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Semi-Analytical Technique to Study The Dynamics Of Low Earth Satellites Under Gravitational Perturbations Using Universal Variables

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Abstract: Low-altitude Earth orbits provide important benefits for space missions. Among these benefits are the high-resolution images and resupply. In the present work, we studied the dynamics of Earth satellites that move in low orbits. The force model comprises the gravitational resonance 13:1, besides the Earth's gravitational potential up to the second degree and order. In order to avoid the appearance of singularities, the quasi-Hamiltonian equations of motion are formulated in terms of non-singular universal variables instead of Delaunay variables. We integrated numerically the equations of motion and performed some numerical simulations. It is found that the variations in eccentricity and inclination are small. In addition, we studied the effect of gravitational resonance 13:1 on the dynamics using Lagrange planetary equations. It is found that the effect of the gravitational resonance 13:1 in low Earth orbits is about tens of meters.

Keywords: Gravitational perturbations, resonance, singularities, non-singular variables, universal variables.

1 Introduction

Low-altitude Earth artificial satellites are satellites that move in close proximity to the Earth's surface. It is commonly utilized for satellite imaging and International Space Station. These orbits can provide several significant benefits over higher altitudes ones. The fact that these satellites are located near the surface of the Earth, helps them to obtain high-resolution images. Also, lower-cost missions can be enabled by operating in this region. Furthermore, it is easy for astronauts to reach the LEO region due to its proximity to Earth. Artificial satellites operating in this region are subject to a lot of disturbing forces. The Earth's gravity field and drag force mainly affect the satellites operating in this region, besides some small disturbing forces like resonance force. Artificial satellites that move in LEO suffer from gravitational resonances. Resonant tesseral harmonics can interact together and produce complex dynamical motions,[1], [2]. The existence of singularities is one of the major issues in celestial mechanics and nonlinear dynamics. Mathematically, singularities are points that when described give an infinite value. This problem is caused due to the presence of small divisors in the analytic integration of the secular terms developed by the averaging process [3], [4]. Singularities appear in artificial satellites theory, like the singularities for small eccentricities appearing in the literal developments of lunar theory [5]. Also, singularities appear due to small eccentricities and inclinations present in Brouwer's theory of an artificial satellite [6]. Furthermore, the critical inclination singularity is presented in the solution of the motion of an orbiter that moves in the gravitational field of an oblate body, [7].
The problem under consideration has been studied by many authors extensively in the literature. Valk et al studied the motion of geosynchronous space objects under the gravitational influence. The influence of the Earth's gravity field and the lunisolar perturbations are taken into account, as well. The authors presented the

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resonant motion and its main characteristics, such as equilibria and stability by comparison with a basic analytical dynamical model [8]. Celletti et al used the bifurcation theory to investigate the secular resonances induced by the combined effect of the Sun and Moon on space objects orbiting the Earth. They concentrated on the secular resonances, that depend only on the orbital inclination of the space debris. They investigated the birth of periodic orbits and determined the energy thresholds at which the bifurcations due to the luni-solar secular resonances occur [9].
Celletti et al studied the dynamics of resonances and the existence of equilibria in the region of the Low Earth orbit. The results of their work are based on a simplified force model which comprises the Earth's geopotential and the atmospheric drag. Utilizing the mentioned force model, they studied qualitatively the resonances and found the equilibrium positions [10]. Alessi et al highlighted the important role that the orbital resonances associated with solar radiation pressure can have in the LEO region. The authors compared the results obtained with the simplified force model with those obtained in the case of a complex dynamical model. For well defined initial conditions, they provided an analytical tool to estimate the maximum eccentricity value which can be achieved [11].
Recently, Celletti et al studied the presence of resonances in space around the Earth. The perturbations considered are the Earth's geopotential and the effects of the Sun and Moon. The authors distinguished different types of resonances. They characterized such resonances, giving precise statements on the space regions where the different types of resonances can be found [2]. Aleksandrova et al identified secular resonances that act on space objects that move in the LEO–MEO regions. The authors gave the distribution maps of the identified secular resonances. Furthermore, they analyzed the orbital evolution of the space objects [12].
The goal of the present work is to study the dynamics of the low-altitude earth satellites under gravitational perturbations. We constructed the equations of motion in terms of non-singular universal coordinates to avoid singularities. Also, we investigate the effects caused by the tesseral gravitational resonance 13:1 on the dynamics of the problem using Lagrange planetary equations. We carried out several numerical investigations to shed light on dynamics.

2 Singularities and universal set of variables

When we use orbital elements to describe the motion of artificial satellites, singularities cause a lot of problems. These problems occur when the satellite moves in a circular orbit ($e = 0$), equatorial orbit ($i = 0$), circular orbit and equatorial orbit ($i = 0, e = 0$). Then, it is useful to choose a new set of variables that remove the singularities that appear in the equations of motion. The scientific literature contains a lot of universal variables which prevent the singularity problem. To mention some, the equinoctial variables, $(a, \lambda = M+\omega+\Omega, e\sin(\omega+\Omega), e\cos(\omega+\Omega), \tan\frac{i}{2}\sin\Omega, \tan\frac{i}{2}\cos\Omega)$. But the equations of motion, in this case, are complex and difficult to deal with, so we will use Poincaré variables as another approach. We have

$$P = L-G,\ p = -\omega-\Omega,\ Q = G-H,\ q = -\Omega.$$

Where λ is the mean longitude and (M,ω,Ω,L,G,H) are the Delaunay elements defined by:

$$L = \sqrt{\mu a},\ G = \sqrt{\mu a(1-e^2)},\ H = \sqrt{\mu a(1-e^2)}\cos i.$$

3 Potential of the Earth

The gravitational field R of the Earth can be expanded in spherical harmonics with respect to spherical coordinates[13]:

$$R = \frac{\mu}{a}\sum_{n=2}^{+\infty}\sum_{m=0}^{n}\left(\frac{R_e}{a}\right)^n\sum_{p=0}^{n}F_{nmp}(i)\sum_{q=-\infty}^{\infty}G_{npq}(e) \times S_{nmpq}(M,\omega,\Omega,\theta) \quad (1)$$

where

$$S_{nmpq} = \begin{bmatrix} C_{nm} \\ -S_{nm}\end{bmatrix}_{n-m\ even}^{n-m\ odd}\cos\Psi_{nmpq} + \begin{bmatrix} C_{nm} \\ S_{nm}\end{bmatrix}_{n-m\ even}^{n-m\ odd}\sin\Psi_{nmpq}$$

and

$$\Psi_{nmpq} = (n-2p)\omega+(n-2p+q)M+m(\Omega-\theta)$$

where $F_{nmp}(i)$ is the inclination function, $G_{npq}(e)$ is the eccentricity functions, μ is the gravitational parameter of the Earth, a is the semi-major axis, R_e is the radius of the Earth, S_{nm} and C_{nm} are the normalized geopotential coefficients, (a,e,i,M,ω,Ω) are the orbital elements and θ is the sidereal time.

Now we proceed to formulate the Earth's gravitational potential in terms of the orbital elements, we have

$$\begin{aligned} \bar{x} &= \cos(f+\omega+\Omega)+2\sin(f+\omega)\sin\Omega\sin^2 i/2 \\ \bar{y} &= \sin(f+\omega+\Omega)-2\sin(f+\omega)\cos\Omega\sin^2 i/2 \\ \bar{z} &= 2\sin(f+\omega)\sin i/2\cos i/2 \end{aligned} \quad (2)$$

and using the developments in powers of the eccentricity truncated at an arbitrary order [3]:

$$\begin{aligned} \frac{r}{a} &= 1+\frac{e^2}{2}-2e\sum_{s=1}^{+\infty}\frac{1}{s^2}\frac{d}{de}J_s(se)\cos sM \\ \sin f &= 2\sqrt{1-e^2}\sum_{s=1}^{+\infty}\frac{1}{s}\frac{d}{de}J_s(se)\sin sM \\ \cos f &= -e+\frac{2(1-e^2)}{e}+\sum_{s=1}^{\infty}\cos sM \end{aligned} \quad (3)$$

where $J_s(se)$ is the Bessel function, f and r are the true anomaly and the geocentric distance of the satellite, respectively.

$$R_{J_2} = \frac{\mu^4 R_e^2}{2L^6}\left(\frac{a^2}{r}\right)\left(1-3\bar{z}^2\right), \qquad R_{J_{22}} = \frac{3\mu^4 R_e^2}{L^6}\left[C_{22}\left(\bar{x}^2-\bar{y}^2\right)+S_{22}(2\bar{x}\bar{y})\right]. \quad (4)$$

where J_2 is the second zonal harmonic, J_{22} is the sectorial harmonic of second degree and order and $L=\sqrt{\mu a}$.

Now we introduce the variables U and V rather than the eccentricity and inclination:

$$U=\sqrt{\tfrac{2P}{L}}, \qquad V=\sqrt{\tfrac{2Q}{L}}$$

Let us note that for small moderate eccentricity and inclination, we have:

$$U \approx e\,, \qquad V \approx 2\sin\tfrac{i}{2} \approx i,$$

$$e = U - \tfrac{1}{8}U^3 - \tfrac{1}{128}U^5 + \mathscr{O}(U^7), \quad (5)$$

$$2\sin\tfrac{i}{2} = V + \tfrac{1}{4}VU^2 + \tfrac{3}{32}VU^4 + \mathscr{O}(U^6).$$

therefore, it is straightforward to deduce the final expansion in the set of non-dimensional Cartesian variables (X_1,Y_1,X_2,Y_2), defined by:

$$X_1 = U\sin p = \sqrt{\tfrac{2P}{L}}\sin p, \quad Y_1 = U\cos p = \sqrt{\tfrac{2P}{L}}\cos p$$
$$X_2 = V\sin q = \sqrt{\tfrac{2Q}{L}}\sin q, \quad Y_2 = V\cos q = \sqrt{\tfrac{2Q}{L}}\cos q \quad (6)$$

After removing the short-period terms (those terms depending on λ) from the disturbing function the Hamiltonian functions due to J_2 and J_{22} can be written as:

$$H_{J_2} = \frac{J_2 R^2\left(1+3X_1^2-24X_2^2+3Y_1^2-24Y_2^2\right)\mu^4}{2L^6}, \quad (7)$$

$$\begin{aligned} H_{J_{22}} = &-\frac{9C_{22}R^2X_1^2\mu^4}{4L^6}+\frac{9C_{22}R^2Y_1^2\mu^4}{4L^6}-\frac{24C_{22}R^2\mu^4X_2^2}{L^6}+ \\ &\frac{24C_{22}R^2\mu^4Y_2^2}{L^6}-\frac{9R^2S_{22}X_1Y_1\mu^4}{2L^6}-\frac{48R^2S_{22}X_2Y_2\mu^4}{L^6} \\ &+\frac{12C_{22}R^2\mu^4Y_1Y_2^2}{L^6}-\frac{12S_{22}R^2\mu^4X_1Y_2^2}{L^6} \\ &-\frac{24S_{22}R^2\mu^4Y_1X_2Y_2}{L^6}-\frac{24C_{22}R^2\mu^4X_1X_2Y_2}{L^6} \\ &-\frac{12C_{22}R^2\mu^4Y_1X_2^2}{L^6}+\frac{12S_{22}R^2\mu^4X_1X_2}{L^6}. \end{aligned} \quad (8)$$

the new differential system of equations of motion are given by [13]:

$$\begin{aligned} \dot{X}_i &= \frac{1}{L}\left(\frac{\partial\mathscr{H}}{\partial Y_i}-\frac{1}{2}X_i\dot{L}\right),\ \dot{Y}_i = -\frac{1}{L}\left(\frac{\partial\mathscr{H}}{\partial X_i}-\frac{1}{2}Y_i\dot{L}\right), \\ \dot{\lambda} &= \frac{\partial\mathscr{H}}{\partial L}-\frac{1}{2L}\left[\sum_{i=1}^{2}\frac{\partial\mathscr{H}}{\partial X_i}X_i+\sum_{i=1}^{2}\frac{\partial\mathscr{H}}{\partial Y_i}Y_i\right], \\ \dot{L} &= -\frac{\partial\mathscr{H}}{\partial\lambda}. \end{aligned} \quad (9)$$

where $(i=1,2)$ and $\mathscr{H}=\mathscr{H}(X_1,Y_1,X_2,Y_2,L)$ is the Hamiltonian function rewritten in the non-dimensional and Cartesian set of variables defined in equation (6).

4 The gravitational resonance 13:1

In this part, we will study the effect of the gravitational resonance 13:1 on the evolution of the orbital elements of an artificial satellite. To handle this dynamical problem, we will use Lagrange planetary equations. we will focus our attention on the variations of the semi-major axis a, inclination i, and eccentricity e. Now Lagrange's equations of motion can be written as [14], [15]

$$\begin{aligned} \frac{da_{nmpq}}{dt} &= \frac{2\mu R_e^n F_{nmp}G_{npq}S'_{nmpq}}{\bar{n}a^{n+2}}(n-2p+q), \\ \frac{di_{nmpq}}{dt} &= \frac{\mu R_e^n F_{nmp}G_{npq}S'_{nmpq}}{\bar{n}a^{n+3}\sqrt{1-e^2}\sin i}((n-2p)\cos i-m), \\ \frac{de_{nmpq}}{dt} &= \frac{\mu R_e^n F_{nmp}G_{npq}S'_{nmpq}}{\bar{n}a^{n+3}e}\times \\ &\left(\left(1-e^2\right)(n-2p+q)-\sqrt{1-e^2}(n-2p)\right). \end{aligned} \quad (10)$$

where, as mentioned, $F_{nmp}(i)$ represents the inclination function, $G_{npq}(e)$ is the eccentricity function and $\bar{n}$ is the mean motion of the spacecraft.

$$S'_{nmpq} = \frac{dS_{nmpq}}{d\Psi_{nmpq}} = -\begin{bmatrix} C_{nm} \\ -S_{nm}\end{bmatrix}_{n-meven}^{n-modd}\sin\Psi_{nmpq} + \begin{bmatrix} C_{nm} \\ S_{nm}\end{bmatrix}_{n-meven}^{n-modd}\cos\Psi_{nmpq}$$

and

$$\dot{\Psi}_{nmpq} = (n-2p)\dot{\omega}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta}) \quad (11)$$

Artificial satellites orbiting near the Earth at low-altitude orbits are affected by different types of perturbing forces, the gravitational resonance 13:1 is one of these forces. This resonance affects the orbital elements of the satellite, especially on the semi-major axis. So, in order to formulate an accurate mathematical theory to describe the dynamics of artificial satellites in low Earth orbits, the effect of resonance should be taken into account.

Considering all the significant resonant terms of the disturbing function due to this resonance is important. Integrating Lagrange planetary equations, the derivative $\dot{\Psi}_{nmpq}$ will appear in the denominator of the resultant solution. When the quantity $\dot{\Psi}_{nmpq} \simeq 0$, the resonance occurs and as a result, we encounter a large perturbation case.

In the gravitational resonance 13:1, the condition for resonance can be written as:

$$\dot{\Psi}_{nmpq} = \beta\dot{\omega} + \dot{M} + 13(\dot{\Omega} - \dot{\theta}) \simeq 0, \quad \beta = 0, 1, .. \qquad (12)$$

5 Results and discussion

Once we formulate the quasi-Hamiltonian system of equations that represents the motion of an Earth satellite that moves in the LEO region, we will be able to investigate its long-period dynamics. As we mentioned, the gravitational potential due to the Earth is taken into account up to second degree and order. In order to make the problem clear, we will carry out some selected cases by integrating numerically the equations of motion. The case of small eccentricity and inclination will be among the selected cases. The time history of the eccentricity and inclination is modeled over three years for different initial conditions.

Figs. 1a and 1b depict the evolutions of the eccentricity and inclination for the initial values $e = 0.0001$ and $i = 2°$. It is clear from fig. 1a that the maximum amplitude of variation of the eccentricity can reach values up to 0.000089, while we can see from fig. 1b that the change in the inclination is small. Figs. 1c and 1d show the variations of the eccentricity and inclination in the case of initial values $e = 0.001$ and $i = 30°$.

Fig.2 illustrates the dynamical evolution of the semi-major axis and the eccentricity of an artificial satellite that moves in low Earth orbit. The satellite moves under the effect of Earth's gravity field up to the second zonal harmonic and perturbation due to the gravitational resonance 13:1. The integration of Lagrange planetary equations is carried out for three years from the initial time. Figs. 2a, 2b, and figs. 2c, 2d represent the time history of the semi-major axis and eccentricity for the initial conditions $i = 55°$ and $i = 60.3°$, respectively, $a = 7230\ km$ in both cases. It is visible from both figures that the variation in the semi-major axis due to the perturbations considered can reach tens of meters. Also, we see that the variation in the eccentricity is small and the maximum amplitude can only reach the value of 0.002.

Fig. 3a shows the location of resonance centers in the space (i, e), of the form $\psi \simeq 0$, where only the effects of the second zonal harmonic perturbation on the argument of perigee and longitude of the node have been

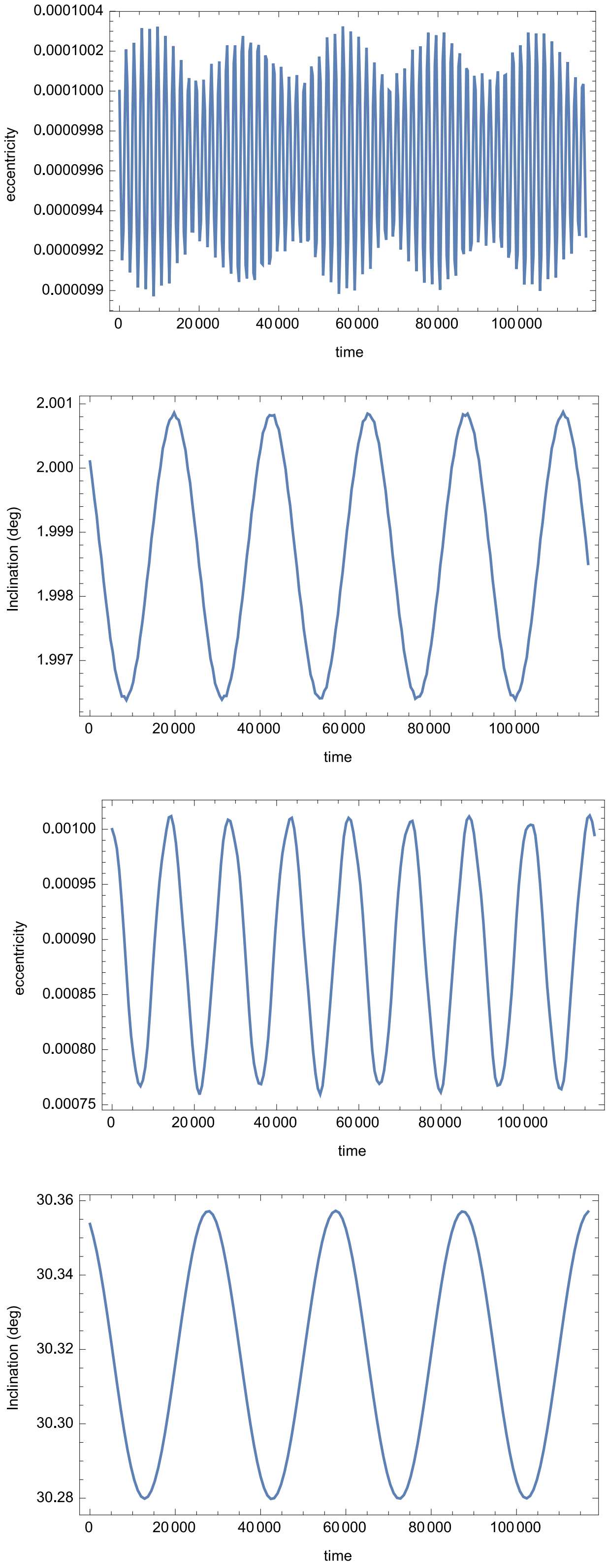

Fig. 1: Time history of the eccentricity and inclination for different initial conditions. In figs. 1a, 1b the initial eccentricity and inclination are $e = 0.0001$ and $i = 2°$, while in figs. 1c and 1d, $e = 0.001$ and $i = 30°$. Time is measured in canonical units.

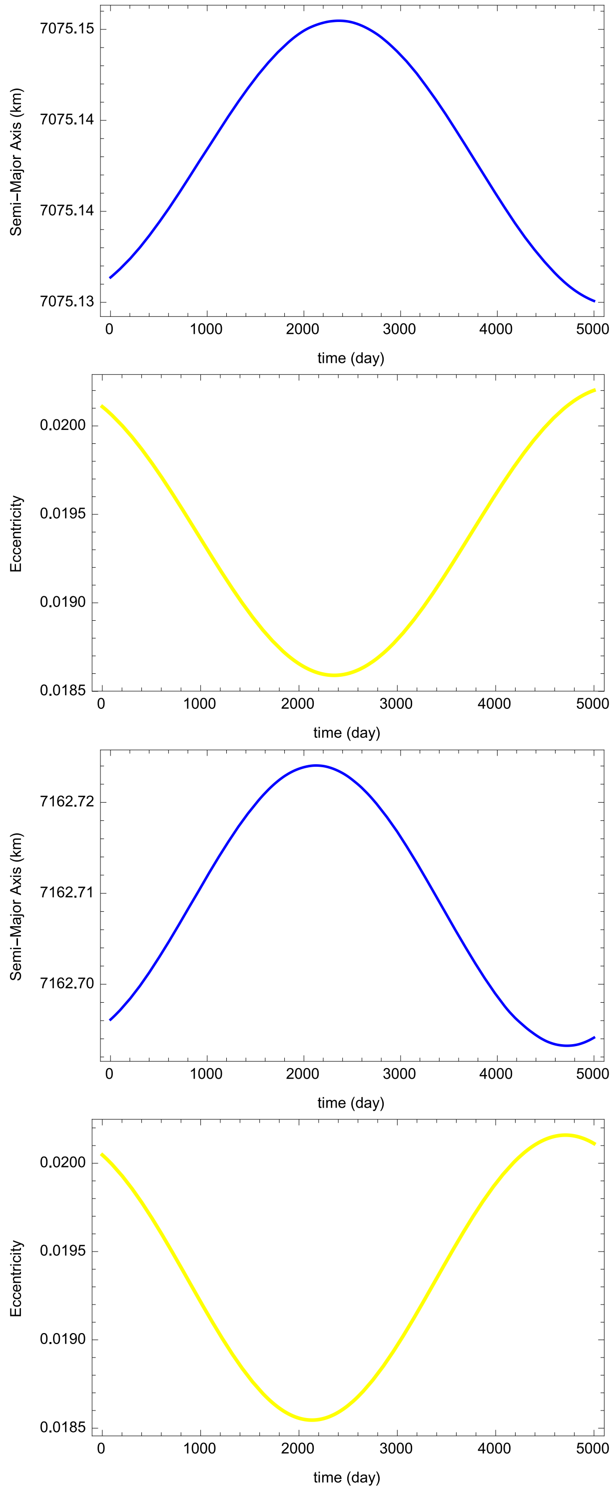

Fig. 2: Time history of the semi-major axis and eccentricity. In figs. 2a, 2b the initial inclination is $i = 55°$. In figs. 2c, 2d the initial inclination is $i = 60.3°$. In both figs. the initial eccentricity and semi-major axis are $e = 0.02$ and $a = 7230$ km.

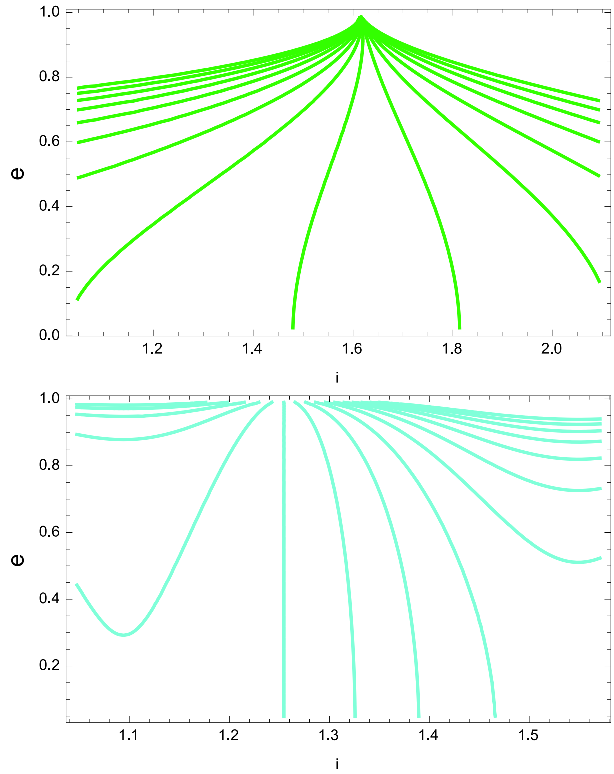

Fig. 3: The phase space contours (e,i) at $a = 7230$ km. (a) represents the structure of the phase space considering only the disturbing function due to the second zonal harmonic, J_2. (b) Represents the contours considering the resonance 13:1 besides the effect of J_2.

considered. The figure represents the curve when the semi-major axis is fixed at $a = 7230$ km. Fig. 3b depicts the phase space structure when the effects due to the gravitational resonance 13:1 are taken into consideration in addition to the oblateness coefficient J_2. It is visible from the figure that the web structure is modified due to the disturbing function of the resonance 13:1. If we consider more combinations of $(n-2p)$ a greater array of resonances are obtained, see equation (11). If we have multiple resonances case, then the different resonances will interact with each other leading to chaotic phase space.

Fig. 4a shows the space (a,e) at fixed $i = 55°$, while fig. 5a shows the space (a,i) at fixed $e = 0.02$. Both figures are obtained only under the influences of the oblateness coefficient J_2. Fig. 4b shows the space (a,e) at fixed $i = 55°$, while fig. 5b shows the space (a,i) at fixed $e = 0.02$. In the latter-mentioned case, the effects of the gravitational resonance are taken into account besides the oblateness coefficient. We can easily see from the two figures the variations in the phase space due to resonance 13:1.

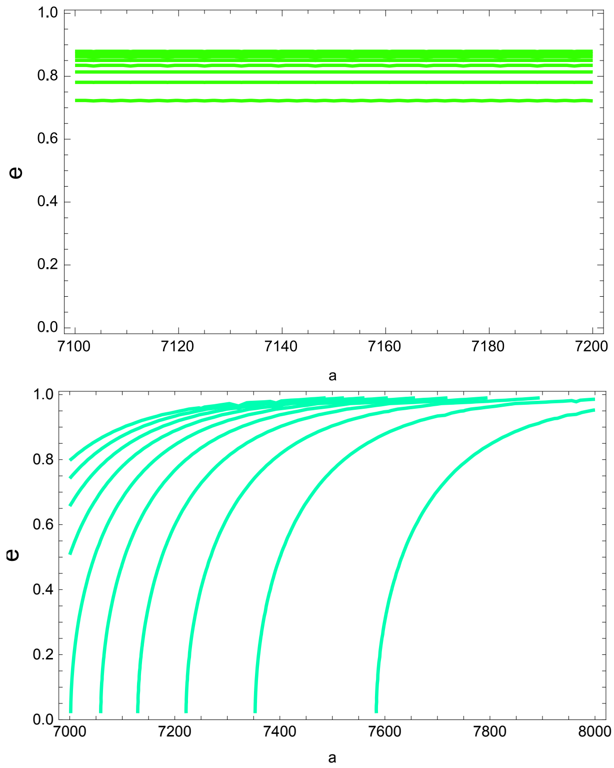

Fig. 4: The phase space contours (e,a) at $i = 55^\circ$. (a) represents the structure of the phase space considering only the disturbing function due to the second zonal harmonic, J_2. (b) Represents the contours considering the resonance 13:1 besides the effect of J_2.

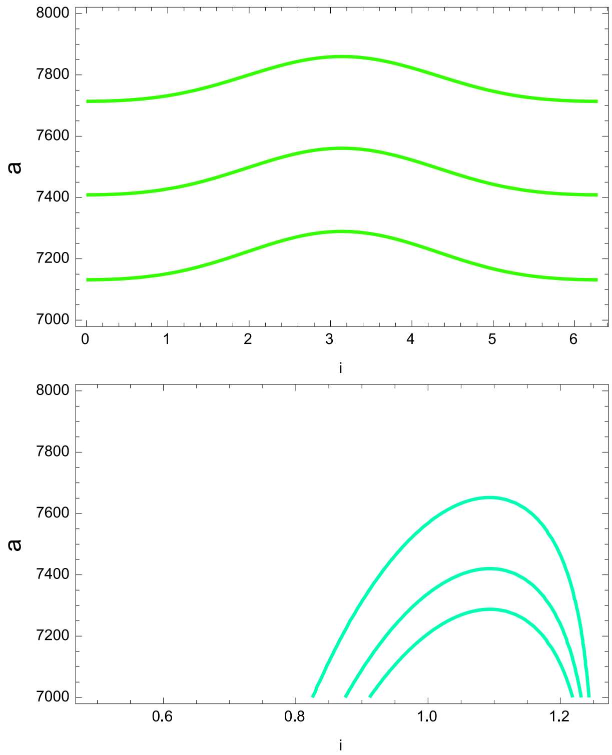

Fig. 5: The phase space contours (a,i) at $e = 0.02$. (a) represents the structure of the phase space considering only the disturbing function due to the second zonal harmonic, J_2. (b) Represents the contours considering the resonance 13:1 besides the effect of J_2.

6 Computational algorithms

Now we design some computational algorithms to explain the sequence of computations.

Computational algorithm (1): An algorithm to integrate the quasi-Hamiltonian system of equations.

- Express the disturbing function $R = R_{J_2} + R_{J_{22}}$ in terms of the orbital elements using equations (2) and (3).
- Use equation (5) and the relations $U = \sqrt{\frac{2P}{L}}$, $V = \sqrt{\frac{2Q}{L}}$ to formulate the disturbing function R in terms of the variables (U,V,q,p,L,λ), *i.e.* $R = R(U,V,q,p,L,\lambda)$.
- Eliminate the short-period terms from the disturbing function to obtain the normalized one, $R' = R'(U,V,L)$.
- Use equations (6) to express R in terms of a set of non-dimensional Cartesian variables (X_1,Y_1,X_2,Y_2), *i.e.* $R' = R'(X_1,Y_1,X_2,Y_2,L)$.
- Construct the Hamiltonian function of the problem, $\mathscr{H} = -\frac{\mu^2}{2L^2} + \Lambda\dot{\theta} + H_{J2} + H_{J22}$. Where Λ is the conjugate momentum corresponding to the sidereal time.
- Construct the quasi-Hamiltonian system of differential equations, equation (9).
- Integrate numerically the system using the initial values $(X_{10},Y_{10},X_{20},Y_{20},L_0)$, initial epoch t_0 and t_f.
- Plot the results to obtain the time history of eccentricity and inclination, fig.1.

Computational algorithm (2): An algorithm to integrate Lagrange planetary equations.

- Use equation (1) to formulate the disturbing function due to the gravitational resonance $13:1, R_{Gr}$, in terms of the orbital element.
- Use equation (1) to formulate the normalized disturbing function, $\overline{R}_{J2}$, due to the Earth's potential up to the second zonal harmonic J_2.
- Evaluate the required eccentricity and inclination functions G_{npq} and F_{nmp}.
- Construct the total disturbing function $\overline{R} = \overline{R}_{J2} + R_{Gr}$
- Use the disturbing function $\overline{R}$ to construct Lagrange planetary equations, equation (10).
- Integrate numerically Lagrange planetary equations using the initial values $(e_0,a_0,i_0,\omega_0,\Omega_0,M_0)$, initial epoch t_0 and t_f.
- Plot the results to obtain the time history of eccentricity and semi-major axis, fig.2.

Computational algorithm (3): An algorithm to plot the phase spaces (e,a), (e,i) and (a,i).

–Construct the quantities $\dot{\omega}, \dot{M}$ and $\dot{\Omega}$ taking into consideration effects due to Earth's oblateness and the gravitational resonance 13:1.
–Construct the condition for resonance, equation (12).
–Plot the required phase spaces, fig.3, fig.4, and fig.5.

7 Conclusion

Investigating the long-period dynamics of artificial satellites in low Earth orbits is crucial. In the present work, we studied the dynamics of orbiters that move in the LEO region under perturbations due to the Earth's gravity field and gravitational resonance 13:1. We used non-singular variables to avoid singularity problems at small inclinations and eccentricities. The non-singular variables allowed us to study the behavior of the dynamical system near small eccentricity and inclination. The gravitational resonance 13:1 causes a small variation in the semi-major axis. However, when formulating a high-accuracy theory of the motion of satellites in this region, we have to take its effect into account.

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Data availability

Data will be available on request.

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