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Kernel Functions and New Applications of an Accurate Technique

Rania Wannan [1], *Ali Akgül* [2,3,4,*], *Esra Karatas Akgül* [3], *Eyad Hasan Hasan*[5] *and Jihad Asad*[6]

[1]Department of Applied Mathematics, Faculty of Applied Sciences, Palestine Technical University- Kadoorie, Tulkarm, West Bank, Palestine
[2]Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon
[3]Department of Mathematics, Art and Science Faculty, Siirt University, TR-56100 Siirt, Turkey
[4]Department of Mathematics, Mathematics Research Center, Near East University, Near East Boulevard, PC: 99138, Nicosia /Mersin 10 – Turkey
[5]Applied Physics Department, Faculty of Science, Tafila Technical University O. Box: 179, Tafila 66110, Jordan
[6]Department of Physics, Faculty of Applied Sciences, Palestine Technical University- Kadoorie, Tulkarm, West Bank, Palestine

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Abstract: In this article, some general reproducing kernel Sobolev spaces was constructed. We find the general functions in these reproducing kernel Sobolev spaces. Many higher order boundary value problems can be investigated by these special functions.

Keywords: Special functions, Sobolev spaces, Inner product.

1 Introduction

Sobolev spaces are the basis of the theory of weak or variational forms of partial differential equations. They are very important spaces in the mathematical analysis. The knowledge of the reproducing kernels is very valuable in the analysis of many problems. In literature explicit formulas for the reproducing kernels of the Sobolev spaces is hard to obtained [1].

Reproducing kernel space is a special Sobolev space and there are many works on the solution of the nonlinear problems with reproducing kernel method [2]. Daniel [3] has studied the reproducing kernel spaces and applications. On the other hand, Niu et al. [4] have investigated the numerical solution of nonlinear singular boundary value problems and Chen et al. [5] have obtained the exact solution of a class of fractional integro-differential equations with the weakly singular kernel by the reproducing kernel method. Furthermore, Xu et al. [6] have worked the simplified reproducing kernel method for fractional differential equations with delay. Also, Mei et al. [7] have researched the simplified reproducing kernel method for impulsive delay differential equations. Whereas, Geng et al. [8] have investigated the piecewise shooting reproducing kernel method for linear singularly perturbed boundary value problems. In addition, Li et al. [9] have studied the space--time spectral method for the Cattaneo equation with time fractional derivative. The reproducing kernel method for the numerical solution of the Brinkman–Forchheimer momentum equation have been researched by Abbasbandy et al. [10] and Li et al. [11] have investigated the novel method for nonlinear singular fourth order four-point boundary value problems. Li et al. [12] have worked the continuous method for nonlocal functional differential equations with delayed or advanced arguments. Mohammadi et al. [13] have given the reproducing kernel method for solving a class of nonlinear system of partial differential equations. Finally, Mohammadi et al. [14] have studied the Galerkin-reproducing kernel method. For more details see [15-23].

2 Preliminaries

Definition 2.1. Let $m \in \mathbb{N} \cup \{0\}$. We set:

*Corresponding author e-mail: j.asad@ptuk.edu.ps

$$\langle u,v\rangle_{H^m(\Omega)} = \int_{\Omega} \sum_{|\alpha|\le m} D^{\alpha}u(x)\overline{D\hat{}\alpha\, v(x)}\, dx.$$

The pair $(H^m(\Omega), \langle .,.\rangle)$ is a Hilbert space.

Definition 2.2. We define the reproducing kernel Sobolev space $S_2^m[a,b]$ as [2]:

$$S_2^m[a,b] = \left\{ \begin{matrix} f|f^{(m-1)} \text{is absolutely continuous function,} \\ f^{(m)} \in L^2[a,b], x \in [a,b] \end{matrix} \right\}$$

We define the inner product and norm for this space as:

$$\langle f,g\rangle_{S_2^m} = \int_a^b \left(\sum_{i=0}^{m} f^{(i)}(x) g^{(i)}(x) \right) dx.$$

And

$$\|f\|_{S_2^m} = \sqrt{\langle f,f\rangle_{S_2^m}}.$$

Definition 2.3. Let $m \in \mathbb{N}$ and $I = [a,b]$ an interval in $\mathbb{R}$. The reproducing kernel of the space $S_2^m(I)$ is a function $R_y: I \to \mathbb{R}$ such that:

$\langle f, R_y\rangle_{S_2^m} = f(y)$, for almost a.e. $y \in I$.

3 Reproducing Kernel Functions in the Special Cases

In this section we introduce the reproducing kernel functions for $m = 1,2$, which have been discussed in [19].

For $m = 1$ the reproducing kernel function:

$$R_y(x) = \begin{cases} \frac{1}{2}\frac{(e^{1-y}+e^{-1+y})(e^{1+x}+e^{1-x})}{e^2-1}, & x \le y \\ \frac{1}{2}\frac{(e^{1+y}+e^{1-y})(e^{-1+x}+e^{1-x})}{e^2-1}, & x > y. \end{cases}$$

For $m = 2$ we have the reproducing kernel function when $x \le y$:

$$\begin{aligned} R_y(x) = \frac{1}{6}\frac{1}{e^{\sqrt{3}}-1} e^{\frac{\sqrt{3}}{2}} \Bigg(& \cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y+1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + 2\cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x+y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} \\ & + 2\cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x+y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + \cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y+1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + \sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y+1)} \sin\left(\frac{x}{2}\right)\sqrt{3} \\ & + 2\sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x+y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} + 2\sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x+y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} + \sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y+1)} \sin\left(\frac{x}{2}\right)\sqrt{3} \\ & - 3\cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y+1)} \sin\left(\frac{x}{2}\right) + 3\cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y+1)} \sin\left(\frac{x}{2}\right) + 3\sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y+1)} \cos\left(\frac{x}{2}\right) \\ & - 3\sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y+1)} \cos\left(\frac{x}{2}\right) \Bigg). \end{aligned}$$

The reproducing kernel function for $x > y$ is obtained as:

$$\begin{aligned} R_y(x) = \frac{1}{6}\frac{1}{e^{\sqrt{3}}-1} e^{\frac{\sqrt{3}}{2}} \Bigg(& \cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + 2\cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x+y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} \\ & + 2\cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x+y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + \cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y-1)} \cos\left(\frac{x}{2}\right)\sqrt{3} + \sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} \\ & + 2\sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x+y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} + 2\sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x+y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} + \sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y-1)} \sin\left(\frac{x}{2}\right)\sqrt{3} \\ & - 3\cos\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y-1)} \sin\left(\frac{x}{2}\right) + 3\cos\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y-1)} \sin\left(\frac{x}{2}\right) + 3\sin\left(\frac{y}{2}\right) e^{\frac{\sqrt{3}}{2}(x-y-1)} \cos\left(\frac{x}{2}\right) \\ & - 3\sin\left(\frac{y}{2}\right) e^{-\frac{\sqrt{3}}{2}(x-y-1)} \cos\left(\frac{x}{2}\right) \Bigg). \end{aligned}$$

4 Some Special Functions in the General Case

We obtain the general functions of the reproducing kernel Sobolev spaces in this section. These special functions are new in the literature.

Theorem 4.1. We obtain the reproducing kernel function of the reproducing kernel Sobolev space $S_2^4[0,1]$ as:

$$B_y(x) = \begin{cases} a_y(x), & x \le y \\ b_y(x), & x > y, \end{cases}$$

Where $b_y(x) = a_x(y)$ and $a_x(y)$ is given in the appendix. **Proof.** We have
$\langle u, B_y \rangle_{S_2^4[0,1]} = \int_0^1 \left[u(x)B_y(x) + u'^{(x)B'}{}_{y(x)} + u''^{(x)B''}{}_{y(x)} + u'''^{(x)B'''}{}_{y(x)} + u^{(4)}(x)B_y^{(4)}(x)\right]dx,$
By the definition of the inner product. We use integration by parts and obtain:
$\langle u, B_y \rangle_{S_2^4[0,1]} = \int_0^1 u(x)B_y(x)dx + u(x)B_y(x)dx + u(1)B'{}_y(1) - u(0)B'{}_y(0) - \int_0^1 u(x)B''{}_y(x)dx + u'(1)B''{}_y(1) - u'(0)B''{}_y(0) - u(1)B'''{}_y(1) + u(0)B'''{}_y(0) + \int_0^1 u(x)B_y^{(4)}(x)dx + u''(1)B'''{}_y(1) - u''(0)B'''{}_y(0) - u'(1)B^{(4)}{}_y(1) + u'(0)B^{(4)}{}_y(0) + u(1)B^{(5)}{}_y(1) - u(0)B^{(5)}{}_y(0) - \int_0^1 u(x)B_y^{(6)}(x)dx + u'''(1)B^{(4)}{}_y(1) - u'''(0)B^{(4)}{}_y(0) - u''(1)B^{(5)}{}_y(1) + u''(0)B^{(5)}{}_y(0) + u'(1)B^{(6)}{}_y(1) - u'(0)B^{(6)}{}_y(0) - u(1)B^{(7)}{}_y(1) + u(0)B^{(7)}{}_y(0) + \int_0^1 u(x)B_y^{(8)}(x)dx.$
If we choose the coefficients of $u^{(i)}(0), u^{(i)}(1), i = 0,1,2,3$ to be zeros we get the following equations:

1) $-B_y'(0) + B_y'''(0) - B_y^{(5)}(0) + B_y^{(7)}(0) = 0,$
2) $-B_y''(0) + B_y^{(4)}(0) - B_y^{(6)}(0) = 0,$
3) $-B_y'''(0) + B_y^{(5)}(0) = 0,$
4) $B_y^{(4)}(0) = 0,$
5) $B_y'(1) - B_y'''(1) + B_y^{(5)}(1) - B_y^{(7)}(1) = 0,$
6) $B_y''(1) - B_y^{(4)}(1) + B_y^{(6)}(1) = 0,$
7) $B_y'''(1) - B_y^{(5)}(1) = 0,$
8) $B_y^{(4)}(1) = 0.$

Then, we get
$\langle u, B_y \rangle_{S_2^4[0,1]} = \int_0^1 u(x)\left[B_y(x) - B''{}_y(x) + B^{(4)}{}_y(x) - B^{(6)}{}_y(x) + B_y^{(8)}(x)\right]dx.$
From the reproducing property we have:
$\langle u, B_y \rangle_{S_2^4[0,1]} = u(y).$
Therefore, we get
$\langle u, B_y \rangle_{S_2^4[0,1]} = \int_0^1 u(x)\left[B_y(x) - B''{}_y(x) + B^{(4)}{}_y(x) - B^{(6)}{}_y(x) + B_y^{(8)}(x)\right]dx = u(y).$
Then, by Dirac-Delta function we obtain
$B_y(x) - B''{}_y(x) + B^{(4)}{}_y(x) - B^{(6)}{}_y(x) + B_y^{(8)}(x)$
$= \delta(x - y).$
When $x \neq y$, we reach to the following homogeneous differential equation
$B_y(x) - B''{}_y(x) + B^{(4)}{}_y(x) - B^{(6)}{}_y(x) + B_y^{(8)}(x) = 0,$
which has the characteristic equation:

$$1 - \lambda^2 + \lambda^4 - \lambda^6 + \lambda^8 = 0,$$

whose roots are:

$$\alpha \pm i\beta, -\alpha \pm i\beta, \gamma \pm i\mu, -\gamma \pm i\mu,$$

where $\alpha = \frac{1}{4}\sqrt{1 + 2\sqrt{5}}$, $\beta = \frac{1}{4}\sqrt{\sqrt{5} - 1}$, $\gamma = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$, $\mu = \frac{1}{4}\sqrt{\sqrt{5} + 1}.$
So, we obtain the reproducing kernel function of this space as:

$$B_y(x) = \begin{cases} A_1 e^{\alpha x}\cos(\beta x) + A_2 e^{\alpha x} sin(\beta x) \\ +A_3 e^{-\alpha x} cos(\beta x) + A_4 e^{-\alpha x} sin(\beta x) \\ +A_5 e^{\gamma x} cos(\mu x) + A_6 e^{\gamma x} sin(\mu x) \\ +A_7 e^{-\gamma x} cos(\mu x) + A_8 e^{-\gamma x} sin(\mu x), x \le y \\ B_1 e^{\alpha x} cos(\beta x) + B_2 e^{\alpha x} sin(\beta x) \\ +B_3 e^{-\alpha x} cos(\beta x) + B_4 e^{-\alpha x} sin(\beta x) \\ +B_5 e^{\gamma x} cos(\mu x) + B_6 e^{\gamma x} sin(\mu x) \\ +B_7 e^{-\gamma x} cos(\mu x) + B_8 e^{-\gamma x} sin(\mu x), x > y. \end{cases}$$

We have sixteen unknown coefficients and eight equations. Eight more equations can be obtained by the properties of the Dirac-Delta function. Therefore, the unknown coefficients can be obtained easily. This completes the proof.

Theorem 4.2. We obtain the reproducing kernel function of the reproducing Sobolev space $S_2^m[a,b]$ as:

$$W_y(x)=\begin{cases}u_y(x), & x\le y\\ v_y(x), & x>y\end{cases}.$$

Proof. We have

$$\langle p, W_y\rangle_{S_2^m}=\int_a^b\left(\sum_{k=0}^{m}p^{(k)}(x)W_y^{(k)}(x)\right)dx$$

by the Definition 2.2. Then, we use integration by parts m times and obtain:

$$\langle p, W_y\rangle_{S_2^m}=A+(-1)^m\int_a^b p(x)W_y^{(2m)}(x)dx.$$

Thus, m equations are obtained by letting $A=0$. Therefore, we reach

$$(-1)^m W_y^{(2m)}(x)=\delta(x-y).$$

When $x\neq y$, we have

$$W_y^{(2m)}=0,$$

whose solution is:

$$W_y(x)=\begin{cases}u_y(x), & x\le y\\ v_y(x), & x>y\end{cases}.$$

By Dirac-Delta function, we will get m more equations. Then we can find the unknown coefficients. This will give us the reproducing kernel function of the reproducing kernel function of the reproducing kernel Sobolev space $S_2^m[a,b]$. In this reproducing kernel function, we know that $u_y(x)=v_x(y)$. this completes the proof.

5 Numerical Examples

In this section, we will give some basic experiments related to our new reproducing kernel functions. Many problems can be used by the obtained reproducing kernel functions. We consider the following general Cauchy problem:

$$y'=f(t,y) \qquad (5.1)$$

with the initial condition

$$y(0)=y_0. \qquad (5.2)$$

We need to homogenize the initial condition to obtain the solutions in the reproducing kernel Sobolev spaces. Therefore, we use the following transformation:

$$u(t)=y(t)-y_0. \qquad (5.3)$$

Then we get:

$$\begin{cases}u'=g(t,u),\\ u(0)=0.\end{cases}$$

Consider the above problem in the reproducing kernel Sobolev space $S_2^2[0,1]$. Use the bounded linear operator P as:

$$Pu=g(t,u) \qquad (5.4)$$

with the initial condition

$$u(0)=0 \qquad (5.5)$$

To construct an orthogonal system $\{\psi_i(t)\}_{i=1}^{\infty}$ of $S_2^2[0,1]$, let
$\psi_i(t)=P^*R_{t_i}(t)$, where P^* is conjugate operator of P. The orthonormal system $\{\hat{\psi}_i(t)\}_{i=1}^{\infty}$ of $S_2^2[0,1]$ can be obtained by Gram-Schmidt orthogonalization operator of $\{\psi_i(t)\}_{i=1}^{\infty}$ as follows:

$$\hat{\psi}_i(t)=\sum_{k=1}^{i}\gamma_{ik}\psi_k(t),\quad (\gamma_{ii}>0, i=1,2,\dots), \qquad (5.6)$$

where γ_{ik} are the orthogonalization coefficients. Now, $\{\psi_i(t)\}_{i=1}^{\infty}$ is the complete system in $S_2^2[0,1]$ which means that $\{\hat{\psi}_i(t)\}_{i=1}^{\infty}$ is the complete orthonormal system in $S_2^2[0,1]$, then the exact solution $u(t)$ of (5.4) can be written as:

$$u(t)=\sum_{i=1}^{\infty}\langle u(t),\hat{\psi}_i(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.7)$$

$$=\sum_{i=1}^{\infty}\langle u(t),\sum_{k=1}^{i}\gamma_{ik}\psi_k(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.8)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\gamma_{ik}\langle u(t),\psi_k(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.9)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\gamma_{ik}\langle u(t),P^*R_{t_k}(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.10)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\gamma_{ik}\langle Pu(t),R_{t_k}(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.11)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\gamma_{ik}\langle g(t,u(t)),R_{t_k}(t)\rangle_{S_2^2}\hat{\psi}_i(t) \qquad (5.12)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\gamma_{ik}\, g(t_k,u_k(t_k))\hat{\psi}_i(t) \qquad (5.13)$$

where $\{t_i\}_{i=1}^{\infty}$ is dense in $[0,1]$.

For numerical computation, we define the n-term numerical solution of (5.4) by truncating the series in (5.13) as:

$$u_n(t)=\sum_{i=1}^{n}\sum_{k=1}^{i}\gamma_{ik}\, g(t_k,u_{k-1}(t_k))\hat{\psi}_i(t), \qquad (5.14)$$

with initial function u_0 (t_k)=u(t_k).

Remark 5.1. There is a relation between the order of the problem and the reproducing kernel Sobolev spaces. Since our problem is first order, we use the reproducing kernel Sobolev space S_2^2 [0,1] in this work. We can generalize this relation. Let's choose the order of the problem m. Then, we need to investigate the solutions of the problem in the reproducing kernel Sobolev space $S_2^{m+1}[0,1]$.

6 Conclusions

In this work, we constructed the general reproducing kernel Sobolev spaces. New reproducing kernel functions have obtained in these spaces. The reproducing kernel functions can be used to solve higher order boundary value problems in the reproducing kernel Sobolev spaces.

7 Acknowledgment

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8 Appendix

$a_y(x) =$
$0.3400761824e^{-0.5877852522x-0.5877852522y}$
$sin(0.8090169942x)sin(0.8090169942y) +$
$2.137269895.10^{-11}e^{-0.5877852522x+0.9510565158y}$
$sin(0.8090169942x)sin(0.3090169942y) -$
$0.03243506685e^{-0.5877852522x+0.587785252y}$
$sin(0.8090169942x)sin(.8090169942y) -$
$1.700664523.10^{-10}e^{-0.5877852522x-0.5877852522y}$
$sin(0.8090169942x)cos(0.8090169942y) +$
$1.157017963.10^{-10}e^{-0.5877852522x-0.9510565162y}$
$sin(0.8090169942x)cos(0.3090169942y) +$
$0.09982487139e^{-0.5877852522x+0.5877852522y}$
$sin(0.8090169942x)cos(0.8090169942y) +$
$4.586013687.10^{-10}e^{-0.9510565162x-0.58778525222y}$
$cos(.3090169942x)sin(0.8090169942y +$
$4.8090169942.10^{-11}e^{-0.5877852522x+0.9510565158y}$
$sin(0.8090169942x)cos(0.3090169942y +$
$2.936252000.10^{-11}e^{0.9510565162x+0.9510565158y}$
$cos(0.3090169942x)sin(0.3090169942y +$
$0.2628358071e^{0.9510565162x-0.9510565162y}$
$cos(0.3090169942x)sin(0.3090169942y +$
$8.644511107.10^{-12}e^{0.9510565162x-0.5877852522y}$
$cos(0.3090169942x)cos(0.8090169942y) +$
$4.303744248.10^{-11}e^{0.9510565162x-0.5877852522y}$
$cos(0.3090169942x)sin(0.8090169942y) -$
$4.422450288.10^{-10}e^{-0.9510565162x-0.5877852522y}$
$cos(0.3090169942x)cos(0.8090169942y) -$
$7.086877017.10^{-11}e^{0.9510565162x+0.5877852522y}$
$cos(0.3090169942x)sin(0.8090169942y) +$
$1.451038806.10^{-10}e^{-0.9510565162x+0.5877852522y}$
$cos(0.3090169942x)cos(0.8090169942y) +$
$3.247193721.10^{-11}e^{-0.9510565162x+0.5877852522y}$
$cos(0.3090169942x)sin(0.8090169942y) +$
$0.05399409870e^{-0.9510565162x+0.9510565158y}$
$cos(0.3090169942x)cos(0.3090169942y) +$
$0.4471629833e^{-0.9510565162x-0.9510565162y}$
$cos(0.3090169942x)cos(0.3090169942y) +$
$0.3617624527e^{0.9510565162x-0.9510565162y}$
$sin(0.3090169942x)sin(0.3090169942y) -$
$2.879138068.10^{-11}e^{0.9510565162x-0.5877852522y}$
$sin(0.3090169942x)sin(0.8090169942y) +$
$0.06674037632e^{0.9510565162x+0.9510565158y}$
$sin(0.3090169942x)sin(0.3090169942y) -$
$3.036424261.10^{-12}e^{0.9510565162x-0.5877852522y}$
$sin(0.3090169942x)cos(0.8090169942y) +$
$6.868389976.10^{-12}e^{0.9510565162x+0.5877852522y}$
$sin(0.3090169942x)cos(0.8090169942y) +$
$6.433164073.10^{-12}e^{0.9510565162x+0.5877852522y}$
$sin(0.3090169942x)sin(0.8090169942y -$
$8.492876296.10^{-11}e^{0.9510565162x+0.9510565158y}$
$sin(0.3090169942x)cos(0.3090169942y) -$
$0.2628358071e^{0.9510565162x-0.9510565162y}$
$sin(0.3090169942x)cos(0.3090169942y) -$
$5.886422410.10^{-11}e^{-0.5877852522x+0.9510565158y}$
$cos(0.8090169942x)sin(0.3090169942y) -$
$1.004274322.10^{-10}e^{-0.5877852522x-0.9510565162y}$
$cos(0.8090169942x)sin(0.3090169942y) +$
$2.186675170.10^{-10}e^{-0.5877852522x-0.5877852522y}$
$cos(0.8090169942x)sin(0.8090169942y) -$
$0.09982487135e^{-0.5877852522x+0.5877852522y}$
$cos(0.8090169942x)sin(0.8090169942y) +$
$0.3400761824e^{-0.5877852522x-0.5877852522y}$
$cos(0.8090169942x)cos(0.8090169942y) -$
$1.069137729.10^{-11}e^{-0.5877852522x-0.9510565162y}$
$cos(0.8090169942x)cos(0.3090169942y) -$
$0.03243506681e^{-0.5877852522x+0.5877852522y}$
$cos(0.8090169942x)cos(0.8090169942y) +$
$2.413273214.10^{-11}e^{0.5877852522x-0.9510565162y}$

$sin(0.8090169942x)sin(0.3090169942y) +$
$4.160287741.10^{-11}e^{-0.5877852522x+0.9510565158y}$
$cos(0.8090169942x)cos(0.3090169942y) -$
$0.1050893199e^{0.5877852522x-0.5877852522y}$
$sin(0.8090169942x)sin(0.8090169942y) +$
$4.428080582.10^{-11}e^{0.5877852522x+0.9510565158y}$
$sin(0.8090169942x)sin(0.3090169942y) +$
$0.1049620813e^{0.5877852522x+0.5877852522y}$
$sin(0.8090169942x)sin(0.8090169942y) -$
$0.3234316694e^{0.5877852522x-0.5877852522y}$
$sin(0.8090169942x)cos(0.8090169942y) -$
$6.418911165.10^{-11}e^{0.5877852522x-0.9510565162y}$
$sin(0.8090169942x)cos(0.3090169942y) -$
$3.245007249.10^{-11}e^{0.5877852522x+0.5877852522y}$
$sin(0.8090169942x)cos(0.8090169942y) +$
$4.754325672.10^{-11}e^{0.5877852522x-0.9510565162y}$
$cos(0.8090169942x)sin(0.3090169942y) -$
$6.529288244.10^{-11}e^{0.5877852522x+0.9510565158y}$
$sin(0.8090169942x)cos(0.3090169942y) +$
$0.3234316694e^{0.5877852522x-0.5877852522y}$
$cos(0.8090169942x)sin(0.8090169942y) +$
$4.788740962.10^{-11}e^{0.5877852522x+0.9510565158y}$
$cos(0.8090169942x)sin(0.3090169942y) -$
$1.165675703.10^{-11}e^{0.5877852522x+0.5877852522y}$
$cos(0.8090169942x)sin(0.8090169942y) -$
$0.1050893198e^{0.5877852522x-0.5877852522y}$
$cos(0.8090169942x)cos(0.8090169942y) +$
$2.453494531.10^{-11}e^{0.5877852522x-0.9510565162y}$
$cos(0.8090169942x)cos(0.3090169942y +$
$0.1049620813e^{0.5877852522x+0.5877852522y}$
$cos(0.8090169942x)cos(0.8090169942y) +$
$0.4471629836e^{-0.9510565162x-0.9510565162y}$
$sin(0.3090169942x)sin(0.3090169942y) +$
$2.342757393.10^{-11}e^{0.5877852522x+0.9510565158y}$
$cos(0.8090169942x)cos(0.3090169942y) -$
$4.211891583.10^{-11}e^{-0.9510565162x-0.5877852522y}$
$sin(0.3090169942x)sin(0.8090169942y) +$
$0.05399409865e^{-0.9510565162x+0.9510565158y}$
$sin(0.3090169942x)sin(0.3090169942y) +$
$4.837714210.10^{-11}e^{-0.9510565162x+0.5877852522y}$
$sin(0.3090169942x)sin(0.8090169942y) -$
$2.720256538.10^{-10}e^{-0.9510565162x-0.5877852522y}$
$sin(0.3090169942x)cos(0.8090169942y) +$
$2.797885789.10^{-10}e^{-0.9510565162x-0.9510565162y}$
$sin(0.3090169942x)cos(0.3090169942y) +$
$9.133894186.10^{-12}e^{-0.9510565162x+0.5877852522y}$
$sin(0.3090169942x)cos(0.8090169942y) +$
$9.190114774.10^{-11}e^{-0.9510565162x-0.9510565162y}$
$sin(0.3090169942x)cos(0.3090169942y) +$
$0.03922900894e^{-0.9510565162x+0.9510565158y}$
$sin(0.3090169942x)cos(0.3090169942y) -$

$cos(0.3090169942x)cos(0.3090169942y) -$
$1.644485746.10^{-11}e^{0.9510565162x+0.5877852522y}$
$cos(0.3090169942x)cos(0.8090169942y) +$
$0.06674037633e^{0.9510565162x+0.9510565158y}$
$cos(0.3090169942x)cos(0.3090169942y) +$
$3.602976585.10^{-12}e^{-0.5877852522x-0.9510565162y}$
$sin(0.8090169942x)sin(0.3090169942y.$

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