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Fractional Calculus Analysis of Tourism Mathematical Model

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Abstract: The authors of this paper take a fractional calculus approach to the Casagrandi and Rinaldi mathematical model of tourists in an area or country. The tourism model uses an ordinary differential equation to investigate the number of tourists in the area, the quality of natural resources, and the amounts invested in tourist infrastructure. For each scenario in the model, the ordinary differential equations are fractionalized using the Caputo derivative of a function with respect to a specific exponential function. In each example, we incorporate the concept of fractionalization in conjunction with a specific exponential function in order to change the model. As a result, various hypotheses are elicited by allowing some adjustments to the initial parameters. The results are further displayed by plotting Mittag-Leffler function graphs for various parameters and comparing them to the original solutions. The graphs' analysis investigates the behaviour of the modified model's solution; in this study, all modified model solutions are of the Mittag – Leffler form, whereas all original models solve using an exponential function. The assumptions and changes in parameters cause minor changes in the behaviour of the solutions.

Keywords: Domestic Tourism; Animation effect; Fractional Calculus, Mittag-Leffler function, Caputo-type derivatives.

1 Introduction

Fractional calculus is the mathematical study of the characteristics of integral and differential operators with real or complex orders [1]. When modeling processes or systems that have memory or are nonlocal, the techniques of fractional calculus may be quite helpful [2]. Different forms of fractional integrals and differential operators have been suggested by mathematicians such as Riemann, Liouville, Grunwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Hadamard, Kober, Erdelyi, Caputo, and many more. Among the peculiar characteristics of the fractional derivatives is the fact that they break the product and chain principles. A key feature of derivatives of non-integer orders is that they enable us to explain complicated features of processes and systems by deviating from the conventional form of the product rule [3].

Broad terms, sustainability refers to any effort made to ensure that a certain resource will be there for future generations [4]. However, the term sustainability truly encompasses four separate domains: the human, social, economic, [5] and environmental [6]. Without the development of mathematical models and ancillary tools to predict the main indicators of sustainable tourism development, identify key development factors, and evaluate the efficacy of management decisions, the economic, socioeconomic, and environmental problems of sustainable tourism development cannot be solved [7]. The key uses of economic and mathematical models for company development include predicting visitor numbers and measuring the economic performance of the tourist sector [8]. The establishment of models detailing the region's economic, environmental, and social sustainability and identifying crucial components of sustainable development may assure long-term good benefits of hotel sector expansion [5]. The established mathematical models take into account the effects of tourist expansion [9] on the economy, the environment, and society[7, 8]. Using factors like the abundance of natural attractions and the level of investment in tourist infrastructure, the Casagrandi and Rinaldi Model [10, 11] can estimate the total number of visitors to a given nation. It has the following form, as suggested in 2002:

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$$\begin{cases} \dfrac{dT}{dt} = T\left(\dfrac{\mu_E E}{E+\varphi_E} + \mu_c \dfrac{\frac{C}{1+T}}{\frac{C}{1+T}+\varphi_c} - \alpha T - a\right) \\ \dfrac{dE}{dt} = zE\left(1-\dfrac{E}{K}\right) - E(\beta C + \gamma T) \\ \dfrac{dC}{dt} = -\delta C + \varepsilon T \end{cases} \quad (1\cdot 1)$$

Above, The No of tourists, T(t), the quality of the area's natural resources, E(t), the amount of money put into the area's tourist infrastructure, C(t), the attractiveness of the environment at E →∞, the attractiveness of the infrastructure at C, the half-saturation constants, → ∞, φ_E and φ_c are all shown in the equation. an is the coefficient of the rate of decline in scenic allure as visitor numbers rise, z is the rate of improvement in environmental quality (assuming small E), and K is the carrying capacity of the ecosystem. The environmental effect of tourism in the area (country) and the availability of capital investments in infrastructure are measured by the metrics γ & β, respectively. Here, ε is the rate of return on investment, and δ is the rate at which physical assets are being worn down.

2 A function's fractional derivative with relation to the another function

Fractional calculus is branch of mathematics that investigates several powers for a particular operator, which can be real or complex integers[12]. Fractional calculus applications have gotten a lot of attention in the last five years in the domains of economics, physics, engineering, and biology [13, 14].

The Riemann-Liouville fractional derivative definition may be recreated by replacing the order of the ordinary derivative with the fractional integral operator [15].The Laplace transformation is now reliant on the beginning circumstances of this new integer order derivative, as opposed to the initial conditions of the fractional order derivative when using the Riemann-Liouville fractional derivative.

The following definitions are used throughout this article to explain the fractional derivative and the integral of a function with respect to another function[16].

Definition1 Let $\theta > 0$. $n \in N. I$ is an interval $-\infty \leq a \leq b \leq \infty$. f is integrable function defined on I and $g \in C^1(I)$such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. Then, with regard to another function g, the fractional integral of function f is provided by

$$I_{a+}^{\theta.g} f(t) := \frac{1}{\Gamma(\theta)} \int_a^t g'(\tau)[g(t)-g(\tau)]^{\theta-1} f(\tau) d\tau. \quad (2.1)$$

Note that if g(t)=t, the above assumption may be reduced to the RiemannLiouville fractional integral.

Using the first definition as a guide, the Caputo derivative of f with respect to g may be thought of in the following way:
Definition2 Let $\theta > 0$. $n = [\theta] \in N. I = (a.b)$ is an interval $-\infty \leq a \leq b \leq \infty. f \in C^n(I). g \in C^1(I)$two functions such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. The right fractional derivatives of f with respect to $g$are respectively given by

$$cD_{a+}^{\theta.g} f(t) := I_{a+}^{n-\theta.g} \left(\frac{1}{g'(t)} \frac{d}{dt}\right)^n f(t). \quad (2.2)$$

Remark 1 If $f(t) = [g(t) - g(a+)]^{\beta-1}$. $\beta > 1$ then:

$$cD_{a+}^{\theta.g} f(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\theta)} [g(t) - g(a+)]^{\beta-\theta-1} \quad (2.3)$$

In this context, we recall the Mittag-Leffler functions, which have been extensively studied by several authors owing to their central role in the study of fractional differential equations and their applications (for examples, [17, 18]).

It was recently proposed by Almeida in [19],that Mittag-Leffler functions may be composed with other functions., and can potentially be advantageous. Integro-differential equations may provide solid theoretical foundations for these investigations. To that purpose, study propose:

Remark 2 If $f(t) = E_\theta\left\{\lambda[g(t) - g(a+)]^{\beta-\theta-1}\right\}.\lambda \in R$

$$cD_{a+}^{\theta;\,g} f(t) = \lambda f(t) \tag{2.4}$$

Proposition 1 The following application is considered in the present technique. Any form model will suffice.

$$e^{xt}\frac{dM(t)}{dt} = yM(t) \tag{2.5}$$

First, we perform a fractionalization of the first order differential equation by including fractional derivative of $M(t)$with regard to the strictly increasing function $g(t) = e^{xt}$. The new model will then be:

$$cD_{a+}^{\theta;\,g} M_\theta(t) = yM_\theta(t). \quad \theta \in (0.1]. \text{ with initial condition } M_\theta(0) = M_0 \tag{2.6}$$

$$cD_{a+}^{\theta;\,g}\ M_\theta(t) = \frac{1}{\Gamma(1-\theta)}\int_0^1 \left(\frac{e^{-x\tau} - e^{-xt}}{x}\right)^{-\theta} \frac{dM_\theta}{d\tau}\, d\tau. \tag{2.7}$$

This is a specific case of the fractional derivative of a function with regard to another function that is of the Caputo type ,as we assumed that $g(t) = e^{xt}$ with the restriction of $\theta \in (0.1]$. It is evident that Equation (2.6) is an Eigenvalue problem for the Caputo-type fractional operator defined in Equation (2.7), whose solution is given by

$$M_\theta(t) = M_0 E_\theta\left[\frac{y}{x^\theta}(1 - e^{-xt})^\theta\right] \tag{2.8}$$

3 Fractionalizing Casagrandi and Rinaldi Model (General)

We consider the case The vector (T, E, C) = (0,0,0) which is a non-movable point that linearizes (1.1) in its vicinity:

$$\begin{cases} \dfrac{dT}{dt} = -aT \\ \dfrac{dE}{dt} = zE \\ \dfrac{dC}{dt} = -\delta C + \varepsilon T \end{cases} \tag{3.1}$$

The solutions to the first two equations (3.1) may be found with relative ease:

$$T(t) = e^{-at}T_0. \tag{3.2}$$

$$E(T) = e^{zt}E_0. \tag{3.3}$$

When the answer to equations (3.2) and (3.3) are swapped into the third equation, we get the following result:

$$\frac{dC}{dt} = -\delta C + \varepsilon T_0 e^{-at}. \tag{3.4}$$

The solution of (3.4) has the form:

$$C = \frac{\varepsilon T_0}{\frac{d}{dt} + \delta} e^{-at} = \frac{\varepsilon T_0}{-a + \delta} e^{-at} \tag{3.5}$$

Despite the exponential increase in environmental quality, the number of visitors and capital investments in infrastructure trend exponentially to zero if a > 0. (3.3). Obviously, the resort won't be operational, so let's assume that. The number of visitors, environmental quality (z > 0), and investment all increase at a constant exponential rate in solutions (3.2), (3.3), and (3.5), respectively. The start of the holiday season may be seen in these decisions.

Proposition 2: "The Caputo fractional derivate of $T(t)$ with respect to e^{xt}".
We let $a = -e^{-xt}$. $x > 1$ in assertion (3.1), that is

$$\frac{dT}{dt} = e^{-xt} T \rightarrow e^{xt} \frac{dT}{dt} = T \tag{3.6}$$

Then we apply *Proposition 1*, we fractionalize the latter by considering the Caputo fractional derivative of $T(t)$ with respect to $g(t) = e^{xt}$. which is strictly increasing with $g'(t) \neq 0$. the model equation will be

$$cD_{0+}^{\theta \cdot g} T_\theta(t) = -T_\theta(t). \qquad \theta \in (0.1) \tag{3.7}$$

With initial condition $T_\theta(0) = T_0 > 0$

$$cD_{0+}^{\theta \cdot g} T_\theta(t) = \frac{1}{\Gamma(1-\theta)} \int_0^t \left(\frac{e^{-x\tau} - e^{-xt}}{x} \right)^{-\theta} \frac{dT_\theta}{d\tau} d\tau. \tag{3.8}$$

The solution is given by

$$T_\theta(t) = T_0 \mathbf{E_\theta} \left[\frac{1}{x^\theta} (1 - e^{-xt})^\theta \right] \tag{3.9}$$

Proposition 3: "The Caputo fractional derivate of $E(t)$ with respect to e^{xt}"
We apply the same modifications as in Proposition 2 , we also consider the case when the rate of growth z is a function of t that is $z = z(t) = e^{-xt}$, then equation 2 in (3.2) can be viewed as

$$\frac{dE}{dt} = e^{-xt} E \rightarrow e^{xt} \frac{dE}{dt} = E. \ \theta \in (0.1). \tag{3.10}$$

This leads to yield a new model of equation

$$cD_{0+}^{\theta \cdot g} E_\theta(t) = E_\theta(t). \qquad \theta \in (0.1) \tag{3.11}$$

Following the similar approach, we receive

$$E_\theta(t) = E_0 \mathbf{E_\theta} \left[\frac{1}{x^\theta} (1 - e^{-xt})^\theta \right] \tag{3.12}$$

Proposition 4: "The Caputo fractional derivate of $C(t)$ with respect to e^{xt}"
We consider the case when the pace of depreciation of the infrastructure is a function of t that is $\delta = \delta(t) = e^{-xt}$, then equation 3 in (3.1) can be viewed as

$$\frac{dC}{dt} = -e^{-xt} C + \varepsilon T_0 e^{-at} \rightarrow e^{xt} \frac{dC}{dt} = C + \varepsilon T_0 e^{(-a+xt)} . T_0 \approx 0 \rightarrow e^{xt} \frac{dC}{dt} = C \tag{3.13}$$

This leads to yield a new model of equation

$$cD_{0+}^{\theta,g}C_{\theta}(t) = C_{\theta}(t). \qquad \theta \in (0.1) \tag{3.14}$$

With solution;

$$C_{\theta}(t) = C_0 \mathbf{E}_{\boldsymbol{\theta}}\left[\frac{1}{x^{\theta}}(1 - e^{-xt})^{\theta}\right] \tag{3.15}$$

4 Fractionalizing Casagrandi and Rinaldi Model (Peak of the holiday season)

We consider the peak season where T >> 1. We can than view (1.1) as:

$$\begin{cases} \dfrac{dT}{dt} \approx T\left(\dfrac{\mu_E E}{E + \varphi_E} - \alpha T - a\right) \\ \dfrac{dE}{dt} \approx z(K - E) - \gamma ET \\ \dfrac{dC}{dt} \approx -\delta C + \varepsilon T \end{cases} \tag{4.1}$$

Now, let's take a look at the situation., when $E \gg \varphi_E$, then the form of (4.1) is as follows:

$$\begin{cases} \dfrac{dT}{dt} \approx T(\mu_E - \alpha T - a) \\ \dfrac{dE}{dt} \approx z(K - E) - \gamma ET \\ \dfrac{dC}{dt} \approx -\delta C + \varepsilon T \end{cases} \tag{4.2}$$

The form of the stationary solution to the first equation of the system (4.2) is as follows:

$$T_0 = \frac{\mu_E - a}{\alpha} \tag{4.3}$$

As a result of extending the first equation (4.2) around the point (4.3), we get:

$$\frac{dT}{dt} \approx (\mu_E - a - 2\alpha T_0)(T - T_0). \tag{4.4}$$

The solution of (4.4) has the form:

$$T(t) = T_0 + \exp\{(\mu_E - a - 2\alpha T_0)t\}(T(o) - T_0). \tag{4.5}$$

When $T_0 > \frac{\mu_E - a}{2\alpha}$, Which is usual throughout the holiday season's peak, $T(t) \to T_0$. Consequently, the two lower equations (4.2) acquire the following form:

$$\begin{cases} \dfrac{dE}{dt} \approx -(z + \gamma T_0)\left(E - \dfrac{zK}{z + \gamma T_0}\right) \\ \dfrac{dC}{dt} \approx -\delta\left(C - \dfrac{\varepsilon}{\delta}T_0\right) \end{cases} \tag{4.6}$$

The following are its potential solutions:

$$E(t) = \frac{zK}{z+\gamma T_0} + \left(E(o) - \frac{zK}{z+\gamma T_0}\right)e^{-(z+\gamma T_0)t} \tag{4.7}$$

$$C(t) = \frac{\varepsilon}{\delta}T_0 + \left(C(o) - \frac{\varepsilon}{\delta}T_0\right)e^{-\delta t}. \tag{4.8}$$

These solutions show that, when $\gg 1 E(t) \approx \frac{zK}{z+\gamma T_0}$. $C(t) \approx \frac{\varepsilon}{\sigma}T_0$.

Proposition 5

We evoke the impact of a significantly low capacity of the ecosystem, that is $z \approx 0$. We also assume that $-(z+\gamma T_0) = e^{xt}$. then equation 1 in (4.6) reduces to

$$\frac{dE}{dt} = -e^{-xt}E \rightarrow e^{xt}\frac{dE}{dt} = -E \quad (4.9)$$

Which after fractionalization solves for

$$E_\theta(t) = E_0\mathbf{E}_\theta\left[\frac{1}{x^\theta}(1-e^{-xt})^\theta\right] \quad (4.10)$$

Proposition 6

We consider a very low initial number of tourists, that is $T_0 \approx 0$, we also view the pace of depreciation of the infrastructure as an exponential function $\delta = e^{xt}$ then equation 2 in (4.6) is

$$\frac{dC}{dt} \approx -e^{-xt}C \rightarrow e^{xt}\frac{dC}{dt} = -C$$

Which after fractionalization solves for

$$C_\theta(t) = C_0\mathbf{E}_\theta\left[\frac{1}{x^\theta}(1-e^{-xt})^\theta\right] \quad (4.11)$$

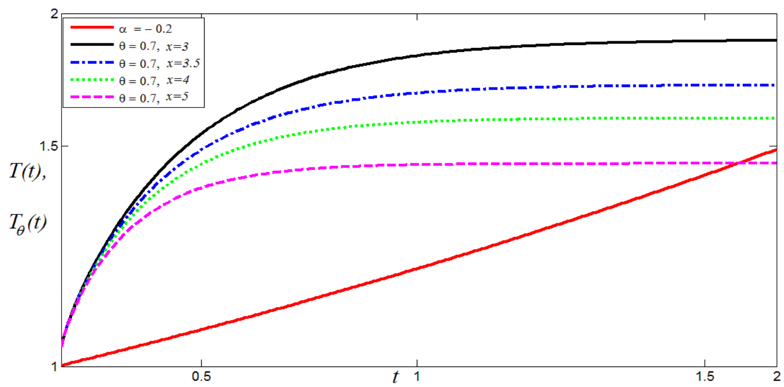

Figure 1: T(t) vs Tɵ (t) T(t) in equation (3.3) for the value a = - 0.8 and Tɵ (t) in equation (3.10) for ϴ = 0.3 and four representative values of x =1.2 (Black), 1.5 (Blue), 2 (Green) and 2.5 (Magenta)

The parameter α is a reference value for attractiveness, and the parameter ∈ describes the investment program being pursued. High value has both of these parameters. Both of these criteria are taken into account while looking at the bifurcation.

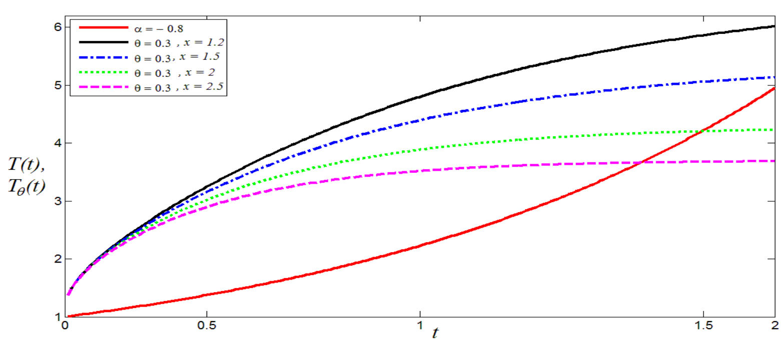

Figure 2: T(t) vs Tɵ (t) T(t) in equation (3.3) for the value a = - 0.2 and Tɵ (t) in equation (3.10) for ϴ = 0.7 and four representative values of x =3 (Black), 3.5 (Blue), 4 (Green) and 5 (Magenta)

5 Graphical Analysis (ODE Vs Fractional Models)

Figure 1: T(t) vs T_{θ} (t) T(t) in equation (3.3) for the value a = - 0.8 and T_{θ} (t) in equation (3.10) for $\Theta = 0.3$ and four representative values of x =1.2 (Black), 1.5 (Blue), 2 (Green) and 2.5 (Magenta)

Figure 2: T(t) vs Tθ (t) T(t) in equation (3.3) for the value a = - 0.2 and Tθ (t) in equation (3.10) for Θ = 0.7 and four representative values of x =3 (Black), 3.5 (Blue), 4 (Green) and 5 (Magenta)

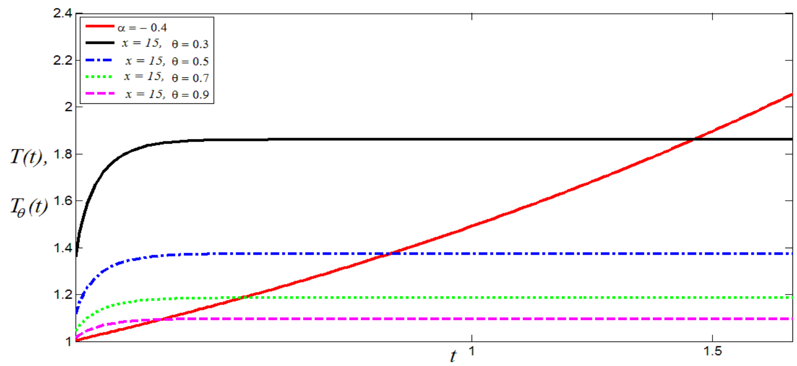

Figure 3: T(t) vs Tθ (t) T(t) in equation (3.3) for the value a = - 0.4 and Tθ (t) in equation (3.10) for x = 15 and four representative valuesof Θ =03 (Black), 0.5 (Blue), 0.7 (Green) and 0.9 (Magenta).

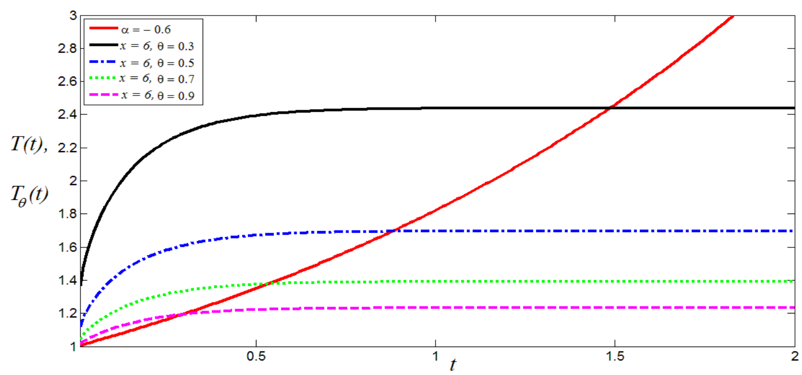

Figure 4: T(t) vs Tθ (t) T(t) in equation (3.3) for the value a = - 0.6 and Tθ (t) in equation (3.10) for x = 0.6 and four representative values of Θ =0.3 (Black), 0.5 (Blue), 0.7 (Green) and 0.9 (Magenta).

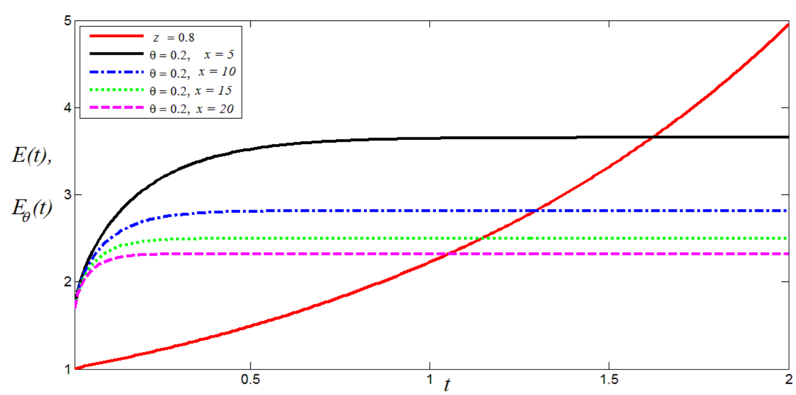

Figure 5: E(t) vs Eθ (t), E(t) in equation (3.4) for the value z = 0.8 and Eθ (t) in equation (3.13) for Θ = 0.2 and four representative values of x =5 (Black), 10 (Blue), 15(Green) and 20 (Magenta).

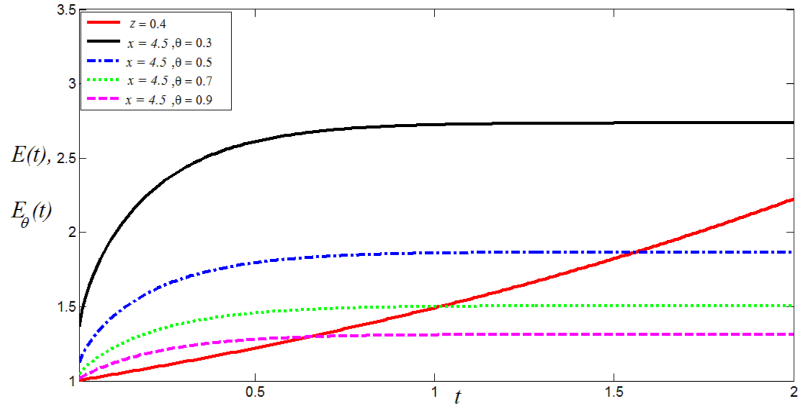

Figure 6: E(t) vs Eɵ (t), E(t) in equation (3.4) for the value z = 0.4 and Eɵ (t) in equation (3.13) for x = 4.5 and four representative values of Ɵ =0.3 (Black),0.5 (Blue), 0.7 (Green) and 0.9 (Magenta).

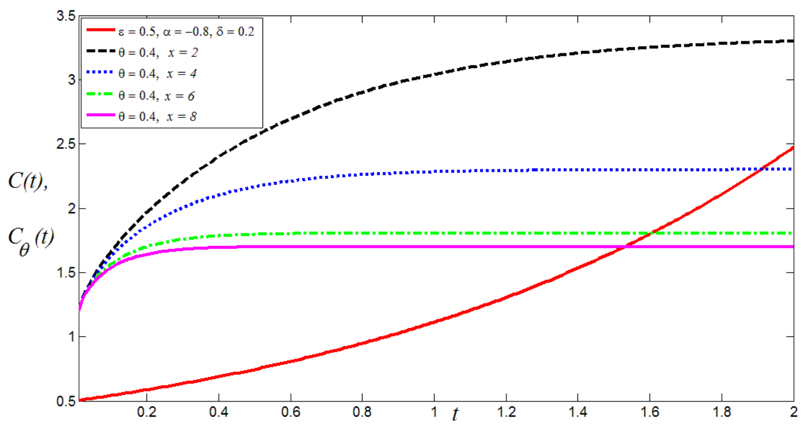

Figure 7: C(t) vs Cɵ (t), C(t) in equation (3.6) for the values ε= 0.5, δ = 0.2, a = - 0.8 and Cɵ (t) in equation (3.16) for Ɵ = 0.4 and four representative values of x =2 (Black), 4 (Blue), 6 (Green) and 8 (Magenta).

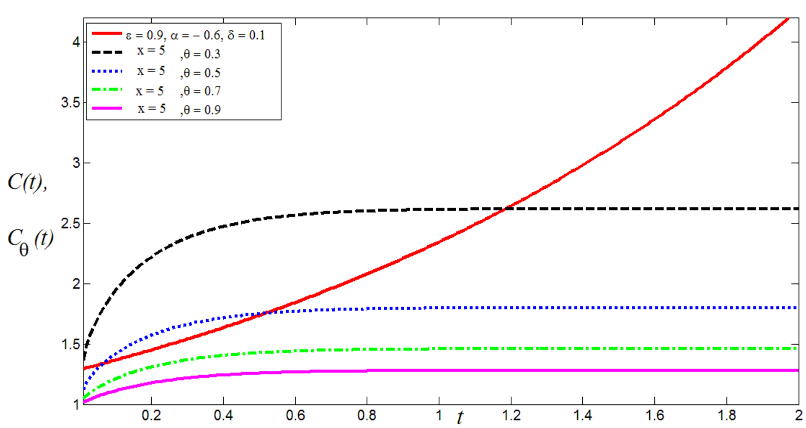

Figure 8:C(t) vs Cɵ (t), C(t) in equation (3.6) for the values ε= 0.9, δ = 0.1, a = - 0.6 and Cɵ (t) in equation (3.16) for x = 5 and four representative values of Ɵ =0.3 (Black), 0.5 (Blue), 0.7 (Green) and 0.9 (Magenta).

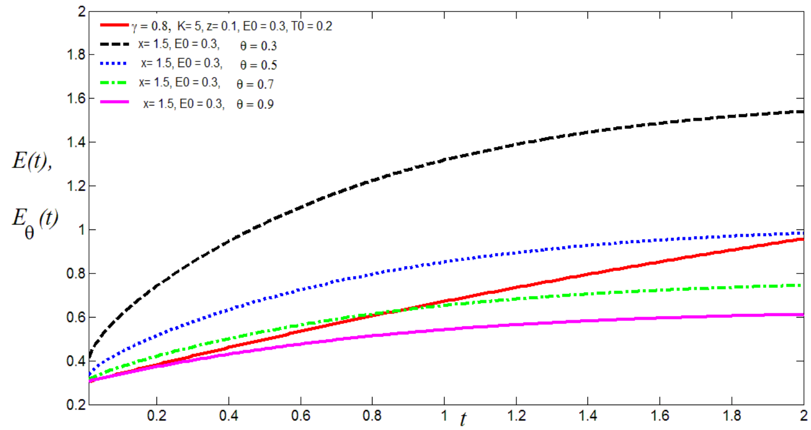

Figure 9: E(t) vs Eɵ (t), E(t) in equation (4.7) for the values $z = 0.1$, $K = 5$, $\gamma = 0.8$, $T_0 = 0.2$ and Eɵ (t) in equation (4.10) for $E_0 = 0.3$, $x = 1.5$, and four representative values of ϴ =0.3 (Black), 0.5 (Blue), 0.7 (Green) and 0.9 (Magenta).

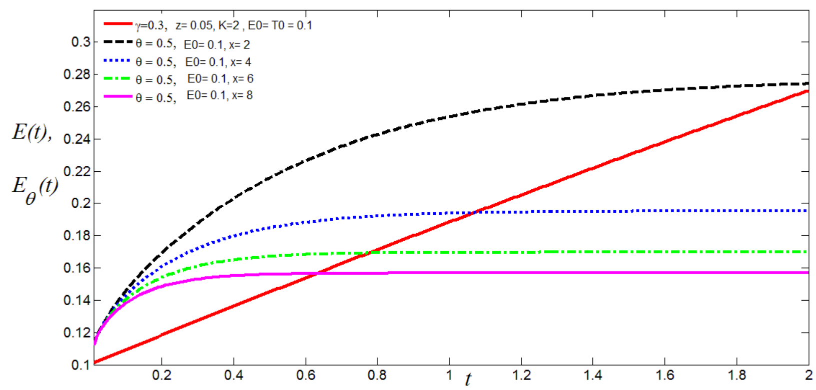

Figure 10: E(t) vs Eɵ (t), E(t) in equation (4.7) for the values of $z = 0.05$, $K = 2$, $\gamma = 0.3$, $T_0 = E_0 = 0.1$ and Eɵ (t) in equation (4.10) for ϴ = 0.5 and four representative values of x =2 (Black), 4 (Blue), 6 (Green) and 8 (Magenta).

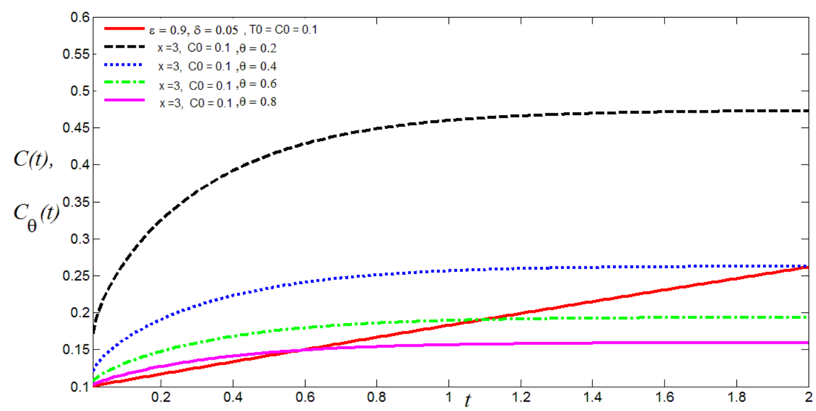

Figure 11:C(t) vs Cɵ (t,) C(t) in equation (4.8) for the values $\varepsilon= 0.9$, $\delta = 0.05$, $C_0 = T_0 = 0.1$ and Cɵ (t) in equation (4.11) for x= 3 and four representative values of ϴ =0.2 (Black), 0.4 (Blue), 0.6 (Green) and 0.8 (Magenta)

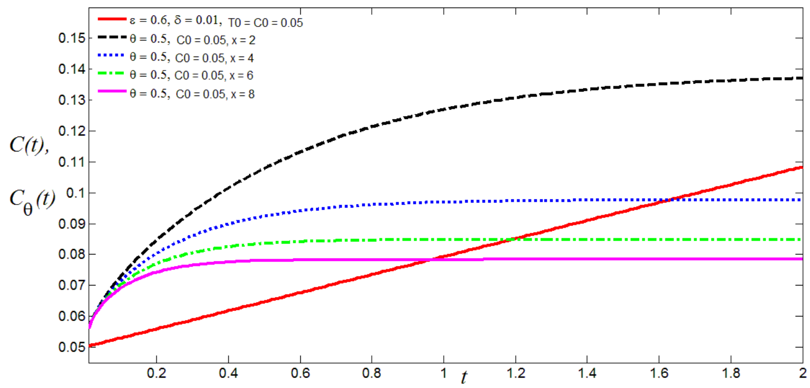

Figure 2: C(t) vs Cϴ (t), C(t) in equation (4.8) for the values ε= 0.6, δ = 0.01, $C_0 = T_0 = 0.05$ and Cϴ (t) in equation (4.11) for ϴ = 0.5 and four representative values of x =2 (Black), 4 (Blue), 6 (Green) and 8 (Magenta).

6 Conclusions

Generally, growth (decay) models are obtained from observations under specific circumstances and within a limited time interval, without denigrating the efficiency of such models, the one may question their accuracy for long run especially that the constant (decay) rate may also be subject to changes. For that, considering growth rates in this study as functions of time notably exponential functions resulted fractional models that solve as Mittag – Leffler functions. Therefore; the figures have shown that all fractional models are also increasing exactly like the original exponential one. However, the growth of fractional models is almost logarithmic which can be a clue that for long term, fractional calculus models seem to be more accurate and realistic than the ODE models.

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