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Two-Term Trace Estimates for Gradually Successive Relativistic Stable Processes

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Abstract: The current study aims at showing trace estimates, following the way of the method proved by Rodrigo Bañuelos, Jebessa B. Mijena and Erkan Nane [1] for the relativistic $(1+\epsilon)$-stable process extending the result of Bañuelos, and Kulczycki [2] in the stable case.

Keywords: Relativistic stable process, Trace Asymptotics.

1 Introduction

Introduction and statement of main results

For $\epsilon \geq 0$, an $\mathbb{R}^{2+\epsilon}$-valued process with independent, stationary increments having the following characteristic function:

$$\mathbb{E}e^{i\xi \cdot X_{1+\epsilon}^{2+\epsilon,1+\epsilon}} = e^{-(1+\epsilon)\left\{\left((1+\epsilon)^{\frac{2}{2+\epsilon}}+|\xi|^2\right)^{\frac{2+\epsilon}{2}}-(1+\epsilon)\right\}}, \quad \xi \in \mathbb{R}^{2+\epsilon},$$

is called relativistic $(2+\epsilon)$-stable process with mass $(1+\epsilon)$. We assume that sample paths of $X_{1+\epsilon}^{2+\epsilon,1+\epsilon}$ are right continuous and have left-hand limits a.s. If we put $\epsilon = -1$ we obtain the symmetric rotation invariant $(2+\epsilon)$-stable process with the characteristic function $e^{-(1+\epsilon)|\xi|^{2+\epsilon}}, \xi \in \mathbb{R}^{2+\epsilon}$. We refer to this process as isotropic $(2+\epsilon)$-stable Lévy process. For the rest of the paper we keep $\epsilon \geq 0$ fixed and drop $2+\epsilon$, in the notation, when it does not lead to confusion. Hence from now on the relativistic $(2+\epsilon)$-stable process is denoted by $X_{1+\epsilon}$ and its counterpart isotropic $(2+\epsilon)$-stable Lévy process by $\tilde{X}_{1+\epsilon}$. We keep this notational convention consistently throughout the paper, e.g., if $p_{1+\epsilon}(\epsilon)$ is the transition density of $X_{1+\epsilon}$, then $\tilde{p}_{1+\epsilon}(\epsilon)$ is the transition density of $\tilde{X}_{1+\epsilon}$.

In Ryznar [3] Green function estimates of the Schödinger operator with the free Hamiltonian of the form

$$\left(-\Delta + (1+\epsilon)^{\frac{2}{1+\epsilon}}\right)^{\frac{1+\epsilon}{2}} - (1+\epsilon),$$

were investigated, where $\epsilon \geq 0$ and Δ is the Laplace operator acting on $L^2(\mathbb{R}^{2+\epsilon})$. Using the estimates in Lemma 2.6 below and proof in Bañuelos and Kulczycki (2008) we provide an extension of the asymptotics in [2]to the relativistic $(1+\epsilon)$-stable processes for any $0 \leq \epsilon < 1$. Brownian motion has a characteristic function

$$\mathbb{E}^0 e^{i\xi \cdot B_{1+\epsilon}} = e^{-(1+\epsilon)|\xi|^2}, \qquad \xi \in \mathbb{R}^{2+\epsilon}.$$

Let $\epsilon \geq 0$. Ryznar showed that $X_{1+\epsilon}$ can be represented as a time-changed Brownian motion. Let $T_{\frac{1+\epsilon}{2}}(1+\epsilon), \epsilon \geq 0$, denote the strictly $\left(\frac{1+\epsilon}{2}\right)$-stable subordinator with the following Laplace transform

$$\mathbb{E}^0 e^{-\lambda T_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon)} = e^{-(1+\epsilon)\lambda^{\left(\frac{1+\epsilon}{2}\right)}}, \quad \lambda > 0. \quad (1.1)$$

Let $\theta_{\frac{1+\epsilon}{2}}(1+\epsilon, u), u > 0$, denote the density function of $T_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon)$. Then the process $B_{T_{\frac{1+\epsilon}{2}}}(1+\epsilon)$ is the standard symmetric $(1+\epsilon)$-stable process.

Ryznar [[3], Lemma 1] showed that we can obtain $X_{1+\epsilon} = B_{T_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon,1+\epsilon)}$, where a subordinator $T_{\frac{1+\epsilon}{2}}(1+\epsilon, 1+\epsilon)$ is a positive infinitely divisible process with stationary increments with probability density function

$$\theta_{\frac{1+\epsilon}{2}}(1+\epsilon, u, 1+\epsilon) = e^{-(1+\epsilon)^{\frac{2}{1+\epsilon}}u+(1+\epsilon)^2}\theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u), \quad u > 0.$$

Transition density of $T_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, 1+\epsilon)$ is given by $\theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u - v, 1+\epsilon)$. Hence the transition density of $X_{1+\epsilon}$ is $p(1+\epsilon, x, x-\epsilon) = p(1+\epsilon, \epsilon)$ given by

$$p(1+\epsilon, x) = e^{(1+\epsilon)^2}\int_0^\infty \frac{1}{(4\pi u)^{\frac{2+\epsilon}{2}}} e^{-\frac{|x|^2}{4u}} e^{-(1+\epsilon)^{\frac{2}{1+\epsilon}}u}\theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u)du. \quad (1.2)$$

Then

$$p(1+\epsilon, x, x) = p(1+\epsilon, 0) = e^{(1+\epsilon)^2}\int_0^\infty \frac{1}{(4\pi u)^{\frac{2+\epsilon}{2}}} e^{-(1+\epsilon)^{\frac{2}{1+\epsilon}}u}\theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u)du.$$

The function $p(1+\epsilon, x)$ is a radially symmetric decreasing and that

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$$p(1+\epsilon,x) \le p(1+\epsilon,0) \le e^{(1+\epsilon)^2} \int_0^\infty \frac{1}{(4\pi u)^{\frac{2+\epsilon}{2}}} \theta_{\frac{1+\epsilon}{2}}(1+\epsilon,u)du = e^{(1+\epsilon)^2}(1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}} \frac{\omega_{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{1+\epsilon}\right)}{(2\pi)^{2+\epsilon}(1+\epsilon)}, \quad (1.3)$$

where $\omega_{2+\epsilon} = \frac{2\pi^{\frac{2+\epsilon}{2}}}{\Gamma\left(\frac{2+\epsilon}{2}\right)}$ is the surface area of the unit sphere in $\mathbb{R}^{2+\epsilon}$. For an open set D in $\mathbb{R}^{2+\epsilon}$ we define the first exit time from D by $\tau_D = inf\{\epsilon \ge -1: X_{1+\epsilon} \notin D\}$.

We set

$$r_D(1+\epsilon,x,x-\epsilon) = \mathbb{E}^x\left[p(1+\epsilon - \tau_D, X_{\tau_D}, x-\epsilon);\ \tau_D < 1+\epsilon\right], \quad (1.4)$$

and

$$p_D(1+\epsilon,x,x-\epsilon) = p(1+\epsilon,x,x-\epsilon) - r_D(1+\epsilon,x,x-\epsilon), \quad (1.5)$$

for any $x, x-\epsilon \in \mathbb{R}^{2+\epsilon}, \epsilon \ge 0$. For a nonnegative Borel function f and $\epsilon \ge 0$,let

$$P_{1+\epsilon}^D f(x) = \mathbb{E}^x\left[f(X_{1+\epsilon}): 1+\epsilon < \tau_D\right] = \int_D p_D(1+\epsilon,x,x-\epsilon) f(x-\epsilon) d(x-\epsilon),$$

be the semigroup of the killed process acting on $L^2(D)$, see, Ryznar [[3], Theorem 1].

Let D be a bounded domain (or of finite volume). Then the operator $P_{1+\epsilon}^D$ maps $L^2(D)$ into $L^\infty(D)$ for every $\epsilon \ge 0$. This follows from (1.3), (1.4), and the general theory of heat semigroups as described in [4]. It follows that there exists an orthonormal basis of eigenfunctions $\{\varphi_n: n = 1,2,3,\dots\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n: n = 1,2,3,\dots\}$ of the generator of the semigroup $P_{1+\epsilon}^D$ satisfying $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$. By definition, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_{1+\epsilon}^D \varphi_n(x) = e^{-\lambda_n(1+\epsilon)} \varphi_n(x), \quad x \in D, \quad \epsilon \ge 0.$$

Under such assumptions we have

$$p_D(1+\epsilon,x,x-\epsilon) = \sum_{n=1}^\infty e^{-\lambda_n(1+\epsilon)} \varphi_n(x)\varphi_n(x-\epsilon). \quad (1.6)$$

In this paper we are interested in the behavior of the trace of this semigroup

$$Z_D(1+\epsilon) = \int_D p_D(1+\epsilon,x,x)dx. \quad (1.7)$$

Because of (1.6) we can write (1.7) as

$$Z_D(1+\epsilon) = \sum_{n=1}^\infty e^{-\lambda_n(1+\epsilon)} \int_D \varphi_n^2(x)dx = \sum_{n=1}^\infty e^{-\lambda_n(1+\epsilon)}. \quad (1.8)$$

We denote $(2+\epsilon)$-dimensional volume of D by $|D|$.

The first result is Weyl's asymptotic for the eigenvalues of the relativistic Laplacian

$$\lim_{\epsilon\to-1} (1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}} e^{-(1+\epsilon)^2} Z_D(1+\epsilon) = C_1|D|, \quad (1.9)$$

where $C_1 = \frac{\omega_{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{1+\epsilon}\right)}{(2\pi)^{2+\epsilon}(1+\epsilon)}$.

Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which do not exceed λ. It follows from (1.9) and the classical Tauberian theorem (see for example [[5], p. 445, Theorem 2]) where $L(1+\epsilon) = C_1|D|e$ is our slowly varying function at infinity that

$$\lim_{\lambda\to\infty} \lambda^{-\frac{2+\epsilon}{1+\epsilon}} e^{-\frac{1+\epsilon}{\lambda}} N(\lambda) = \frac{C_1|D|}{\Gamma\left(\frac{3+2\epsilon}{1+\epsilon}\right)}. \quad (1.10)$$

This is the analogue for the relativistic stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

Remark 1.2. The first author of [1] presented (1.10) at a conference in Vienna at the Schrödinger Institute in 2009 (see [6]) and at the 34th conference in stochastic processes and their applications in Osaka in 2010 (see [7]). Thus this result has been known to the authors [1], and perhaps to others, for a number of years.

The author in [1] obtains the second term in the asymptotics of $Z_D(1+\epsilon)$ under some additional assumptions on the smoothness of D. The result is inspired by the result for trace estimates for stable processes by Bañuelos and Kulczycki [2]. To state our main result we need the following property of the domain D (see [1]).

Definition 1.3. The boundary, ∂D, of an open set D in $\mathbb{R}^{2+\epsilon}$ is said to be $(1+2\epsilon)$-smooth if for each point $x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii $(1+2\epsilon)$ such that $B_1 \subset D, B_2 \subset \mathbb{R}^{2+\epsilon}\backslash(D \cup \partial D)$ and $\partial B_1 \cap \partial B_2 = x_0$.

Theorem 1.4. Let $D \subset \mathbb{R}^{2+\epsilon}, \epsilon \ge 0$, be an open bounded set with $(1+2\epsilon)$-smooth boundary. Let $|D|$ denote the volume ($(2+\epsilon)$-dimensional Lebesgue measure) of D and $|\partial D|$ denote its surface area ($(1+\epsilon)$-dimensional Lebesgue measure) of its boundary. Suppose $0 \le \epsilon < 1$. Then

$$\left|Z_D(1+\epsilon)-\frac{C_1(1+\epsilon)e^{(1+\epsilon)^2}|D|}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}+C_2(1+\epsilon)|\partial D|\right| \leq \frac{C_3 e^{2(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}, \quad \epsilon \geq 0, \qquad (1.11)$$

Where

$$C_1(1+\epsilon)=\frac{1}{(4\pi)^{\frac{2+\epsilon}{2}}}\int_0^\infty (x-2\epsilon)^{-\frac{2+\epsilon}{2}} e^{-((1+\epsilon)^2)^{\frac{2}{1+\epsilon}}(x-2\epsilon)}\theta_{\frac{1+\epsilon}{2}}(1,x-2\epsilon)d(x-2\epsilon) \to C_1 = \frac{\omega_{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{1+\epsilon}\right)}{(2\pi)^{2+\epsilon}(1+\epsilon)}, \quad as\ \epsilon \to -1,$$

$$C_2(1+\epsilon)=\int_0^\infty r_H\big(1+\epsilon,(x_1,0,\dots,0),(x_1,0,\dots,0)\big)\,dx_1 \leq \frac{C_4 e^{2(1+\epsilon)^2}(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}, \quad \epsilon \geq 0,$$

$$C_4 = \int_0^\infty \tilde{r}_H\,(1,(x_1,0,\dots,0),(x_1,0,\dots,0))dx_1,$$

$C_3 = C_3(2+\epsilon,1+\epsilon), H = \{(x_1,\dots,x_{2+\epsilon}) \in \mathbb{R}^{2+\epsilon}: x_1 > 0\}$ and r_H is given by (1.4).

Remark 1.5. When $0 \leq \epsilon \leq 1, C_2(1+\epsilon) = C_4(1+\epsilon)^{\frac{1}{1+\epsilon}}/(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}$. Then the result in Theorem 1.4 becomes, for bounded domains with $(1+2\epsilon)$-smooth boundary,

$$\left|Z_D(1+\epsilon)-\frac{C_1|D|}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}+\frac{C_4|\partial D|(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\right| \leq \frac{C_7|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}, \qquad (1.12)$$

where C_1, C_4 are as in Theorem 1.4. This was established by Bañuelos and Kulczycki [2] recently.

The asymptotic for the trace of the heat kernel when $\epsilon = 1$ (the case of the Laplacian with Dirichlet boundary condition in a domain of $\mathbb{R}^{2+\epsilon}$), has been extensively studied by many authors. For Brownian motion van den Berg [8], proved that under the $(1+2\epsilon)$-smoothness condition

$$\left|Z_D(1+\epsilon) - (4\pi(1+\epsilon))^{-\frac{2+\epsilon}{2}}\left(|D| - \frac{\sqrt{\pi(1+\epsilon)}}{2}|\partial D|\right)\right| \leq \frac{C_{2+\epsilon}|D|(1+\epsilon)^{-\frac{\epsilon}{2}}}{(1+2\epsilon)^2}, \quad \epsilon \geq 0. \, (1.13)$$

For domains with C^1 boundaries the result

$$Z_D(1+\epsilon) = (4\pi(1+\epsilon))^{-\frac{2+\epsilon}{2}}\left(|D| - \frac{\sqrt{\pi(1+\epsilon)}}{2}|\partial D| + o\left((1+\epsilon)^{\frac{1}{2}}\right)\right), \quad as\ \epsilon \to -1, (1.14)$$

was proved by Brossard and Carmona [9], for Brownian motion.

2 Preliminaries

Let the ball in $\mathbb{R}^{2+\epsilon}$ with center at x and radius $r, \{x-\epsilon: |\epsilon| < r\}$, be denoted by $B(x,r)$. We will use $\delta_D(x)$ to denote the Euclidean distance between x and the boundary, ∂D, of D. That is, $\delta_D(x) = \mathrm{dist}(x, \partial D)$. Define

$$\psi(\theta) = \int_0^\infty e^{-v}\, v^{1+\epsilon}(\theta + v/2)^{1+\epsilon}\, dv, \qquad \theta \geq 0,$$

We put $\mathcal{R}(1+\epsilon, 2+\epsilon) = \mathcal{A}(-(1+\epsilon), 2+\epsilon)/\psi(0)$, where $\mathcal{A}(v, 2+\epsilon) = (\Gamma\,((2+\epsilon-v)/2))/(\pi^{\frac{2+\epsilon}{2}} 2^v\,|\Gamma\,(v/2)|)$. Let $v(x), \tilde{v}(x)$ be the densities of the Lévy measures of the relativistic $(1+\epsilon)$-stable process and the standard $(1+\epsilon)$-stable process, respectively. These densities are given by

$$v(x) = \frac{\mathcal{R}(1+\epsilon,2+\epsilon)}{|x|^{3+2\epsilon}}\, e^{-(1+\epsilon)^{\frac{1}{1+\epsilon}}|x|}\,\psi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}|x|\right), \qquad (2.1)$$

And

$$\tilde{v}(x) = \frac{\mathcal{A}(-(1+\epsilon), 2+\epsilon)}{|x|^{3+2\epsilon}}.$$

We need the following estimate of the transition probabilities of the process $X_{1+\epsilon}$ which is given in [[10], Lemma 2.2]: For any $x, x-\epsilon \in \mathbb{R}^{2+\epsilon}$ and $\epsilon \geq 0$ there exist constants $\epsilon \geq 0$,

$$p(1+\epsilon, x, x-\epsilon) \leq (1+\epsilon)e^{(1+\epsilon)^2}\min\left\{\frac{1+\epsilon}{|\epsilon|^{3+2\epsilon}}\, e^{-(1+2\epsilon)|\epsilon|}, (1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}\right\}. \qquad (2.3)$$

We will also use the fact [[11], Lemma 6] that if $D \subset \mathbb{R}^{2+\epsilon}$ is an open bounded set satisfying a uniform outer cone condition, then $P^x(X(\tau_D) \in \partial D) = 0$ for all $x \in D$. For the open bounded set D we will denoted by $G_D(x, x-\epsilon)$ the Green function for the set D equal to,

$$G_D(x, x-\epsilon) = \int_0^\infty p_D(1+\epsilon, x, x-\epsilon)d(1+\epsilon), \qquad x, x-\epsilon \in \mathbb{R}^{2+\epsilon}.$$

For any such D the expectation of the exit time of the processes $X_{1+\epsilon}$ from D is given by the integral of the Green function over the domain. That is

$$E^x(\tau_D) = \int_D G_D(x, x-\epsilon) d(x-\epsilon).$$

Lemma 2.1. Let $D \subset \mathbb{R}^{2+\epsilon}$ be an open set. For any $x, x-\epsilon \in D$ we have

$$r_D(1+\epsilon, x, x-\epsilon) \leq (1+\epsilon)e^{(1+\epsilon)^2}\left(\frac{1+\epsilon}{\delta_D^{3+2\epsilon}(x)} e^{-(1+2\epsilon)\delta_D(x)} \wedge (1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}\right).$$

Proof. Using (1.4) and (2.3) we have

$$r_D(1+\epsilon, x, x-\epsilon) = E^{x-\epsilon}\left(p(1+\epsilon - \tau_D, X(\tau_D), x) \; ; \tau_D < 1+\epsilon\right)$$

$$\leq (1+\epsilon)e^{(1+\epsilon)^2} E^{x-\epsilon}\left(\frac{1+\epsilon}{|x - X(\tau_D)|^{3+2\epsilon}} e^{-(1+2\epsilon)|x-X(\tau_D)|} \wedge (1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}\right)$$

$$\leq (1+\epsilon)e^{(1+\epsilon)^2}\left(\frac{1+\epsilon}{\delta_D^{3+2\epsilon}(x)} e^{-(1+2\epsilon)\delta_D(x)} \wedge (1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}\right).$$

We need the following result for the proof of Proposition 1.1.

Lemma 2.2.

$$\lim_{\epsilon \to -1} p(1+\epsilon, 0)e^{-(1+\epsilon)^2}(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}} = C_1, \qquad (2.4)$$

Where

$$C_1 = (4\pi)^{\frac{2+\epsilon}{2}} \int_0^\infty u^{-\frac{2+\epsilon}{2}} \theta_{\left(\frac{1+\epsilon}{2}\right)}(1,u) du = \frac{\omega_{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{1+\epsilon}\right)}{(2\pi)^{2+\epsilon}(1+\epsilon)}.$$

Proof. By (1.2) we have

$$p(1+\epsilon, x, x) = p(1+\epsilon, 0) = e^{(1+\epsilon)^2} \int_0^\infty \frac{1}{(4\pi u)^{\frac{2+\epsilon}{2}}} e^{-(1+\epsilon)^{\frac{2}{1+\epsilon}} u} \theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u) du.$$

Now using the scaling of stable subordinator $\theta_{\left(\frac{1+\epsilon}{2}\right)}(1+\epsilon, u) = (1+\epsilon)^{-\frac{2}{1+\epsilon}} \theta_{\left(\frac{1+\epsilon}{2}\right)}(1, u(1+\epsilon)^{-\frac{2}{1+\epsilon}})$ and a change of variables we get

$$p(1+\epsilon, 0) = \frac{e^{(1+\epsilon)^2}}{(4\pi)^{\frac{2+\epsilon}{2}}(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}} \int_0^\infty (x-2\epsilon)^{-\frac{2+\epsilon}{2}} e^{-(1+\epsilon)^{\frac{2}{1+\epsilon}}(1+\epsilon)^{\frac{2(x-2\epsilon)}{1+\epsilon}}} \theta_{\left(\frac{1+\epsilon}{2}\right)}(1, x-2\epsilon) d(x-2\epsilon) = \frac{C_1(1+\epsilon)e^{(1+\epsilon)^2}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}},$$

then by dominated convergence theorem, we obtain

$$\lim_{\epsilon \to -1} p(1+\epsilon, 0)e^{-(1+\epsilon)^2}(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}} = \frac{1}{(4\pi)^{\frac{2+\epsilon}{2}}} \int_0^\infty (x-2\epsilon)^{-\frac{2+\epsilon}{2}} \theta_{\left(\frac{1+\epsilon}{2}\right)}(1, x-2\epsilon) d(x-2\epsilon),$$

and this last integral is equal to the density of $(1+\epsilon)$-stable process at time 1 and $x = 0$ which was calculated in [2] to be

$$\frac{\omega_{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{1+\epsilon}\right)}{(2\pi)^{2+\epsilon}(1+\epsilon)}.$$

We next give the proof of Proposition 1.1.

Proof of Proposition 1.1. By (1.4) we see that

$$\frac{p_D(1+\epsilon, x, x)}{C_1 e^{(1+\epsilon)^2}(1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}} = \frac{p(1+\epsilon, 0)}{C_1 e^{(1+\epsilon)^2}(1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}} - \frac{r_D(1+\epsilon, x, x)}{C_1 e^{(1+\epsilon)^2}(1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}}. \qquad (2.5)$$

Since the first term tend to 1 as $\epsilon \to -1$ by (2.4), in order to prove (1.9), we show that

$$\frac{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}{C_1 e^{(1+\epsilon)^2}} \int_D r_D(1+\epsilon, x, x) dx \to 0, \quad as\ \epsilon \to -1. \qquad (2.6)$$

For $\epsilon \geq -1$, we define $D_{1+\epsilon} = \{x \in D: \delta_D(x) \geq 1+\epsilon\}$. Then for $0 < \epsilon < 1$, consider the subdomain $D^C_{(1-\epsilon)^{1/2(1+\epsilon)}} = \{x \in D: \delta_D(x) \geq (1-\epsilon)^{\frac{1}{2(1+\epsilon)}}\}$ and its complement $D^C_{(1-\epsilon)^{1/2(1+\epsilon)}} = \{x \in D: \delta_D(x) < (1-\epsilon)^{\frac{1}{2(1+\epsilon)}}\}$. Recalling that $|D| < \infty$, by Lebesgue dominated convergence theorem we get $|D^C_{(1-\epsilon)^{1/2(1+\epsilon)}}| \to 0$, as $\epsilon \to 1$. Since $p_D(1-\epsilon, x, x) \leq p(1-\epsilon, x, x)$, by (1.3) we see that

$$\frac{r_D(1-\epsilon,x,x)}{C_1 e^{1-\epsilon^2}(1-\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}} \leq 1,$$

for all $x \in D$. It follows that

$$\frac{(1-\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}{C_1 e^{1-\epsilon^2}} \int_{D^C_{(1-\epsilon)^{\frac{1}{2(1+\epsilon)}}}} r_D(1-\epsilon,x,x)dx \to 0, \quad as\ \epsilon \to 1. \quad (2.7)$$

On the other hand, by Lemma 2.2 in [10] we obtain

$$\frac{r_D(1-\epsilon,x,x)}{C_1 e^{1-\epsilon^2}(1-\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}}} = \frac{\mathbb{E}^x\left[p\left(1-\epsilon - \tau_D, X_{\tau_D}, x\right); 1-\epsilon \geq \tau_D\right]}{C_1 e^{1-\epsilon^2}(1-\epsilon)^{-(2+\epsilon)(1+\epsilon)}}$$

$$\leq c\mathbb{E}^{x-\epsilon} \min\left\{\frac{(1-\epsilon)^{\frac{3+2\epsilon}{1+\epsilon}}}{|x - X(\tau_D)|^{3+2\epsilon}} e^{-(1+2\epsilon)|x-X(\tau_D)|}, 1\right\}$$

$$\leq c \min\left\{\frac{(1-\epsilon)^{\frac{3+2\epsilon}{1+\epsilon}}}{\delta_D(x)^{3+2\epsilon}} e^{-(1+2\epsilon)\delta_D(x)}, 1\right\}. \quad (2.8)$$

For $x \in D_{(1-\epsilon)^{\frac{1}{2(1+\epsilon)}}}$ and $0 < \epsilon < 1$, the right-hand side of (2.8) is bounded above by $c(1-\epsilon)^{\frac{3+2\epsilon}{2(1+\epsilon)}}$ and hence

$$\frac{(1-\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}{C_1 e^{1-\epsilon^2}} \int_{D_{1-\epsilon}^{\frac{1}{2(1+\epsilon)}}} r_D(1-\epsilon,x,x)dx \leq c(1-\epsilon)^{\frac{3+2\epsilon}{2(1+\epsilon)}}|D|, \quad (2.9)$$

and this last quantity goes to 0 as $\epsilon \to 1$.

For an open set $D \subset \mathbb{R}^{2+\epsilon}$ and $x \in \mathbb{R}^{2+\epsilon}$, the distribution $P^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic $(1+\epsilon)$-harmonic measure for D. The following Ikeda–Watanabe formula recovers the relativistic $(1+\epsilon)$-harmonic measure for the set D from the Green function.

Proposition 2.3. (See [10].) Assume that D is an open, nonempty, bounded subset of $\mathbb{R}^{2+\epsilon}$, and A is a Borel set such that $\mathrm{dist}(D,A) > 0$. Then

$$P^x(X(\tau_D) \in A, \tau_D < \infty =) \int_D G_D(x,x-\epsilon) \int_A v(\epsilon)d(x-2\epsilon)d(x-\epsilon), x \in D. (2.10)$$

Here we need the following generalization already stated and used in [2].

Proposition 2.4. (See [12], [[10], Proposition 2.5].) Assume that D is an open, nonempty, bounded subset of $\mathbb{R}^{2+\epsilon}$, and A is a Borel set such that $A \subset D^c \backslash \partial D$ and $0 \leq \epsilon < \infty, x \in D$. Then we have

$$P^x(X(\tau_D) \in A, 1+\epsilon < \tau_D < 1+2\epsilon) = \int_D \int_{1+\epsilon}^{1+2\epsilon} p_D(s,x,x-\epsilon)ds \int_A v(\epsilon)d(x-2\epsilon)d(x-\epsilon).$$

The following proposition holds for a large class of Lévy processes

Proposition 2.5. (See [[2], Proposition 2.3].) Let D and F be open sets in $\mathbb{R}^{2+\epsilon}$ such that $\subset F$. Then for any $x, x-\epsilon \in \mathbb{R}^{2+\epsilon}$ we have

$$p_F(1+\epsilon,x,x-\epsilon) - p_D(1+\epsilon,x,x-\epsilon) = E^x(\tau_D < 1+\epsilon, X(\tau_D) \in F/D;\ p_F(1+\epsilon-\tau_D, X(\tau_D), x-\epsilon)).$$

Lemma 2.6. (See [[3], Lemma 5].) Let $D \subset \mathbb{R}^{2+\epsilon}$ be an open set. For any $x, x-\epsilon \in D$ and $\epsilon \geq 0$ the following estimates hold

$$p_D(1+\epsilon,x,x-\epsilon) \leq e^{(1+\epsilon)^2}\tilde{p}_D(1+\epsilon,x,x-\epsilon),$$

$$r_D(1+\epsilon,x,x-\epsilon) \leq e^{2(1+\epsilon)^2}\tilde{r}_D(1+\epsilon,x,x-\epsilon). \quad (2.11)$$

We need the following lemma given by van den Berg in [8].

Lemma 2.7. (See [[8], Lemma 5].) Let D be an open bounded set in $\mathbb{R}^{2+\epsilon}$ with $(1+2\epsilon)$-smooth boundary ∂D and for $\epsilon \geq 0$ denote the area of boundary of $\partial D_{1+\epsilon}$ by $|\partial D_{1+\epsilon}|$.Then

$$\left(\frac{\epsilon}{1+2\epsilon}\right)^{1+\epsilon}|\partial D| \leq |\partial D_{1+\epsilon}| \left(\frac{1+2\epsilon}{\epsilon}\right)^{1+\epsilon}|\partial D|, \quad \epsilon \geq 0. \quad (2.12)$$

Corollary 2.8. (See [[2], Corollary 2.14].) Let D be an open bounded set in $\mathbb{R}^{2+\epsilon}$ with $(1+2\epsilon)$-smooth boundary. For any $\epsilon \geq 0$ we have

(i) $2^{-(1+\epsilon)}|\partial D| \leq |\partial D_{1+\epsilon}| \leq 2^{1+\epsilon}|\partial D|$,

(ii) $|\partial D| \leq \frac{2^{2+\epsilon}|D|}{1+2\epsilon}$,

(iii) $|\partial D_{1+\epsilon}| - |\partial D| \leq \frac{2^{2+\epsilon}(2+\epsilon)(1+\epsilon)|\partial D|}{1+2\epsilon} \leq \frac{2^{2(2+\epsilon)}(2+\epsilon)(1+\epsilon)|D|}{(1+2\epsilon)^2}$.

3 Proof of the main result

Proof of Theorem 1.4. (See [1])For the case $(1+\epsilon)^{\frac{1}{1+\epsilon}} > \frac{1+2\epsilon}{2}$ the theorem holds trivially. Indeed, by Eq. (1.3)

$$Z_D(1+\epsilon) \le \int_D p(1+\epsilon,x,x)dx \le \frac{(1+\epsilon)e^{(1+\epsilon)^2}|D|}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}$$
$$\le \frac{(1+\epsilon)e^{(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}.$$

By Corollary 2.8 and Lemma 2.6 we also have

$$C_2(1+\epsilon)|\partial D| \le \frac{C_4 e^{2(1+\epsilon)^2}|\partial D|(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}$$
$$\le \frac{2^{2+\epsilon}C_4 e^{2(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+2\epsilon)(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}$$
$$\le \frac{2^{3+\epsilon}C_4 e^{2(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}},$$

$$\frac{C_1(1+\epsilon)e^{(1+\epsilon)^2}|D|}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}} \le \frac{C_1 e^{(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}.$$

Therefore for $(1+\epsilon)^{\frac{1}{1+\epsilon}} > \frac{1+2\epsilon}{2}$ (1.11) holds. Here and in sequel we consider the case $(1+\epsilon)^{\frac{1}{1+\epsilon}} \le \frac{1+2\epsilon}{2}$. From (1.5) and the fact that $p(1+\epsilon,x,x) = \frac{C_1(1+\epsilon)e^{(1+\epsilon)^2}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}$, we have that

$$Z_D(1+\epsilon) - \frac{C_1(1+\epsilon)e^{(1+\epsilon)^2}|D|}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}$$
$$= \int_D p_D(1+\epsilon,x,x)dx - \int_D p(1+\epsilon,x,x)dx$$
$$= -\int_D r_D(1+\epsilon,x,x)dx, \qquad (3.1)$$

where $C_1(1+\epsilon)$ is as stated in the theorem. Therefore we must estimate (3.1). We break our domain into two pieces, $D_{\frac{1+2\epsilon}{2}}$ and its complement $D^C_{\frac{1+2\epsilon}{2}}$. We will first consider the contribution of $D_{\frac{1+2\epsilon}{2}}$.

Claim 1. For $(1+\epsilon)^{\frac{1}{1+\epsilon}} \le \frac{1+2\epsilon}{2}$ we have

$$\int_{D_{\frac{1+2\epsilon}{2}}} r_D(1+\epsilon,x,x)dx \le \frac{(1+\epsilon)e^{2(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}. \qquad (3.2)$$

Proof. By Lemma 2.6 we have

$$\int_{D_{\frac{1+2\epsilon}{2}}} r_D(1+\epsilon,x,x)dx \le e^{2(1+\epsilon)^2}\int_{D_{\frac{1+2\epsilon}{2}}} \tilde{r}_D(1+\epsilon,x,x)dx, \qquad (3.3)$$

and by scaling of the stable density the right-hand side of (3.3) equals

$$\frac{e^{2(1+\epsilon)^2}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\int_{D_{\frac{1+2\epsilon}{2}}} \tilde{r}_{D/(1+\epsilon)^{\frac{1}{1+\epsilon}}}\left(1,\frac{x}{(1+\epsilon)^{\frac{1}{1+\epsilon}}},\frac{x}{(1+\epsilon)^{\frac{1}{1+\epsilon}}}\right)dx.$$

For $x \in D_{\frac{1+2\epsilon}{2}}$ we have $\delta_{D/(1+\epsilon)^{\frac{1}{1+\epsilon}}}\left(x/(1+\epsilon)^{\frac{1}{1+\epsilon}}\right) \ge \frac{1+2\epsilon}{2(1+\epsilon)^{\frac{1}{1+\epsilon}}} \ge 1$. By [[2], Lemma 2.1], we get

$$\tilde{r}_{D/(1+\epsilon)^{\frac{1}{1+\epsilon}}}\left(1,\frac{x}{(1+\epsilon)^{\frac{1}{1+\epsilon}}},\frac{x}{(1+\epsilon)^{\frac{1}{1+\epsilon}}}\right)$$
$$\le \frac{c}{\delta^{3+2\epsilon}_{D/(1+\epsilon)^{\frac{1}{1+\epsilon}}}\left(x/(1+\epsilon)^{\frac{1}{1+\epsilon}}\right)}$$
$$\le \frac{c}{\delta^{2}_{D/(1+\epsilon)^{\frac{1}{1+\epsilon}}}\left(x/(1+\epsilon)^{\frac{1}{1+\epsilon}}\right)}$$
$$\le \frac{c(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2}.$$

Using the above inequality, we get

$$\int_{D_{\frac{1+2\epsilon}{2}}} r_D(1+\epsilon,x,x)dx$$
$$\le \frac{e^{2(1+\epsilon)^2}}{(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\int_{D_{\frac{1+2\epsilon}{2}}} \frac{c(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2}dx$$
$$\le \frac{ce^{2(1+\epsilon)^2}|D|(1+\epsilon)^{\frac{2}{1+\epsilon}}}{(1+2\epsilon)^2(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}},$$

which proves (3.2).

Now we will introduce the following notation. Since D has $(1+2\epsilon)$-smooth boundary, for any point $x-\epsilon \in \partial D$ there are two open balls B_1 and B_2 both of radius $(1+2\epsilon)$ such that $B_1 \subset D, B_2 \subset \mathbb{R}^{2+\epsilon}\backslash(D \cup \partial D), \partial B_1 \cap \partial B_2 = x-\epsilon$. For any $x \in D_{\frac{1+2\epsilon}{2}}$ there exists a unique point $x_* \in \partial D$ such that $\delta_D(x) = |x - x_*|$. Let $B_1 = B(x_1-2\epsilon, 1+2\epsilon), B_2 = B(x_2-2\epsilon,1+2\epsilon)$ be inner/outer balls for the point x_*. Let $H(x)$ be the half-space containing B_1 such that $\partial H(x)$ contains x_* and is perpendicular to the segment $\overline{(x_1-2\epsilon)(x_2-2\epsilon)}$.

We will need the following very important proposition in the proof of Theorem 1.4. Such a proposition has been proved for the stable process in [[2], Proposition 3.1] (see

[1]).

Proposition 3.1. Let $D \subset \mathbb{R}^{2+\epsilon}, \epsilon \geq 0$, be an open bounded set with $(1+2\epsilon)$-smooth boundary ∂D. Then for any $x \in D^C_{\frac{1+2\epsilon}{2}}$ and $\epsilon \geq 0$ such that $(1+\epsilon)^{\frac{1}{1+\epsilon}} \leq \frac{1+2\epsilon}{2}$ we have

$$\left|r_D(1+\epsilon,x,x) - r_{H(x)}(1+\epsilon,x,x)\right| \leq \frac{ce^{2(1+\epsilon)^2}(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+2\epsilon)(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\left(\left(\frac{(1+\epsilon)^{\frac{1}{1+\epsilon}}}{\delta_D(x)}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right). \quad (3.5)$$

Proof. Exactly as in [2], let $x_* \in \partial D$ be a unique point such that $|x - x_*| = \text{dist}(x, \partial D)$ and B_1 and B_2 be balls with radius $(1+2\epsilon)$ such that $B_1 \subset D, B_2 \subset \mathbb{R}^{2+\epsilon} \setminus (D \cup \partial D), \partial B_1 \cap \partial B_2 = x_*$. Let us also assume that $x_* = 0$ and choose an orthonormal coordinate system $(x_1, x_2, \ldots, x_{2+\epsilon})$ so that the positive axis $0x_1$ is in the direction of $\overrightarrow{0p}$ where p is the center of the ball B_1. Note that x lies on the interval $0p$ so $x = (|x|, 0, 0, \ldots, 0)$. Note also that $B_1 \subset D \subset (\overline{B_2})^c$ and $B_1 \subset H(x) \subset (\overline{B_2})^c$. For any open sets A_1, A_2 such that $A_1 \subset A_2$ we have $r_{A_1}(1+\epsilon, x, x-\epsilon) \geq r_{A_2}(1+\epsilon, x, x-\epsilon)$ so

$$\left|r_D(1+\epsilon,x,x) - r_{H(x)}(1+\epsilon,x,x)\right| \leq r_{B_1}(1+\epsilon,x,x) - r_{(\overline{B_2})^c}(1+\epsilon,x,x).$$

So in order to prove the proposition it suffices to show that

$$r_{B_1}(1+\epsilon,x,x) - r_{(\overline{B_2})^c}(1+\epsilon,x,x) \leq \frac{ce^{2(1+\epsilon)^2}(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+2\epsilon)(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\left(\left(\frac{(1+\epsilon)^{\frac{1}{1+\epsilon}}}{\delta_D(x)}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right),$$

for any $x = (|x|, 0, \ldots, 0), |x| \in (0, \frac{1+2\epsilon}{2}]$. Such an estimate was proved for the case $\epsilon = -1$ in [2]. In order to complete the proof it is enough to prove that

$$r_{B_1}(1+\epsilon,x,x) - r_{(\overline{B_2})^c}(1+\epsilon,x,x) \leq ce^{2(1+\epsilon)^2}\{\tilde{r}_{B_1}(1+\epsilon,x,x) - \tilde{r}_{(B_2)^c}(1+\epsilon,x,x)\}.$$

To show this given the ball B_2, we set $U = (\overline{B_2})^c$. Now using the generalized Ikeda–Watanabe formula, Proposition 2.5 and Lemma 2.6 we have

$$\begin{aligned} r_{B_1}(1+\epsilon,x,x) - r_U(1+\epsilon,x,x) &= E^x\left[1+\epsilon > \tau_{B_1}, X(\tau_{B_1}) \in U\setminus B_1; p_U\left(1+\epsilon-\tau_{B_1}, X(\tau_{B_1}), x\right)\right] \\ &= \int_{B_1}\int_0^{1+\epsilon} p_{B_1}(s,x,x-\epsilon)ds \int_{U\setminus B_1} v(\epsilon)p_U(1+\epsilon-s,x-2\epsilon,x)d(x-2\epsilon)d(x-\epsilon) \\ &\leq e^{2(1+\epsilon)^2}\int_{B_1}\int_0^{1+\epsilon} \tilde{p}_{B_1}(s,x,x-\epsilon)ds \int_{U\setminus B_1} \tilde{v}(\epsilon)\tilde{p}_U(1+\epsilon-s,x-2\epsilon,x)d(x-2\epsilon)d(x-\epsilon) \\ &\leq ce^{2(1+\epsilon)^2}E^x\left[1+\epsilon > \tilde{\tau}_{B_1}, \tilde{X}(\tau_{B_1}) \in U\setminus B_1; \tilde{p}_U\left(1+\epsilon-\tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}), x\right)\right] \\ &= ce^{2(1+\epsilon)^2}\tilde{r}_{B_1}(1+\epsilon,x,x) - \tilde{r}_U(1+\epsilon,x,x) \\ &\leq \frac{ce^{2(1+\epsilon)^2}(1+\epsilon)^{\frac{1}{1+\epsilon}}}{(1+2\epsilon)(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}}\left(\left(\frac{(1+\epsilon)^{\frac{1}{1+\epsilon}}}{\delta_D(x)}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right). \end{aligned}$$

The last inequality followed by Proposition 3.1 in [2].

Now using this proposition we estimate the contribution from $D \setminus D_{\frac{1+2\epsilon}{2}}$ to the integral of $r_D(1+\epsilon,x,x)$ in (3.1).

Claim 2. For $(1+\epsilon)^{\frac{1}{1+\epsilon}} \leq \frac{1+2\epsilon}{2}$ we get

$$\left|\int_{D\setminus D_{\frac{1+2\epsilon}{2}}} r_D(1+\epsilon,x,x)dx - \int_{D\setminus D_{\frac{1+2\epsilon}{2}}} r_{H(x)}(1+\epsilon,x,x)dx\right| \leq \frac{ce^{2(1+\epsilon)^2}|D|(1+\epsilon)^{2(1+\epsilon)}}{(1+2\epsilon)^2(1+\epsilon)^{(2+\epsilon)(1+\epsilon)}}. \quad (3.6)$$

Proof. By Proposition 3.1 the left-hand side of (3.6) is bounded above by

$$\frac{ce^{2(1+\epsilon)^2}}{(1+2\epsilon)(1+\epsilon)}\int_0^{\frac{1+2\epsilon}{2}} |\partial D_{1+\epsilon}|\left(\left((1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right) d(1+\epsilon).$$

By Corollary 2.8, (i), the last quantity is smaller than or equal to

$$\frac{ce^{2(1+\epsilon)^2}|\partial D|}{(1+2\epsilon)(1+\epsilon)}\int_0^{\frac{1+2\epsilon}{2}} \left(\left((1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right) d(1+\epsilon).$$

The integral in the last quantity is bounded by $c(1+\epsilon)^{\frac{1}{1+\epsilon}}$. To see this observe that since $(1+\epsilon)^{\frac{1}{1+\epsilon}} \leq \frac{1+2\epsilon}{2}$ the above integral is equal to

$$\int_0^{(1+\epsilon)^{\frac{1}{1+\epsilon}}} \left(\left((1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right) d(1+\epsilon) + \int_{(1+\epsilon)^{\frac{1}{1+\epsilon}}}^{\frac{1+2\epsilon}{2}} \left(\left((1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+2\epsilon}{2}} \wedge 1\right) d(1+\epsilon)$$
$$= \int_0^{(1+\epsilon)^{\frac{1}{1+\epsilon}}} 1\, d(1+\epsilon) + \int_{(1+\epsilon)^{\frac{1}{1+\epsilon}}}^{\frac{1+2\epsilon}{2}} \left((1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+2\epsilon}{2}} d(1+\epsilon) \le c(1+\epsilon)^{\frac{1}{1+\epsilon}}.$$

Using this and Corollary 2.8, (ii), we get (3.6).

Recall that $H = \{(x_1, \dots, x_{2+\epsilon}) \in \mathbb{R}^{2+\epsilon} : x_1 > 0\}$. For abbreviation let us denote

$$f_H\,(1+\epsilon, 1+2\epsilon) = r_H\left(1+\epsilon, (1+2\epsilon, 0, \dots, 0), (1+2\epsilon, 0, \dots, 0)\right), \qquad \epsilon \ge 0.$$

Of course we have $r_H(x)(1+\epsilon, x, x) = f_H\,(1+\epsilon, \delta_H\,(x))$. In the next step we will show that

$$\left| \int_{D \setminus D_{\frac{1+2\epsilon}{2}}} r_{H(x)}(1+\epsilon, x, x) dx - |\partial D| \int_0^{\frac{1+2\epsilon}{2}} f_H\,(1+\epsilon, 1+2\epsilon) d(1+2\epsilon) \right| \le \frac{c e^{2(1+\epsilon)^2} |D|}{(1+2\epsilon)^2 (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}}. \tag{3.7}$$

We have

$$\int_{D \setminus D_{\frac{1+2\epsilon}{2}}} r_H(x)(1+\epsilon, x, x) dx = \int_0^{\frac{1+2\epsilon}{2}} |\partial D_{1+2\epsilon}| f_H\,(1+\epsilon, 1+2\epsilon) d(1+2\epsilon).$$

Hence the left-hand side of (3.7) is bounded above by

$$\int_0^{\frac{1+2\epsilon}{2}} |\partial D_{1+2\epsilon}| - |\partial D| f_H\,(1+\epsilon, 1+2\epsilon) d(1+2\epsilon).$$

By Corollary 2.8, (iii), this is smaller than

$$\frac{c|D|}{(1+2\epsilon)^2} \int_0^{\frac{1+2\epsilon}{2}} (1+2\epsilon) f_H(1+\epsilon, 1+2\epsilon) d(1+2\epsilon) \le \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2} \int_0^{\frac{1+2\epsilon}{2}} (1+2\epsilon)\, \tilde{f}_H\,(1+\epsilon, 1+2\epsilon) d(1+2\epsilon)$$
$$= \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2} \int_0^{\frac{1+2\epsilon}{2}} (1+2\epsilon)(1+\epsilon)^{-\frac{2+\epsilon}{1+\epsilon}} \tilde{f}_H \left(1, (1+2\epsilon)(1+\epsilon)^{-\frac{1}{1+\epsilon}}\right) d(1+2\epsilon)$$
$$= \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2 (1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}} \int_0^{\frac{1+2\epsilon}{2}(1+\epsilon)^{\frac{1}{1+\epsilon}}} (1+2\epsilon)(1+\epsilon)^{\frac{2}{1+\epsilon}} \tilde{f}_H\,(1, 1+2\epsilon) d(1+2\epsilon)$$
$$\le \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2 (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}} \int_0^{\infty} (1+2\epsilon)\left((1+2\epsilon)^{-(3+2\epsilon)} \wedge 1\right) d(1+2\epsilon) \le \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2 (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}}.$$

This shows (3.7). Finally, we have

$$\left| |\partial D| \int_0^{\frac{1+2\epsilon}{2}} f_H(1+\epsilon, 1+2\epsilon) d(1+2\epsilon) - |\partial D| \int_0^{\infty} f_H(1+\epsilon, 1+2\epsilon) d(1+2\epsilon) \right|$$
$$\le |\partial D| \int_{\frac{1+2\epsilon}{2}}^{\infty} f_H(1+\epsilon, 1+2\epsilon) d(1+2\epsilon)$$
$$\le \frac{c|D|}{1+2\epsilon} \int_{\frac{1+2\epsilon}{2}}^{\infty} f_H(1+\epsilon, 1+2\epsilon) d(1+2\epsilon) \text{ by Corollary 2.8, (ii)}$$
$$\le \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)(1+\epsilon)^{\frac{2+\epsilon}{1+\epsilon}}} \int_{\frac{1+2\epsilon}{2}}^{\infty} f_H\left(1, (1+2\epsilon)(1+\epsilon)^{-\frac{1}{1+\epsilon}}\right) d(1+2\epsilon)$$
$$= \frac{c|D| e^{2(1+\epsilon)^2}}{(1+2\epsilon)(1+\epsilon)} \int_{\frac{1+2\epsilon}{2}(1+\epsilon)^{\frac{1}{1+\epsilon}}}^{\infty} \tilde{f}_H\,(1, 1+2\epsilon) d(1+2\epsilon).$$

Since $\frac{(1+2\epsilon)(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2} \ge 1$, so for $1+2\epsilon \ge \frac{(1+2\epsilon)(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2} \ge 1$ we have $\tilde{f}_H\,(1, 1+2\epsilon) \le c(1+2\epsilon)^{-(3+2\epsilon)} \le c(1+2\epsilon)^{-2}$. Therefore,

$$\int_{\frac{(1+2\epsilon)(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2}}^{\infty} \tilde{f}_H\,(1,1+2\epsilon)d(1+2\epsilon)$$

$$\leq c\int_{\frac{(1+2\epsilon)(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2}}^{\infty} \frac{d(1+2\epsilon)}{(1+2\epsilon)^2}$$

$$\leq \frac{c(1+\epsilon)^{\frac{1}{1+\epsilon}}}{1+2\epsilon}.$$

Hence,

$$\Bigg| |\partial D|\int_0^{\frac{1+2\epsilon}{2}} f_H(1+\epsilon,1+2\epsilon)d(1+2\epsilon) - |\partial D|\int_0^{\infty} f_H(1+\epsilon,1+2\epsilon)d(1+2\epsilon)\Bigg| \leq \frac{c|D|e^{2(1+\epsilon)^2}}{(1+2\epsilon)^2(1+\epsilon)}. \qquad (3.8)$$

Note that the constant $C_2(1+\epsilon)$ which appears in the formulation of Theorem 1.4 satisfies $C_2(1+\epsilon) = \int_0^\infty f_H(1+\epsilon,1+2\epsilon)d(1+2\epsilon)$. Now Eqs. (3.1), (3.2), (3.6), (3.7), (3.8) give (1.11).

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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