$$
$$\le Ch^2|\lambda|^2_{L^\infty(0,T;H^1)} + \frac{a_1}{3} \parallel \varepsilon_h^n \parallel^2 . \tag{36}$$

Combining (29)-(36) with (28) gives the recursive relation

$$\frac{1}{2\triangle t}[(e_h^n,e_h^n)-(e_h^{n-1},e_h^{n-1})] \le C\triangle t \parallel \frac{\partial^2 c}{\partial\tau^2} \parallel^2_{L^2(t^{n-1},t^n,L^2)}$$
$$+\tfrac{C}{\triangle t}\int_{t^{n-1}}^{t^n} \parallel \rho_t \parallel^2 ds + (C+1) \parallel e_h^n \parallel^2$$
$$+C \parallel e_h^{n-1} \parallel^2 + C \parallel \rho \parallel^2_{L^\infty(0,T;L^2)} + C \parallel \zeta_h^n \parallel^2$$
$$+Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)} + |c|^2_{L^\infty(0,T;H^2)} + |\sigma|^2_{L^\infty(0,T;H^1)}). \tag{37}$$

Multiplying (37) by $2\triangle t$, summing in time and noting that $e_h^0=0$, we get

$$\parallel e_h^n \parallel^2 \le C\triangle t^2 \parallel \frac{\partial^2 c}{\partial\tau^2} \parallel^2_{L^2(0,T;L^2)} + C \parallel \rho_t \parallel_{L^2(0,T;L^2)}$$
$$+C\triangle t\sum_{i=1}^n \parallel e_h^i \parallel^2 + Ch^2\triangle t(|\lambda|^2_{L^\infty(0,T;H^1)}$$
$$+|c|^2_{L^\infty(0,T;H^2)} + |\sigma|^2_{L^\infty(0,T;H^1)})$$
$$+C\triangle t\sum_{i=1}^n \parallel \zeta_h^i \parallel^2 + C\triangle t \parallel \rho \parallel^2_{L^\infty(0,T;L^2)} . \tag{38}$$

By (13), we obtain

$$\sum_{i=1}^n \parallel \zeta_h^i \parallel^2 \le C\sum_{i=1}^n \parallel \varepsilon_h^i \parallel^2 + Ch^2|\lambda|^2_{L^\infty(0,T;H^1)} + Ch^2|\sigma|^2_{L^\infty(0,T;H^1)}. \tag{39}$$

Then it follows from (26) that

$$\sum_{i=1}^n \parallel \varepsilon_h^i \parallel^2 \le Ch^2(|\lambda|^2_{L^\infty(0,T;H^1)} + |c|^2_{L^\infty(0,T;H^2)} + |\sigma|^2_{L^\infty(0,T;H^1)})$$
$$+C \parallel \rho \parallel^2_{L^\infty(0,T;L^2)} + Ch^2(|c_t|^2_{L^2(0,T;H^2)} + |\lambda_t|^2_{L^2(0,T;H^1)})$$
$$+C\sum_{i=1}^{n-1} \parallel e_h^i \parallel^2 + C \parallel \rho_t \parallel^2_{L^2(0,T;L^2)} + C\triangle t^2 \parallel \frac{\partial^2 c}{\partial\tau^2} \parallel^2_{L^2(0,T;L^2)} . \tag{40}$$

So

$$C\triangle t\sum_{i=1}^n \parallel \zeta_h^i \parallel^2 \le C\triangle t\sum_{i=1}^{n-1} \parallel e_h^i \parallel^2 + C\triangle t^3 \parallel \frac{\partial^2 c}{\partial\tau^2} \parallel^2_{L^2(0,T;L^2)}$$
$$+C\triangle t \parallel \rho_t \parallel^2_{L^2(0,T;L^2)} + Ch^2\triangle t(|\lambda|^2_{L^\infty(0,T;H^1)} + |c|^2_{L^\infty(0,T;H^2)}$$
$$+|\sigma|^2_{L^\infty(0,T;H^1)}) + C\triangle t \parallel \rho \parallel^2_{L^\infty(0,T;L^2)}$$
$$+Ch^2\triangle t(|c_t|^2_{L^2(0,T;H^2)} + |\lambda_t|^2_{L^2(0,T;H^1)}). \tag{41}$$

Substituting (41) into (38) gives

$$\parallel e_h^n \parallel^2 \le C \parallel \rho_t \parallel^2_{L^2(0,T;L^2)} + Ch^3(|\lambda|^2_{L^\infty(0,T;H^1)} + |c|^2_{L^\infty(0,T;H^2)}$$
$$+|\sigma|^2_{L^\infty(0,T;H^1)}) + C\triangle t \parallel \rho \parallel^2_{L^\infty(0,T;L^2)}$$
$$+Ch^2\triangle t(|c_t|^2_{L^2(0,T;H^2)} + |\lambda_t|^2_{L^2(0,T;H^1)})$$
$$+C\triangle t^2 \parallel \tfrac{\partial^2 c}{\partial\tau^2} \parallel^2_{L^2(0,T;L^2)} + C\triangle t\sum_{i=1}^n \parallel e_h^i \parallel^2 . \tag{42}$$

By Gronwall's lemma, it follows that

$$\parallel e_h^n \parallel \le C \parallel \rho_t \parallel_{L^2(0,T;L^2)} + Ch^{\frac12} \parallel \rho \parallel_{L^\infty(0,T;L^2)}$$
$$+Ch^{\frac32}(|c_t|_{L^2(0,T;H^2)} + |\lambda_t|_{L^2(0,T;H^1)})$$
$$+Ch^{\frac32}(|\lambda|_{L^\infty(0,T;H^1)} + |c|_{L^\infty(0,T;H^2)} + |\sigma|_{L^\infty(0,T;H^1)})$$
$$+C\triangle t \parallel \tfrac{\partial^2 c}{\partial\tau^2} \parallel_{L^2(0,T;L^2)}$$
$$\le Ch \mid c_t \mid_{L^2(0,T;H^1)} + Ch^{\frac32} \mid c \mid_{L^\infty(0,T;H^1)}$$
$$+Ch^{\frac32}(|c_t|_{L^2(0,T;H^2)} + |\lambda_t|_{L^2(0,T;H^1)})$$
$$+Ch^{\frac32}(|\lambda|_{L^\infty(0,T;H^1)} + |c|_{L^\infty(0,T;H^2)} + |\sigma|_{L^\infty(0,T;H^1)})$$
$$+C\triangle t \parallel \tfrac{\partial^2 c}{\partial\tau^2} \parallel_{L^2(0,T;L^2)} . \tag{43}$$

Note that $c_h^n - c^n = e_h^n + \rho^n$. By the interpolation theory, (43) and the triangle inequality, we get (11a).

On the other hand, from (13), (26) and (43), we can derive that

$$\parallel \varepsilon_h^n \parallel \le Ch \mid c_t \mid_{L^2(0,T;H^1)}$$
$$+Ch \mid c \mid_{L^\infty(0,T;H^1)} + Ch(|c_t|_{L^2(0,T;H^2)} + |\lambda_t|_{L^2(0,T;H^1)})$$
$$+Ch(|\lambda|_{L^\infty(0,T;H^1)} + |c|_{L^\infty(0,T;H^2)}$$
$$+|\sigma|_{L^\infty(0,T;H^1)}) + C\triangle t \parallel \tfrac{\partial^2 c}{\partial\tau^2} \parallel_{L^2(0,T;L^2)},$$

$$\parallel \zeta_h^n \parallel \le Ch \mid c_t \mid_{L^2(0,T;H^1)} + Ch \mid c \mid_{L^\infty(0,T;H^1)}$$
$$+Ch(|c_t|_{L^2(0,T;H^2)} + |\lambda_t|_{L^2(0,T;H^1)})$$
$$+Ch(|\lambda|_{L^\infty(0,T;H^1)} + |c|_{L^\infty(0,T;H^2)}$$
$$+|\sigma|_{L^\infty(0,T;H^1)}) + C\triangle t \parallel \tfrac{\partial^2 c}{\partial\tau^2} \parallel_{L^2(0,T;L^2)} .$$

Similarly, we can get (11b) and (11c). The proof is completed.

Remark 1 The above finite element spaces V_h and M_h have been used to deal with second order elliptic problems and Navier-Stokes problems by mixed finite element methods in [15] and [16], respectively.

Remark 2 When the finite element space V_h is replaced by the constrained Q_1^{rot} element space [17]-[19] or P_1-nonconforming finite element space [20] on rectangular meshes, the results obtained in the present work are also valid. But how to extend the results of this paper to arbitrary quadrilateral meshes still remains open.

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