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A New Generalization of Power Garima Distribution with Applications in Blood Cancer and Relief Times

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Abstract: Abstract: The present study deals with the weighted version of power Garima distribution and its various statistical properties have been obtained. For estimating its parameters, the technique of maximum likelihood estimation have been used and also observed its Fisher's information matrix. Finally, the two real lifetime data sets from medical sciences have been used to discuss the superiority of new distribution.

Keywords: Weighted distribution, Power Garima distribution, Reliability measures, Order statistics, Maximum likelihood estimation.

1 Introduction

The application of weighted distributions in the areas of research related to reliability, bio-medicine, ecology and other areas are of great practical importance in mathematics, probability and statistics. Adding extra parameter to an existing distribution brings the classical distribution in a more flexible situation and the distribution becomes useful for data analysis purpose. As a result weighted distributions arise naturally generated from a stochastic process and are recorded with some weight function. The concept of weighted distributions has been employed in wide variety applicatons in many fields of real life such as medicine, reliability, survival analysis, analysis of family data, ecology and forestry. Weighted distributions were firstly introduced by Fisher [11] to model the ascertainment bias then later Rao [32] formulize in a unifying theory. Weighted distributions occur in a natural way in specifying probabilities of events as observed and recorded by making adjustments to probabilities of actual occurrence of events taking into account the method of ascertainment. Failure to make such adjustments can lead to wrong conclusions. The concept of weighted probability models attracted a lot of researchers to contemplate on and to carry out research on this topic. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox [5] and Zelen [45].

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Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Warren [43] was the first to apply the weighted distributions in connection with sampling wood cells. Patil and Rao [31] studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Gupta and Keating [17] introduced the relations for reliability measures under length biased sampling. Van Deusen [42] arrived at size biased distribution theory independently and applied it in fitting assumed distributions to data arising from horizontal point sampling. Khatree [24] obtained the characteristics of many length biased distributions, preservation stability results and comparisons for weighted and length biased distributions. Gupta and Kirmani [18] discussed the weighted distributions role in stochastic modeling. Castillo and Casany [4] discussed on the weighted Poisson distribution for under dispersion and over dispersion solutions. Van Gove [15] reviewed some of the more recent results on weighted distributions pertaining to parameter estimation in forestry. Fay et al. [10] presented the new metric weighted spectral distribution. Gupta and Kundu [19] discussed on the weighted exponential distribution along with its different structural properties. Kersey [23] introduced the weighted inverse weibull distribution and beta inverse weibull distribution. Roman [35] presented the theoretical properties and estimation in weighted Weibull and its related distributions. Ghitany, Alqallaf, Al-Mutairi and Husain [14] introduced a two-parameter weighted Lindley distribution with applications to analyse survival data. Ye et al. [44] obtained the weighted generalized beta distribution of second kind along with its related distributions. Shia et al. [41] proposed the theoretical properties of weighted generalized Rayleigh and its related distributions. Mahdy [28] have introduced the new model of Weibull distribution and its properties. Odubote and Oluyede [29] discussed on a new class of six parameters of distribution called as weighted Feller-Pareto distribution (WFPD). Bashir and Rasul [3] studied the weighted Lindley distribution and discuss its various statistical properties. Alqallaf et al. [2] obtained the two parameter weighted exponential distribution. Das and Kundu [8] discussed on various statistical properties of the weighted exponential distribution and its length biased version. Kilany [26] constructed the weighted Lomax distribution and obtain its several structural properties. Fatima and Ahmad [9] presented the weighted inverse Rayleigh distribution along with its properties and applications. Khan et al. [25] discussed the weighted modified weibull distribution. Para and Jan [30] introduced the three parameter weighted Pareto type II distribution with properties and applications in medical sciences. Dar et al. [7] introduced the weighted transmuted power distribution and discuss its properties and applications. Hassan et al. [20, 21, 22] introduced three new weighted probability models with application to handle life time data in engineering and medical sciences. Rather and Subramanian [33] discussed a new class of sushila distribution known as weighted Sushila distribution with properties and Applications. Ganaie, Rajagopalan and Rather [12] introduced the weighted new quasi Lindley distribution with properties and Applications. Dar, Ahmad and Reshi [6] introduced the new model of gamma-Pareto distribution known as weighted gamma-Pareto distribution and its applications. Rather and Ozel [34] introduced the weighted power Lindley distribution with applications on the life time data. Recently, Ganaie, Rajagopalan and Aldulaimi [13] discussed on the weighted power Shanker distribution with characterizations and applications of the real lifetime data, which shows better fit than the existing classical distribution.

Power Garima distribution is a newly introduced two parametric lifetime model proposed by Abebe et al. [1] The proposed distribution is a two parametric lifetime distribution and the Garima distribution is a particular case of the two parameter power Garima distribution. The two parameter power Garima distribution is discussed with several statistical properties such as moments, hazard rate function and mean residual life function. The parameters of the two-parameter power Garima distribution are estimated by using the method of maximum likelihood estimation. Shanker [40] also discussed on

Garima distribution and obtained its several mathematical and statistical properties. Shanker has also determined its parameters by employing the maximum likelihood estimation technique and method of moments. Shanker [37] also introduced the discrete Poisson-Garima distribution and discusses its various statistical properties and estimate its parameters and also discuss its applications. Shanker and Shukla [38] also have studied the size-biased poisson Garima distribution with applications and also discussed its various mathematical and statistical properties. Shanker and Shukla [39] also discussed the Zero-truncated poisson Garima distribution and discuss its various statistical properties. The goodness of fit of power Garima distribution has also been discussed by executing the real data sets and the fit has been found quite satisfactory over one parameter Garima, Lindley and exponential distributions and the two parameter power Lindley and weibull distribution.

2 Weighted Power Garima (WPG) Distribution

The probability density function of power Garima (PG) distribution is given by

$$f(x;\theta,\alpha)=\frac{\alpha\theta}{\theta+2}\left(1+\theta+\theta x^{\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\;;\; x>0,\theta>0,\ \alpha>0 \tag{1}$$

and the cumulative distribution function of power Garima distribution is given by

$$F(x;\theta,\alpha)=1-\frac{\left(\theta x^{\alpha}+\theta+2\right)}{\theta+2};x>0,\theta>0,\alpha>0 \tag{2}$$

Let X be a non-negative random variable with probability density function $f(x)$.Let $w(x)$ be the non-negative weight function, then, the probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Where $w(x)$ be the non-negative weight function and $E(w(x))=\int w(x)f(x)dx<\infty.$

In this paper, we have to obtain the weighted version of power Garima distribution. We have considered the weight function as $w(x)=x^c$ to obtain the weighted power Garima distribution. The probability density function of weighted power Garima distribution is given by

$$f_w(x;\theta,\alpha,c)=\frac{x^c f(x;\theta,\alpha)}{E(x^c)} \tag{3}$$

Where $E(x^c)=\int_0^{\infty}x^c f(x;\theta,\alpha)dx$

$$E(x^c)=\frac{\alpha\theta}{\theta+2}\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{\alpha+c}{\alpha}}}\right) \tag{4}$$

Substitute the equations (1) and (4) in equation (3), we will obtain the probability density function of weighted power Garima distribution

$$f_w(x;\theta,\alpha,c)=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}} \tag{5}$$

and the cumulative distribution function of weighted power Garima distribution is obtained as

$$F_w(x;\theta,\alpha,c)=\int_0^x f_w(x;\theta,\alpha,c)dx$$

$$F_w(x;\theta,\alpha,c)=\int_0^x\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}dx$$

$$F_w(x;\theta,\alpha,c)=\frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\int_0^x x^{\alpha+c-1}\alpha\theta^{\frac{\alpha+c}{\alpha}}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}dx$$

$$F_w(x;\theta,\alpha,c)=\frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}$$

$$\times\left(\alpha\theta^{\frac{\alpha+c}{\alpha}}\int_0^x x^{\alpha+c-1}e^{-\theta x^{\alpha}}dx+\alpha\theta^{\frac{2\alpha+c}{\alpha}}\int_0^x x^{\alpha+c-1}e^{-\theta x^{\alpha}}dx+\alpha\theta^{\frac{2\alpha+c}{\alpha}}\int_0^x x^{2\alpha+c-1}e^{-\theta x^{\alpha}}dx\right) \tag{6}$$

Put $\theta x^{\alpha}=t\Rightarrow x^{\alpha}=\frac{t}{\theta}\Rightarrow x=\left(\frac{t}{\theta}\right)^{\frac{1}{\alpha}}$

Also $\alpha\theta x^{\alpha-1}dx=dt\Rightarrow dx=\frac{dt}{\alpha\theta x^{\alpha-1}}\Rightarrow dx=\frac{dt}{\alpha\theta\left(\frac{t}{\theta}\right)^{\frac{\alpha-1}{\alpha}}}$

After the simplification of equation (6), we obtain the cumulative distribution function of weighted power Garima distribution

$$F_w(x;\theta,\alpha,c) = \frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right) \tag{7}$$

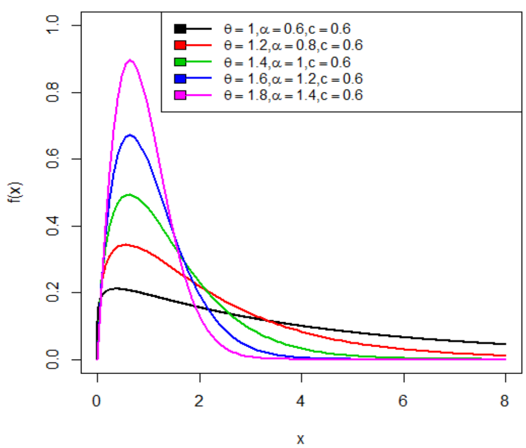

Fig.1:Pdf plot of Weighted Power Garima Distribution

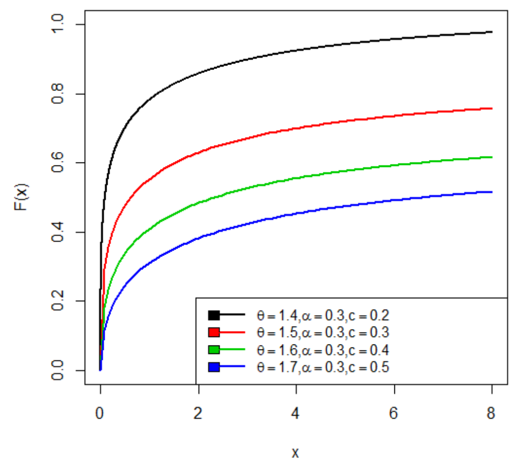

Fig.2:Cdf plot of Weighted Power Garima distribution

3 Reliability Measures

In this section, we have obtained the Reliability function, hazard rate and Reverse hazard rate functions of the proposed weighted power Garima distribution.

3.1 Reliability function

The reliability function is also known as survival function. it can be computed as complement of the cumulative distribution function. The reliability function or the survival function of weighted power Garima distribution is given by

$$R(x) = 1 - F_w(x;\theta,\alpha,c)$$

$$R(x) = 1 - \frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)$$

3.2 Hazard function

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given by

$$h(x)=\frac{f_w(x;\theta,\alpha,c)}{R(x)}$$

$$h(x)=\frac{x^{\alpha+c-1}\,\alpha\theta^{\frac{\alpha+c}{\alpha}}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}}{\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)-\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}$$

3.3 Reverse hazard function

The reverse hazard function of weighted power Garima distribution is given by

$$h_r(x)=\frac{f_w(x;\theta,\alpha,c)}{F_w(x;\theta,\alpha,c)}$$

$$h_r(x)=\frac{x^{\alpha+c-1}\,\alpha\theta^{\frac{\alpha+c}{\alpha}}(1+\theta+\theta x^{\alpha})\,e^{-\theta x^{\alpha}}}{\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\,\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}$$

3.4 Mills Ratio

$$M.R=\frac{1}{h_r(x)}=\frac{\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}{x^{\alpha+c-1}\,\alpha\theta^{\frac{\alpha+c}{\alpha}}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}}$$

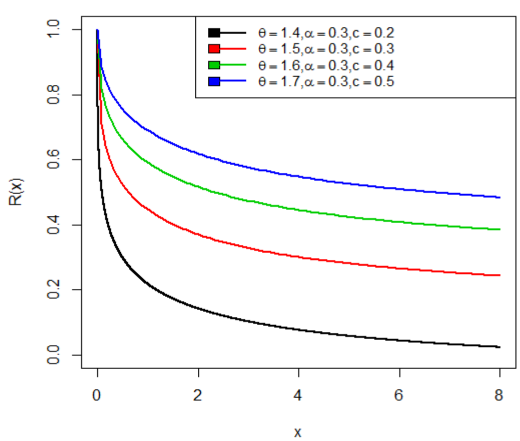

Fig.3:Reliability function of Weighted Power Garima distribution

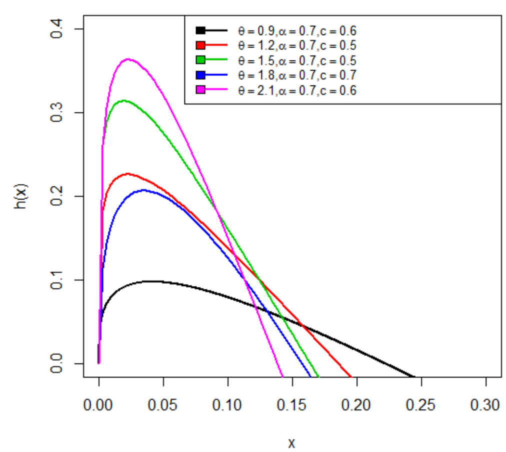

Fig.4:Hazard function of Weighted Power Garima distribution

4 Structural Properties

In this portion, we will discuss the various statistical properties of weighted power Garima distribution especially its moments, harmonic mean, MGF and characteristic function.

4.1 Moments

Let X be the random variable following weighted power Garima distribution with parameters θ, α and c, then the r^{th} order moment $E(X^r)$ of weighted power Garima distribution can be obtained as

$$E(X^r)=\mu_r' = \int_0^\infty x^r f_w(x;\theta,\alpha,c)dx$$

$$E(X^r)=\int_0^\infty \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c+r-1}(1+\theta+\theta x^\alpha)\, e^{-\theta x^\alpha} dx$$

$$=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\int_0^\infty x^{\alpha+c+r-1}(1+\theta+\theta x^\alpha)\, e^{-\theta x^\alpha} dx$$

$$=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\int_0^\infty x^{\alpha+c+r-1} e^{-\theta x^\alpha} dx+\theta\int_0^\infty x^{\alpha+c+r-1} e^{-\theta x^\alpha} dx+\theta\int_0^\infty x^{2\alpha+c+r-1} e^{-\theta x^\alpha} dx\right) \tag{8}$$

Put $x^\alpha = t \quad \Rightarrow x=(t)^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx = dt \quad \Rightarrow dx = \frac{dt}{\alpha x^{\alpha-1}} = \frac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After simplifying equation (8) becomes

$$E(X^r)=\mu_r' = \frac{\Gamma\frac{(\alpha+c+r)}{\alpha}+\theta\Gamma\frac{(\alpha+c+r)}{\alpha}+\Gamma\frac{(2\alpha+c+r)}{\alpha}}{\alpha\theta^{\frac{r}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)} \tag{9}$$

By putting r = 1, 2, 3 and 4 in equation (9), we will obtain the first four moments of weighted power Garima distribution.

$$E(X)=\mu_1'=\frac{\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^2)=\mu_2'=\frac{\Gamma\frac{(\alpha+c+2)}{\alpha}+\theta\Gamma\frac{(\alpha+c+2)}{\alpha}+\Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^3)=\mu_3'=\frac{\Gamma\frac{(\alpha+c+3)}{\alpha}+\theta\Gamma\frac{(\alpha+c+3)}{\alpha}+\Gamma\frac{(2\alpha+c+3)}{\alpha}}{\alpha\theta^{\frac{3}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^4)=\mu_4'=\frac{\Gamma\frac{(\alpha+c+4)}{\alpha}+\theta\Gamma\frac{(\alpha+c+4)}{\alpha}+\Gamma\frac{(2\alpha+c+4)}{\alpha}}{\alpha\theta^{\frac{4}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$\text{Variance}=\frac{\Gamma\frac{(\alpha+c+2)}{\alpha}+\theta\Gamma\frac{(\alpha+c+2)}{\alpha}+\Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}-\left(\frac{\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)^2$$

$$S.D(\sigma)=\sqrt{\left(\frac{\Gamma\frac{(\alpha+c+2)}{\alpha}+\theta\Gamma\frac{(\alpha+c+2)}{\alpha}+\Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}-\left(\frac{\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)^2\right)}$$

$$C.V\left(\frac{\sigma}{\mu_1'}\right)=\sqrt{\left(\frac{\frac{\Gamma(\alpha+c+2)}{\alpha}+\theta\Gamma\frac{(\alpha+c+2)}{\alpha}+\Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\frac{\Gamma(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}-\left(\frac{\frac{\Gamma(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)^2\right)}$$

$$\times\frac{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}{\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}$$

4.2 Harmonic mean

The harmonic mean for the proposed weighted power Garima distribution can be obtained as

$$H.M=E\left(\frac{1}{x}\right)=\int_0^\infty\frac{1}{x}f_W(x;\theta,\alpha,c)dx$$

$$=\int_0^\infty\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-2}\left(1+\theta+\theta x^\alpha\right)e^{-\theta x^\alpha}dx$$

$$=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\int_0^\infty x^{\alpha+c-2}(1+\theta+\theta x^\alpha)e^{-\theta x^\alpha}dx$$

$$=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\int_0^\infty x^{\alpha+c-2}e^{-\theta x^\alpha}dx+\theta\int_0^\infty x^{\alpha+c-2}e^{-\theta x^\alpha}dx+\theta\int_0^\infty x^{2\alpha+c-2}e^{-\theta x^\alpha}dx\right)\tag{10}$$

Put $x^\alpha=t \quad \Rightarrow x=t^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx=dt \quad \Rightarrow dx=\frac{dt}{\alpha x^{\alpha-1}} \quad \Rightarrow dx=\frac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After the simplification of equation (10), we obtain the harmonic mean

$$H.M = \frac{\theta^{\frac{(\alpha+c)}{\alpha}}}{\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\left(\gamma\left(\frac{(\alpha+c-1)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c-1)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(2\alpha+c-1)}{\alpha},\theta x^{\alpha}\right)\right)$$

4.3 Moment generating function and characteristic function

Suppose the random variable X following weighted power Garima distribution with parameters θ, α and c, then the moment generating function of X can be obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f_w(x;\theta,\alpha,c)dx$$

Using Taylor's series, we get

$$= \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+....\right)f_w(x;\theta,\alpha,c)dx$$

$$= \int_0^{\infty}\sum_{j=0}^{\infty}\frac{t^j}{j!}x^j f_w(x;\theta,\alpha,c)dx$$

$$= \sum_{j=0}^{\infty}\frac{t^j}{j!}\mu_j{}'$$

$$= \sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\frac{\Gamma\frac{(\alpha+c+j)}{\alpha}+\theta\Gamma\frac{(\alpha+c+j)}{\alpha}+\Gamma\frac{(2\alpha+c+j)}{\alpha}}{\alpha\theta^{\frac{j}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)$$

$$M_X(t) = \frac{1}{\alpha\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{t^j}{j!\theta^{\frac{j}{\alpha}}}\left(\Gamma\frac{(\alpha+c+j)}{\alpha}+\theta\Gamma\frac{(\alpha+c+j)}{\alpha}+\Gamma\frac{(2\alpha+c+j)}{\alpha}\right)$$

Characteristic function is defined as the function of any real-valued random variable completely defines the probability distribution of a random variable. Similarly, the characteristic function of weighted power Garima distribution can be obtained as

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\alpha\left(\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)} \sum_{j=0}^{\infty} \frac{it^j}{j!\theta^{\frac{j}{\alpha}}} \left(\Gamma\frac{(\alpha+c+j)}{\alpha} + \theta\Gamma\frac{(\alpha+c+j)}{\alpha} + \Gamma\frac{(2\alpha+c+j)}{\alpha}\right)$$

5 Order Statistics

Assume $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represents the order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size n drawn from a continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_X(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \quad (11)$$

Substitute the equations (5) and (7) in equation (11), we will obtain the probability density function of r^{th} order statistics of weighted power Garima distribution which is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c-1}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}\right)$$

$$\times\left(\frac{1}{\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \theta\gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \gamma\left(\frac{(2\alpha+c)}{\alpha}, \theta x^{\alpha}\right)\right)\right)^{r-1}$$

$$\times\left(1 - \frac{1}{\Gamma\frac{(\alpha+c)}{\alpha} + \theta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \theta\gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \gamma\left(\frac{(2\alpha+c)}{\alpha}, \theta x^{\alpha}\right)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of weighted power Garima distribution can be obtained as

$$f_{x(n)}(x)=\frac{n\alpha\,\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}$$

$$\times\left(\frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{n-1} \text{and}$$

the probability density function of first order statistic $X_{(1)}$ of weighted power Garima distribution can be obtained as

$$f_{x(1)}(x)=\frac{n\alpha\,\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}$$

$$\times\left(1-\frac{1}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{n-1}$$

6 Likelihood Ratio Test

Suppose the random sample $X_1, X_2, \ldots, X_n$ of size n drawn from the weighted power Garima distribution. To test the hypothesis, we set up the null and alternative hypothesis.

$$H_o : f(x)=f(x;\theta,\alpha) \text{ against } H_1 : f(x)=f_w(x;\theta,\alpha,c)$$

In order to investigate, whether the random sample of size n comes from the power Garima distribution or weighted power Garima distribution, the following test statistic rule is applied

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\frac{f_w(x;\theta,\alpha,c)}{f(x;\theta,\alpha)}$$

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\left(\frac{x_i^{\,c}\,\alpha\theta^{\frac{c}{\alpha}}\,(\theta+2)}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)$$

$$\Delta=\frac{L_1}{L_o}=\left(\frac{\alpha\theta^{\frac{c}{\alpha}}(\theta+2)}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^n\prod_{i=1}^{n}x_i^{\ c}$$

We should reject the null hypothesis, if

$$\Delta=\left(\frac{\alpha\theta^{\frac{c}{\alpha}}(\theta+2)}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^n\prod_{i=1}^{n}x_i^{\ c}>k$$

Equivalently, we also reject null hypothesis, where

$$\Delta^*=\prod_{i=1}^{n}x_i^{\ c}>k\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{c}{\alpha}}(\theta+2)}\right)^n$$

$$\Delta^*=\prod_{i=1}^{n}x_i^{\ c}>k^*,\ \text{Where}\ k^*=k\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{c}{\alpha}}(\theta+2)}\right)^n$$

When the sample size *n* is large, 2log *Δ* is distributed as chi-square distribution with one degree of freedom and also *p* value is obtained from the chi-square distribution. Also, we refuse to accept the null hypothesis, when the probability value is given by $p\left(\Delta^*>\gamma^*\right)$, Where $\gamma^*=\prod_{i=1}^{n}x_i^{\ c}$ is lower than a specified level of significance and $\prod_{i=1}^{n}x_i^{\ c}$ is the observed value of the statistic Δ^*.

7 Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are also known as income distribution curves or classical curves and are employed in economics to study how wealth is distributed. The bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^q x\,f_w(x;\theta,\alpha,c)dx$$

and $L(p) = pB(p) = \frac{1}{\mu_1'}\int_0^q x\, f_w(x;\theta,\alpha,c)dx$

Where $\mu_1' = \dfrac{\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$ and $q = F^{-1}(p)$

$$B(p) = \frac{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}{p\left(\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}\right)}\int_0^q \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}dx$$

$$B(p) = \frac{\alpha\theta^{\frac{\alpha+c+1}{\alpha}}}{p\left(\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}\right)}\int_0^q x^{\alpha+c}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}dx$$

$$B(p) = \frac{\alpha\theta^{\frac{\alpha+c+1}{\alpha}}}{p\left(\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}\right)}\left(\int_0^q x^{\alpha+c}e^{-\theta x^{\alpha}}dx+\theta\int_0^q x^{\alpha+c}e^{-\theta x^{\alpha}}dx+\theta\int_0^q x^{2\alpha+c}e^{-\theta x^{\alpha}}dx\right) \tag{12}$$

After the simplification of equation (12), we obtain

$$B(p) = \frac{\theta^{\frac{(\alpha+c+1)}{\alpha}}}{p\left(\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}\right)}\left(\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\,\gamma\left(\frac{(2\alpha+c+1)}{\alpha},\theta q\right)\right)$$

$$L(p) = \frac{\theta^{\frac{(\alpha+c+1)}{\alpha}}}{\left(\Gamma\frac{(\alpha+c+1)}{\alpha}+\theta\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(2\alpha+c+1)}{\alpha}\right)}\left(\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\,\gamma\left(\frac{(2\alpha+c+1)}{\alpha},\theta q\right)\right)$$

8 Maximum Likelihood Estimation and Fisher's Information matrix

The method of maximum likelihood estimation is largely applied for estimating the parameters of the distribution. In this section, we will discuss the parameter estimation of weighted power Garima distribution and also discuss its Fisher's information matrix. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from the weighted power Garima distribution, then the likelihood function can be written as

$$L(x)=\prod_{i=1}^{n} f_W(x;\theta,\alpha,c)$$

$$L(x)=\prod_{i=1}^{n}\left(\frac{\alpha\,\theta^{\frac{\alpha+c}{\alpha}}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x_i^{\alpha+c-1}\left(1+\theta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

$$L(x)=\frac{\alpha\theta^{n\left(\frac{\alpha+c}{\alpha}\right)}}{\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)^{n}}\prod_{i=1}^{n}\left(x_i^{\alpha+c-1}\left(1+\theta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

The log likelihood function is given by

$$\log L=n\left(\frac{\alpha+c}{\alpha}\right)\log\alpha\theta-n\log\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$+(\alpha+c-1)\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log(1+\theta+\theta x_i^{\alpha})-\theta\sum_{i=1}^{n}x_i^{\alpha} \qquad (13)$$

The maximum likelihood estimates of θ, α and c can be obtained by differentiating equation (13) with respect to θ, α and c and must satisfy the following normal equations

$$\frac{\partial\log L}{\partial\theta}=\frac{n\alpha}{\alpha\theta}\left(\frac{\alpha+c}{\alpha}\right)-n\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)+\sum_{i=1}^{n}\left(\frac{\left(1+x_i^{\alpha}\right)}{\left(1+\theta+\theta x_i^{\alpha}\right)}\right)-\sum_{i=1}^{n}x_i^{\alpha}=0$$

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n\theta c}{\alpha^{3}\theta}-n\psi\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)+\sum_{i=1}^{n}\log x_i$$

$$+\sum_{i=1}^{n}\left(\frac{\theta x_i^{\alpha}\log x_i}{\left(1+\theta+\theta x_i^{\alpha}\right)}\right)-\theta\sum_{i=1}^{n}x_i^{\alpha}\log x_i=0$$

$$\frac{\partial\log L}{\partial c}=\frac{n}{\alpha}\log\alpha\theta-n\psi\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)+\sum_{i=1}^{n}\log x_i=0$$

Where ψ (.) is the digamma function.

It is important to mention here that the analytical solution of the above non-linear system of equations are unknown. The likelihood equations are too complicated to solve it algebraically, therefore we use numerical technique like Newton-Raphson method for estimating the parameters of the proposed distribution.

To obtain the confidence interval, we use the asymptotic normality results. We have that if $\hat{\lambda}=\left(\hat{\theta},\hat{\alpha},\hat{c}\right)$ denotes the MLE of $\lambda=\left(\theta,\alpha,c\right)$, we can state the result as follows

$$\sqrt{n}(\hat{\lambda}-\lambda)\rightarrow N_3\left(0,I^{-1}(\lambda)\right)$$

Where $I(\lambda)$ is Fisher's information matrix is given by

$$I(\lambda)=-\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2\log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2\log L}{\partial\theta\,\partial\alpha}\right) & E\left(\frac{\partial^2\log L}{\partial\theta\,\partial c}\right) \\ E\left(\frac{\partial^2\log L}{\partial\alpha\,\partial\theta}\right) & E\left(\frac{\partial^2\log L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2\log L}{\partial\alpha\,\partial c}\right) \\ E\left(\frac{\partial^2\log L}{\partial c\,\partial\theta}\right) & E\left(\frac{\partial^2\log L}{\partial c\,\partial\alpha}\right) & E\left(\frac{\partial^2\log L}{\partial c^2}\right) \end{pmatrix}$$

Here we have to show that

$$E\left(\frac{\partial^2\log L}{\partial\theta^2}\right)=-\frac{n\alpha^2}{(\alpha\theta)^2}\left(\frac{\alpha+c}{\alpha}\right)+n\left(\frac{\left(\Gamma\frac{(\alpha+c)}{\alpha}\right)^2}{\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)^2}\right)-\sum_{i=1}^{n}\left(\frac{\left(1+x_i^{\alpha}\right)^2}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right)$$

$$E\left(\frac{\partial^2\log L}{\partial\alpha^2}\right)=\frac{3n\alpha^2\theta^2 c}{(\alpha^3\theta)^2}-n\psi'\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$+\sum_{i-1}^{n}\left(\frac{\left(1+\theta+\theta x_i^{\alpha}\right)\theta x_i^{\alpha}(\log x_i)^2-\left(\theta x_i^{\alpha}\log x_i\right)^2}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right)-\theta\sum_{i=1}^{n}\left(x_i^{\alpha}\log x_i\right)^2$$

$$E\left(\frac{\partial^2\log L}{\partial c^2}\right)=-n\psi'\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial c}\right) = \frac{n\alpha}{\alpha^2\theta} - n\psi\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial\alpha\,\partial c}\right) = -\frac{n\theta}{\alpha^3\theta} - n\psi'\left(\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial\alpha}\right) = -n\psi\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}}{\Gamma\frac{(\alpha+c)}{\alpha}+\theta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right) + \sum_{i=1}^{n}\left(\frac{\left(1+\theta+\theta x_i^{\alpha}\right)x_i^{\alpha}\log x_i - \theta x_i^{\alpha}\log x_i\left(1+x_i^{\alpha}\right)}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right)$$

$$-\sum_{i=1}^{n} x_i^{\alpha}\log x_i$$

where $\psi(.)'$ is the first order derivative of diagamma function. Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence interval for θ, α, c.

9 Simulation Study

In this section, we study the performance of ML estimators for different sample sizes (n=80, 120, 160, 200, 280, 560). We have employed the inverse CDF technique for data simulation for weighted power Garima distribution using R software. The process was repeated 560 times for calculation of bias, variance and MSE. For different values of parameters of weighted power Garima distribution, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite well, consistent in case of weighted power Garima distribution.

Table 1: Bias, Variance and MSE for ML estimators for different sample sizes.

Sample size n	$\theta = 0.3$			$\alpha = 1.2$			$c = 0.2$		
	Bias	**variance**	**MSE**	**Bias**	**variance**	**MSE**	**Bias**	**variance**	**MSE**
80	0.9532365	0.3139122	1.222572	0.05867648	0.2510553	0.2544982	1.053236	0.3139122	1.423219
120	0.7915065	0.03813913	0.6646217	0.4070199	0.692372	0.1657222	0.8915065	0.03813913	0.8329231
160	0.7440034	0.006274372	0.5598154	0.4047172	0.061505	0.1638166	0.8440034	0.006274372	0.7186161

200	0.7749686	0.008642473	0.6092189	0.4031533	0.0062341	0.1625689	0.8749686	0.008642473	0.7742126
280	0.748591	0.007921779	0.5683103	0.4056607	0.00074154	0.164598	0.848591	0.007921779	0.7280285
560	0.733438	0.001847261	0.5397786	0.4058674	0.0006909	0.1647395	0.833438	0.001847261	0.6964662
	$\theta = 0.5$			$\alpha = 1.4$			$c = 0.3$		
	Bias	**variance**	**MSE**	**Bias**	**variance**	**MSE**	**Bias**	**variance**	**MSE**
80	1.478625	0.04976946	2.2361	0.4520027	0.0003681144	0.2046745	1.678625	0.04976946	2.86755
120	0.662273	0.0187689	1.881922	0.4530738	0.000219362	0.2033641	1.562273	0.1187689	1.586831
160	0.635395	0.03877655	1.713294	0.4526063	0.000147532	0.2049199	1.535395	0.03877655	1.407452
200	0.529199	0.02530183	1.383753	0.4536643	0.00013819	0.2058528	1.529199	0.004530183	1.035433
280	0.634507	0.03594834	0.70756	0.4514655	0.00017355	0.2038528	1.514507	0.003594834	1.00401363
560	0.557719	0.01684757	0.443335	0.4513908	0.0000184343	0.2037755	1.457719	0.001684757	1.00106423
	$\theta = 0.8$			$\alpha = 1.7$			$c = 0.5$		
	Bias	**variance**	**MSE**	**Bias**	**variance**	**MSE**	**Bias**	**variance**	**MSE**
80	1.817406	0.5148398	1.08743	0.5190564	0.0002460776	0.2696656	1.117406	0.00148398	1.00046787
120	1.468621	0.4440497	1.047538	0.5097031	0.0001156451	0.2599129	1.1168621	0.00040497	1.00034655
160	1.458263	0.435114	1.0038641	0.5110745	0.0002014048	0.2513986	1.115826	0.00035114	1.00006259
200	1.3867615	0.4340399	1.0039248	0.5131615	0.0002764351	0.2636112	1.167615	0.004340399	1.000010305

280	1.378769	0.1648802	1.0011602	0.5125025	0.00020519	0.2628458	1.166769	0.00164880	1.00001028
560	1.245795	0.02298	1.0003148	0.513904	0.00017580	0.261126	1.1445795	0.02298	1.00000228
	$\theta = 1.0$			$\alpha = 1.9$			$c = 0.7$		
	Bias	**variance**	**MSE**	**Bias**	**variance**	**MSE**	**Bias**	**variance**	**MSE**
80	1.151041	0.0186171	1.00042085	0.5534586	0.0005376269	0.3068541	1.151041	0.186171	1.00002147
120	1.1852845	0.0543703	1.0007995	0.5631392	0.0003983134	0.3055241	1.1510845	0.143703	0.0004012
160	1.131413	0.0459895	1.0030899	0.5630787	0.0002931603	0.3173508	1.031413	0.1429895	0.0003784
200	1.129185	0.0156614	1.0018573	0.5620905	0.000253111	0.3172873	1.021854	0.141614	0.0002685
280	1.1268762	0.01462277	1.0013484	0.5582505	0.0001964522	0.3131051	1.018762	0.140622	0.0002661
560	1.1255107	0.0136976	1.0012701	0.5574596	0.0001196933	0.3030347	1.0175107	0.1383697	0.0001165

10 Data Investigation

In this section, we have analysed and evaluated the two real-life data sets for fitting weighted power Garima distribution in order to show that the weighted power Garima distribution fits better over power Garima, Garima, exponential and Lindley distributions. The two real life data sets are listed below as

Data set 1: The following data set in table 2 represents the bladder cancer patients (n = 128) of the remission times (in months) reported by Lee and Wang [27].

Table 2: Data represents the blood cancer patients reported by Lee and Wang [27]

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	6.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33

5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64
17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40
5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Data set 2: The data set in table 3 represents lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clarke [16].

Table 3: Data reported by Gross and Clarke [16] represents the relief times (in minutes) of 20 patients

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3	1.7	2.3	1.6	2

To compute the model comparison criterion values along with the unknown parameters, the technique of R software is employed. In order to compare the weighted power Garima distribution with power Garima, Garima, exponential and Lindley distributions, we apply the criterions like Bayesian Information criterion (*BIC*), Akaike Information Criterion (*AIC*), Akaike Information Criterion Corrected (*AICC*), Consistent Akaike Information Criterion (*CAIC*), *-2logL* and Shannon's entropy *H(X)*. The better distribution is which corresponds to the lesser values of *AIC, BIC, AICC, CAIC, -2logL* and *H(X)*. For the calculation of *AIC*, *BIC*, *AICC*, *CAIC*, *-2logL* and *H(X)*, the most convenient formulas are used

{EMBED Equation.3}

Where k is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of the log-likelihood function under the considered model.

Table 4 shows values of ML Estimates and S. E estimates for Data Set 1 and Data set 2

Data sets	**Distributions**	**MLE**	**S. E**
	Weighted Power Garima	$\hat{\alpha} = 0.67329566$ $\hat{\theta} = 0.43560496$ $\hat{c} = 0.40310408$	$\hat{\alpha} = 0.03274985$ $\hat{\theta} = 0.08670316$ $\hat{c} = 0.17003955$
	Power Garima	$\hat{\alpha} = 0.94497220$ $\hat{\theta} = 0.18324081$	$\hat{\alpha} = 0.05965188$ $\hat{\theta} = 0.03212646$

1	**Garima**	$\hat{\theta} = 0.15819553$	$\hat{\theta} = 0.01223297$
	Exponential	$\hat{\theta} = 0.108588071$	$\hat{\theta} = 0.009597106$
	Lindley	$\hat{\theta} = 0.1991393$	$\hat{\theta} = 0.0125326$
2	**Weighted Power Garima**	$\hat{\alpha} = 1.055664$ $\hat{\theta} = 4.655777$ $\hat{c} = 7.936450$	$\hat{\alpha} = 0.116922$ $\hat{\theta} = 1.813480$ $\hat{c} = 2.973528$
	Power Garima	$\hat{\alpha} = 2.48197053$ $\hat{\theta} = 0.23373012$	$\hat{\alpha} = 0.37233074$ $\hat{\theta} = 0.09193612$
	Garima	$\hat{\theta} = 0.7395722$	$\hat{\theta} = 0.1405294$
	Exponential	$\hat{\theta} = 0.5263164$	$\hat{\theta} = 0.1176875$
	Lindley	$\hat{\theta} = 0.8161188$	$\hat{\theta} = 0.1360929$

Table 5 shows criterion values (AIC, BIC, AICC, CAIC), -2logL & Shannon's entropy H(X) and Performance of fitted distributions.

Data sets	**Distributions**	**-2logL**	**AIC**	**BIC**	**AICC**	**CAIC**	**H(X)**
1	**Weighted Power Garima**	733.7792	739.7792	748.3353	739.9727	739.9727	5.7326
	Power Garima	825.6614	829.6614	835.3654	829.7574	829.7574	6.4504
	Garima	826.4914	828.4914	831.3434	828.5231	828.5231	6.4569
	Exponential	824.3768	826.3768	829.2289	826.4085	826.4085	6.4404
	Lindley	833.7925	835.7925	838.6445	835.8242	835.8242	6.5140
	Weighted Power Garima	35.47266	41.47266	44.45986	42.9726	42.9726	1.7736

2	**Power Garima**	41.90287	45.90287	47.89434	46.6087	46.6087	2.0951
	Garima	63.21155	65.21155	66.20728	65.4337	65.4337	3.1605
	Exponential	65.67416	67.67416	68.66989	67.8963	67.8963	3.2837
	Lindley	60.4991	62.4991	63.49483	62.7213	62.7213	3.0249

From table 5 given above, it can be clearly observed and seen from the results that the weighted power Garima distribution have the lesser *AIC*, *BIC*, *AICC*, *CAIC*, *-2logL* and *H(X)* values as compared to the power Garima, Garima, Exponential and Lindley distributions, which indicates that the weighted power Garima distribution fits better than the power Garima, Garima, exponential and Lindley distributions. Hence, it can be concluded that the weighted power Garima distribution leads to a better fit over power Garima, Garima, exponential and Lindley distribution

11 Conclusions

In this manuscript, we have introduced a new distribution namely weighted power Garima distribution and the executed new distribution has three parameters. The proposed model is obtained by using the weighted technique given by the Fisher. Its various structural properties including its moments, harmonic mean, moment generating function, characteristic function, reliability function, hazard rate function, reverse hazard rate function, order statistics, bonferroni and lorenz curves have been presented. The estimation of model parameters of the new distribution have been studied by employing the technique of maximum likelihood estimation. The simulation investigation has been carried out to describe the performance on the basis of different values of parameters and the result shows that the bias, variance and MSE decreases, whenever the sample size increases. The application of the weighted power Garima distribution have also been demonstrated by using the two real life data sets from medical sciences and finally the results are compared over power Garima, Garima, exponential and Lindley distributions and the results revealed that the weighted power Garima distribution provides a better fit over power Garima, Garima, exponential and Lindley distributions.

References

[1] Abebe, B., Tesfay, M., Eyob, T. and Shanker, R. A Two Parameter Power Garima distribution with Properties and Applications, Annals of Biostatistics and Biometric Applications, 1(3), 1-7 (2019).

[2] Alqallaf, F., Ghitany, M. E. and Agostinelli, C. Weighted Exponential distribution: Different methods of Estimations, Applied Mathematical and Information Science, 9(3), 1167-1173 (2015).

[3] Bashir, S. and Rasul, M. Some Properties of the Weighted Lindley distribution, EPRA International Journal of Economic and Business Review, 3(8), 11-17 (2015).

[4] Castillo, J. D. and Casany, M. P. Weighted Poison Distribution for over dispersion and under dispersion Situations, Annals of the Institute of Statistical Mathematics, 50(3), 567-585 (1998).

[5] Cox, D. R. Some sampling problems in technology. In: Johnson, N. L. and Smith, H., Eds., New Developments in Survey Sampling, John Wiley & Sons, New York, 506-527 (1969).

[6] Dar, A. A., Ahmed, A. and Reshi, J. A. Weighted Gamma-Pareto distribution and its Application, Pak.J.Statist., 36(4), 287-304 (2020).

[7] Dar, A. A., Ahmed, A. and Reshi, J. A. An extension of power distribution, Revista Investigation Operacional, 39(4), 626-638 (2018).

[8] Das, S. and Kundu, D. On weighted exponential distribution and its length biased version, Journal of the Indian Society for Probability and Statistics, 17(1), 57-77 (2016).

[9] Fatima, K. and Ahmad, S. P. Weighted Inverse Rayleigh distribution, International Journal of Statistics and Systems, 12(1), 119-137 (2017).

[10] Fay, D., Haddadi, H., Uhlig, S., Moore, A. W., Mortier, R. and Jamakovic, A. Weighted spectral distribution, Technical Report, UCAM-CL-TR-729, ISSN: 1476-2986 (2008).

[11] Fisher, R. A. The effects of methods of ascertainment upon the estimation of frequencies, Annals of Eugenics, 6, 13-25 (1934).

[12] Ganaie, R. A., Rajagopalan, V. and Rather, A. A. Weighted new quasi Lindley distribution with Properties and Applications, Journal of Xi'an University of Architecture and Technology, 12(2), 561- 9575 (2020).

[13] Ganaie, R. A., Rajagopalan, V. and Aldulaimi, S. The Weighted Power Shanker distribution with Characterizations and Applications of real life time data, Journal of Statistics Applications and Probability, 10(1), 245-265 (2021).

[14] Ghitany, M. E., Alqallaf, F., Al-Mutairi, D. K. and Husain, H. A. A Two-Parameter Weighted Lindley distribution and its applications to Survival data, Mathematics and Computers in Simulation, 81(6), 1190-1201 (2011). doi:10.1016/j.matcom.2010.11.005

[15] Gove, J. H. Environmental and Ecological Statistics, 10(4), 455-467 (2003).

[16] Gross, A. J. and Clark, V. A. Survival Distributions: Reliability Applications in the Biometrical Sciences, John Wiley, New York. (1975).

[17] Gupta, R. C. and Keating, J. P. Relations for Reliability Measures under Length Biased Sampling, Scandinavian Journal of Statistics, 13, 49-56 (1985).

[18] Gupta, R. C. and Kirmani, S. N. The role of weighted distributions in stochastic modeling, Communication in Statistics - Theory and Methods, 19(9), 3147-3162 (1990).

[19] Gupta, R. D. and Kundu, D. A. A New class of weighted exponential distributions, Statistics, 43(6), 621-634 (2009).

[20] Hassan, A., Dar, M. A., Peer, B. A. and Para, B. A. A New Generalization of Pranav distribution using weighted technique, International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(1), 25-32 (2019).

[21] Hassan, A., Shalbaf, G. A. and Para, B. A. On three Parameter Weighted Quasi Akash Distribution: Properties and Applications, IOSR Journal of Engineering (IOSRJEN), 08(11), 01-10 (2018).

[22] Hassan, A., Wani, S. A. and Para, B. A. On three Parameter Weighted Quasi Lindley Distribution: Properties and Applications, International Journal of Scientific Research in Mathematical and Statistical Sciences, 5(5), 210-224 (2018).

[23] Kersey, J. X. Weighted Inverse Weibull and Beta-Inverse Weibull distribution, M.SC. Thesis, university of Georgia Southern (2010).

[24] Khatree, R. Characterization of Inverse-Gaussian and Gamma distributions through their length biased distribution, IEEE Transactions on Reliability, 38(5), 610-611 (1989).

[25] Khan, M. N., Saeed, A. and Alzaatreh, A. Weighted Modified Weibull distribution, Journal of Testing and Evaluation, 47(5), 3751-3764 (2019).

[26] Kilany, N. M. Weighted Lomax distribution, SpringerPlus, 5(1) (2016).

[27] Lee, E. T. and Wang, J. W. Statistical Methods for Survival Data Analysis. 3rd ed. New York, NY, USA: John Wiley & Sons, 512 (2003).

[28] Mahday, M. A class of Weighted Weibull distributions and its Properties, Studies in Mathematical Science, 6, 35-45 (2013).

[29] Odubote, O. and Oluyede, B. O. Theoretical properties of the weighted Feller-Pareto and Related Distributions, Asian Journal of Mathematics and Applications, ama0173, 1-12 (2014).

[30] Para, B. A. and Jan, T. R. On three Parameter Weighted Pareto Type II Distribution: Properties and Applications in Medical Sciences, Applied Mathematics and Information Sciences Letters, 6(1), 13-26 (2018).

[31] Patil, G. P. and Rao, C. R. Weighted Distributions and Size-Biased Sampling with Applications to Wildlife Populations and Human Families, Biometrics, 34(2), 179-189 (1978).

[32] Rao, C. R. On discrete distributions arising out of method of ascertainment, in classical and Contagious Discrete, G.P. Patil. ed; Pergamon Press and Statistical publishing Society,Calcutta. 320-332 (1965).

[33] Rather, A. A. and Subramanian, C. On Weighted Sushila distribution with properties and its applications, International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(1), 105-117 (2019).

[34] Rather, A. A. and Ozel, G. The Weighted Power Lindley distribution with applications on the life time data, Pak.j.stat.oper.res., 16(2), 225-237 (2020).

[35] Roman, R. Theoretical Properties and Estimation in Weighted Weibull and Related Distributions, Thesis submitted to Graduate faculty of Georgia southern university, (2010).

[36] R core team. R version 3.5.3: A language and environment for statistical computing. R Foundation for statistical computing, Vienna, Austria.URL https:// www.R-project .org/ (2019).

[37] Shanker, R. The discrete Poisson-Garima distribution, Biometrics and Biostatistics International Journal, 5(2), 48-53 (2017).

[38] Shanker, R and Shukla, K. K. Size-biased Poisson-Garima Distribution with Applications, Biometrics and Biostatistics International Journal, 6(3), 335-340 (2017).

[39] Shanker, R. and Shukla, K. K. Zero-Truncated Poisson-Garima Distribution with Applications, Biostatistics and Biometrics, 3(1), 0014-019 (2017).

[40] Shanker, R. Garima distribution and Its Application to Model Behavioral Science Data, Biometrics and Biostatistics International Journal, 4(7), 1-9 (2016).

[41] Shi, X., Oluyede, B. O. and Pararai, M. Theoretical Properties of Weighted Generalized Rayleigh and Related Distributions, Theoretical Mathematics and Applications, 2(2), 45-62 (2012).

[42] Van Deusen, P.C. Fitting assumed distributions to horizontal point sample diameters, Forest Science, 32, 146-148 (1986).

[43] Warren, W. G. Statistical distributions in forestry and forest products research. In: G.P. Patil, S. Kotz, J. K. Ord, (eds) Statistical Distributions in Scientific Work, 2, 369-384 (1975).

[44] Ye, Y., Oluyede, B. O. and Pararai, M. Weighted Generalized Beta Distribution of the Second Kind and Related Distributions, Journal of Statistical and Econometric Methods, 1(1), 13-31 (2012).

[45] Zelen, M. Problems in cell kinetic and the early detection of disease, Reliability and Biometry, 56(3), 701-726 (1974).